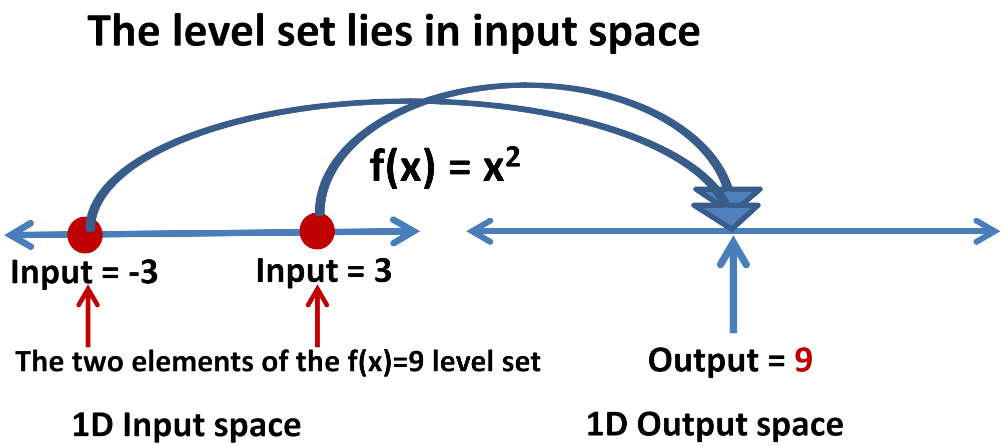
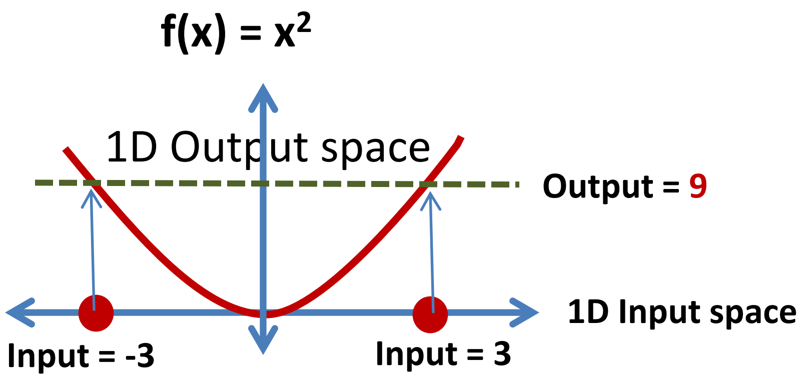
Section 14.1: Review of the Idea of Level Sets

Definition : **Given f(x), we call the "k" level set the set of all inputs that get sent to the output value of k.**





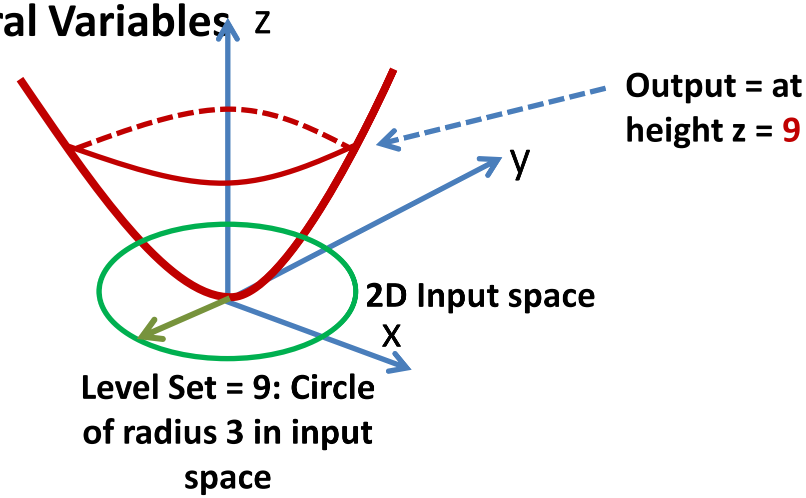
Section 14.1: Functions of Several Variables

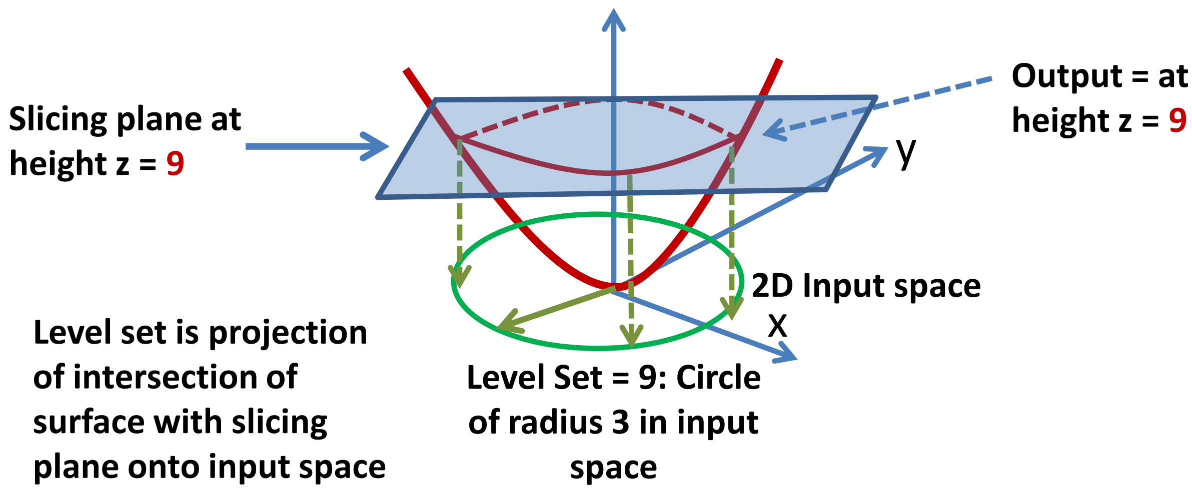
$f(x,y) = x^2 + y^2$ What is the level set $f(x,y) = 9$?

Analytic answer:

it is the set of all input points that get sent to the output 9

Geometric answer: if you slice the graph with a plane at height k, and project the result down to the input plane, the points you get are the k level set.

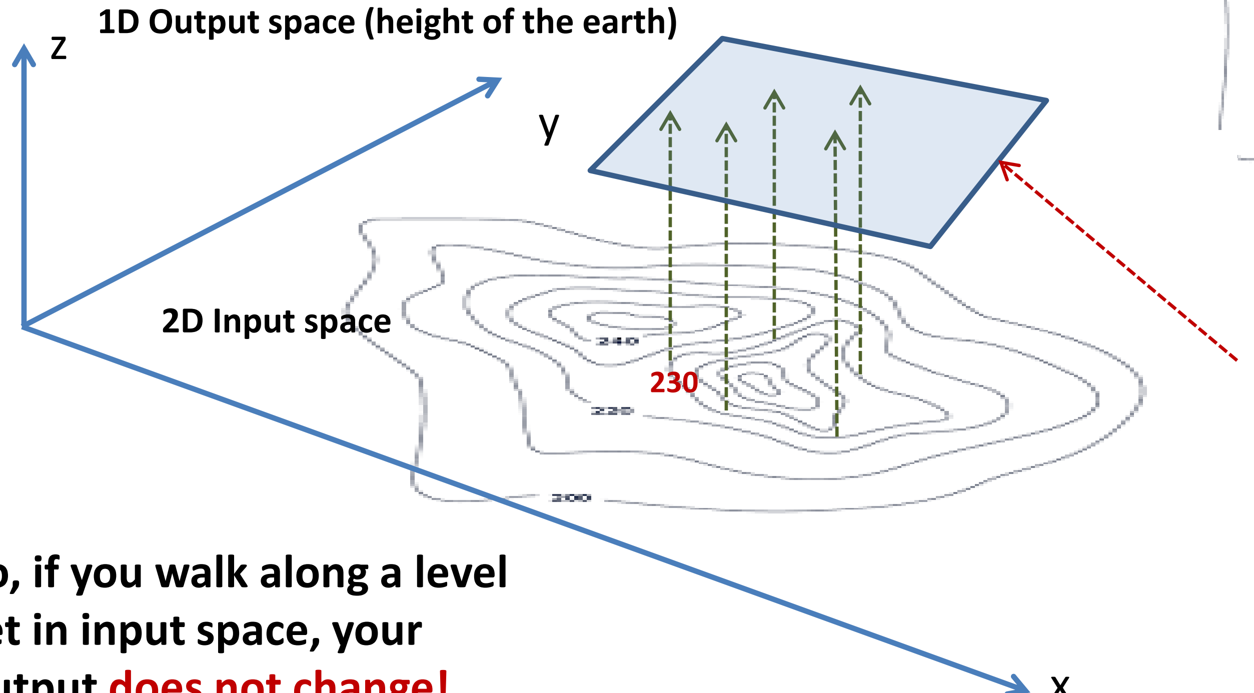


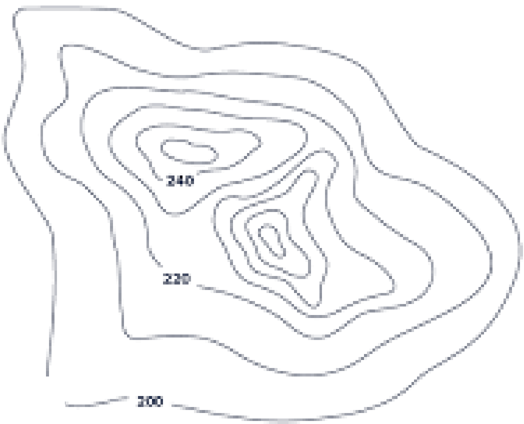


Section 14.1: Functions of Several Variables

And that's what a topographic map is!

A topographic gives you different k-level sets in the input space, showing what the height of the earth would be. Contour lines show points of the same elevation.





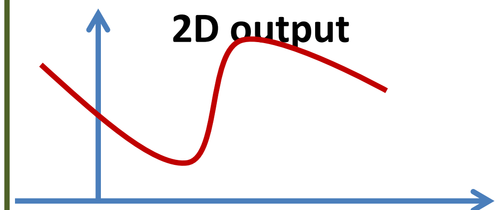
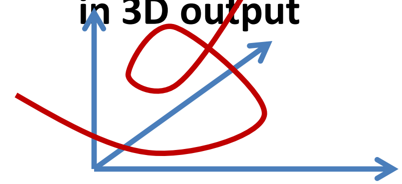
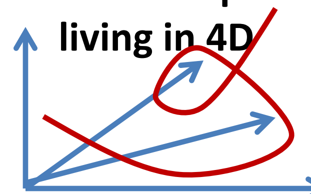
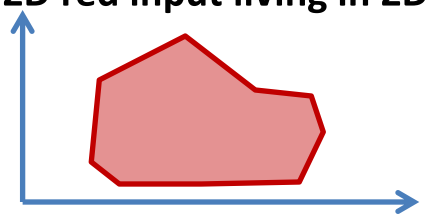
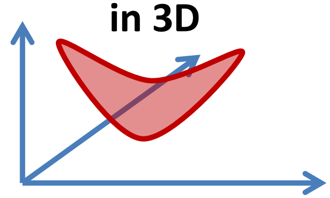
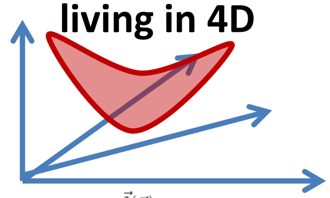
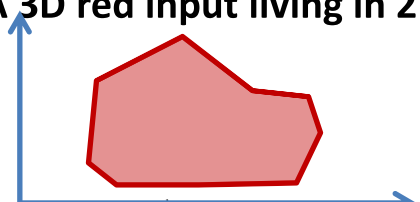
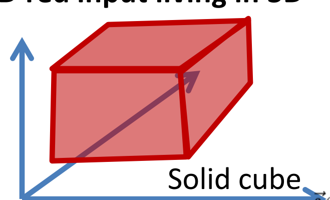
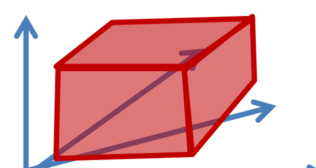
All sent to same height z = 230

So, if you walk along a level set in input space, your output does not change!

Topo map from https://datavizproject.com/data-type/topographic-map/

Let's make sure we agree about what we mean about dimension

Dimension of output space (columns) / **Dimension of input space** (rows)

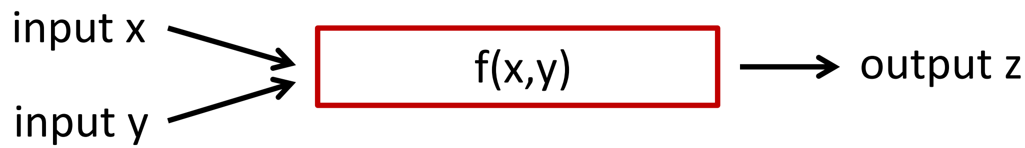
	1D output space	2D output space	3D output space	4D output space
1D input space	A 1D red input living in 1D output  $f(x) \equiv f(x) : \mathbb{R}^1 \to \mathbb{R}^1$	A 1D red input living in 2D output  $\vec{f}(x) \equiv (f(x), g(x)) : \mathbb{R}^1 \to \mathbb{R}^2$	A 1D red input living in 3D output  $\vec{f}(x) \equiv (f(x), g(x), h(x)) : \mathbb{R}^1 \to \mathbb{R}^3$	A 1D red input living in 4D  $\vec{f}(x) \equiv (f(x), g(x), h(x), i(x))$ $\mathbb{R}^1 \to \mathbb{R}^4$
2D input space	A 2D red input living in 1D  $f(\vec{x}) \equiv f(x_1, x_2) :$ $\mathbb{R}^2 \to \mathbb{R}^1$	A 2D red input living in 2D  $\vec{f}(\vec{x}) \equiv (f(x_1, x_2), g(x_1, x_2))) :$ $\mathbb{R}^2 \to \mathbb{R}^2$	A 2D red input living in 3D  $\vec{f}(\vec{x}) \equiv (f(x_1, x_2), g(x_1, x_2), h(x_1, x_2))) :$ $\mathbb{R}^2 \to \mathbb{R}^3$	A 2D red input living in 4D  $\vec{f}(\vec{x}) \equiv$ $(f(x_1, x_2), g(x_1, x_2), h(x_1, x_2) i(x_1, x_2)) :$ $\mathbb{R}^2 \to \mathbb{R}^4$
3D input space	A 3D red input living in 1D  $f(\vec{x}) \equiv f(x_1, x_2, x_3) :$ $\mathbb{R}^3 \to \mathbb{R}^1$	A 3D red input living in 2D  $\vec{f}(\vec{x}) \equiv$ $(f(x_1, x_2, x_3), g(x_1, x_2, x_3))) :$ $\mathbb{R}^3 \to \mathbb{R}^2$	A 3D red input living in 3D  Solid cube $\vec{f}(\vec{x}) \equiv (f(x_1, x_2, x_3), g(x_1, x_2, x_3), h(x_1, x_2, x_3)))$ $\mathbb{R}^3 \to \mathbb{R}^3$	A 3D red input in 4D  $\vec{f}(\vec{x}) \equiv (f(x_1, x_2, x_3), g(x_1, x_2, x_3),$ $h(x_1, x_2, x_3), i(x_1, x_2, x_3)) :$ $\mathbb{R}^3 \to \mathbb{R}^4$

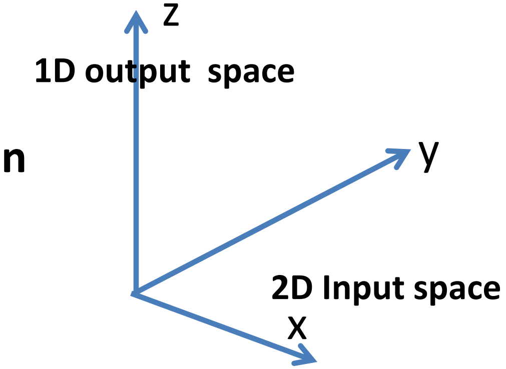
Section 14.1: Functions of Several Variables

Another example: $f(x,y) = x^2 + 4y^2$ This is a function from 2D to 1D

$$f(\vec{x}) \equiv f(x,y) = z : \mathbb{R}^2 \rightarrow \mathbb{R}^1$$

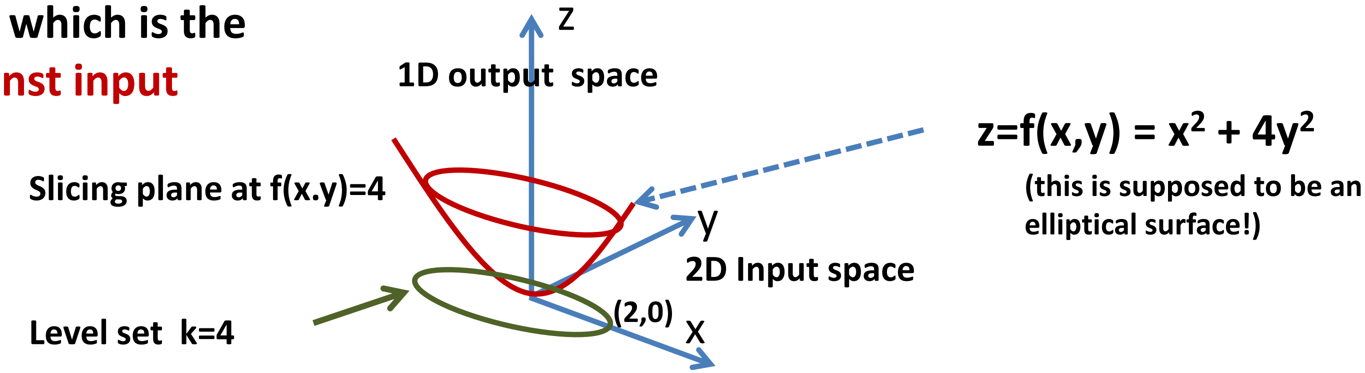
There are two free input variables:
x and y,
which together live in two-dimensional input space





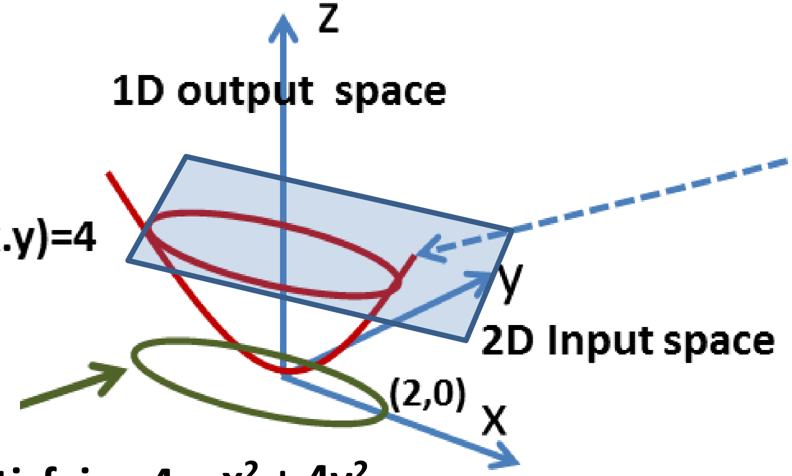
We can plot the output using a third dimension

and then use the equation to link output space to input space, generating a surface which is the graph of output against input



Section 14.1: Functions of Several Variables

The level set k=4



$z=f(x,y) = x^2 + 4y^2$

(this is supposed to be an elliptical surface!)

=Curve in x-y plane satisfying $4 = x^2 + 4y^2$

Let's really talk about *dimensions*

For a mapping $f(\vec{x}) \equiv f(x,y) = z : \mathbb{R}^2 \rightarrow \mathbb{R}^1$ **from 2D input to 1D output**

- **It requires three dimensions to contain the graph of output against input**
- **The graph *itself* is a 2D surface living in 3D space**
- **The level set is a 1D curve in 2D input space**
- **Again—the level curve satisfying $4= x^2 + 4y^2$ is 1D curve, because it only has 1 free variable (as x changes, that nails down y).**

Again, for a mapping $f(\vec{x}) \equiv f(x,y) = z : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

We graph it in 3D. The thing itself is 2D. The level set is 1D.

Section 14.1: Functions of Several Variables

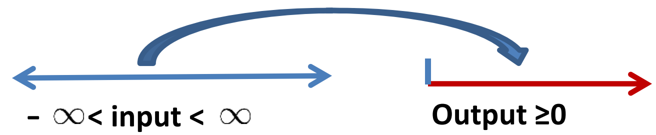
Again, for a mapping $f(\vec{x}) \equiv f(x,y) = z : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

We graph it in 3D. The thing itself is 2D. The level set is 1D.

Let's go down a dimension to see if this is still true:

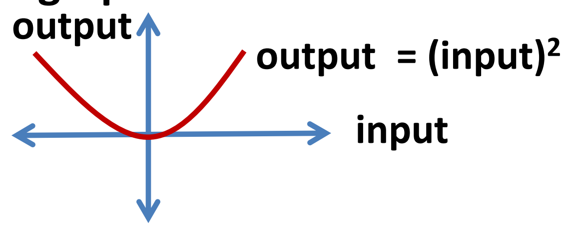
Example: f(x) = x^2

(1) This is mapping from 1D to 1D $f(x) \equiv x^2 : \mathbb{R}^1 \rightarrow \mathbb{R}^1$



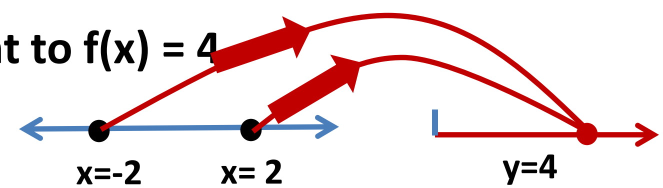
(2) There is one free input variable, so the graph itself is 1D.

(3) We can add a second dimension y and graph it in $\mathbb{R}^2$



(4) The level set k=4 is the set of all points that get sent to f(x) = 4

Those two points are zero-dimensional

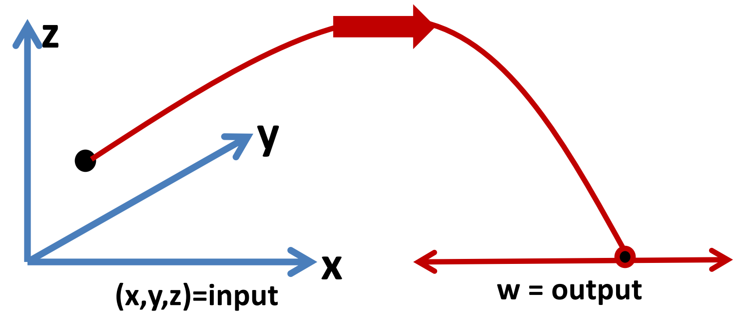


So again: We graph it in 2D. The thing itself is 1D. The level set is 0D.

Section 14.1: Functions of Several Variables

Okay—now I am getting excited—let's go up a dimension

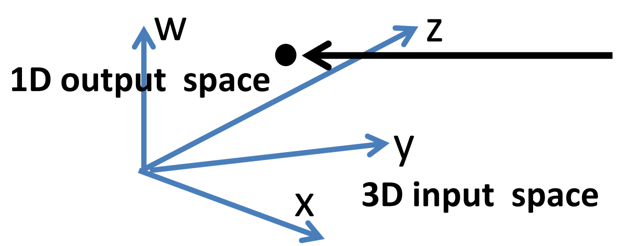
Example: $f(x,y,z) = x^2 + y^2 + z^2$



(1) This is a 3D function

$$f(\vec{x}) \equiv f(x,y,z) = w : \mathbb{R}^3 \rightarrow \mathbb{R}^1$$

(2) We need a fourth dimension to graph it



This point is part of the graph of the 3D surface (x,y,z,f(x,y,z)) which lives in 4D space

(3) And the k level set is the set of all points (x,y,z) such that $k= f(x,y,z) = x^2 + y^2 + z^2$ which is a sphere of radius $\sqrt{k}$

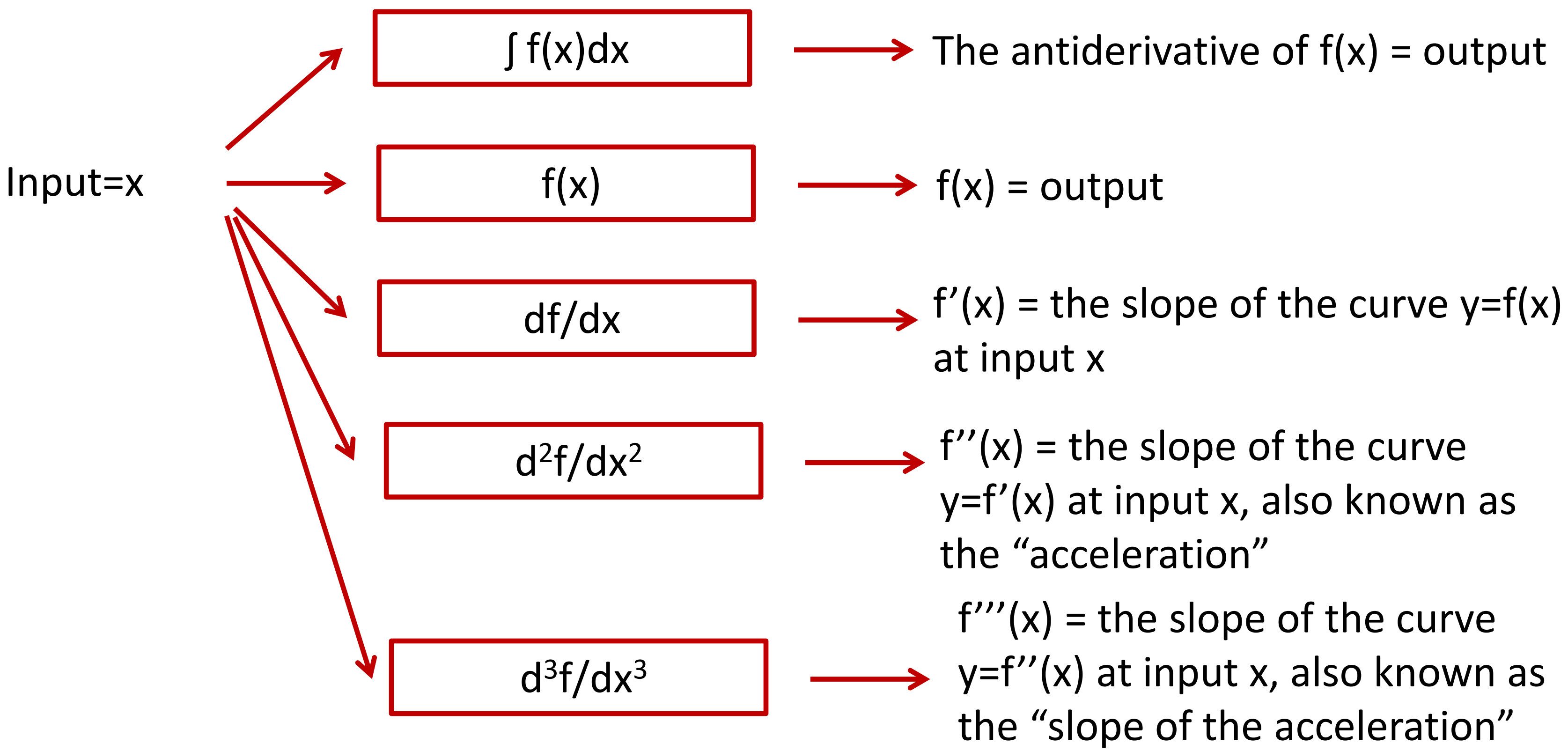
(4) this is a 2D object, since there are only 2 free variables

So again: We graph it in 4D. The thing itself is 3D. The level set is 2D.

Read Section 14.2 on your own---Section 14.3: On to derivatives!

Definition: First, let's remember 1D calculus: Given f(x), we define the derivative of f(x) with respect to x

$$\frac{df}{dx} \equiv f' \equiv \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$



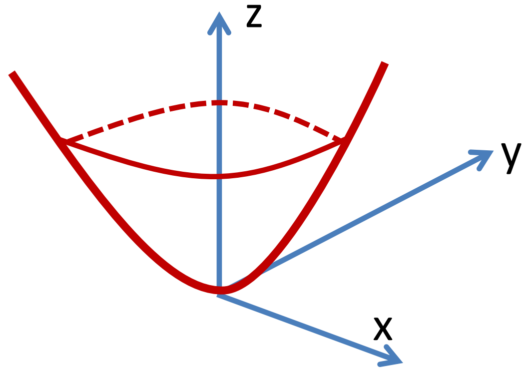
Section 14.3: On to derivatives--Multivariables!

Definition: Given f(x,y), we define the partial derivative of f(x,y) with respect to x as the derivative of f as x changes, assuming y is constant.

Notation: partial derivative $\frac{\partial f(x,y)}{\partial x} \equiv \frac{\partial f}{\partial x} \equiv f_x(x,y)$

Example: f(x,y) = $x^2 + 3y^2$

$$f_x = (2x)$$
$$f_y = (6y)$$



Example: f(x,y) = $12x^3y + x^2 + 3y^2$

$$f_x = (36x^2y + 2x)$$
$$f_y = (12x^3 + 6y)$$

Example: f(x,y) = sin (xy)

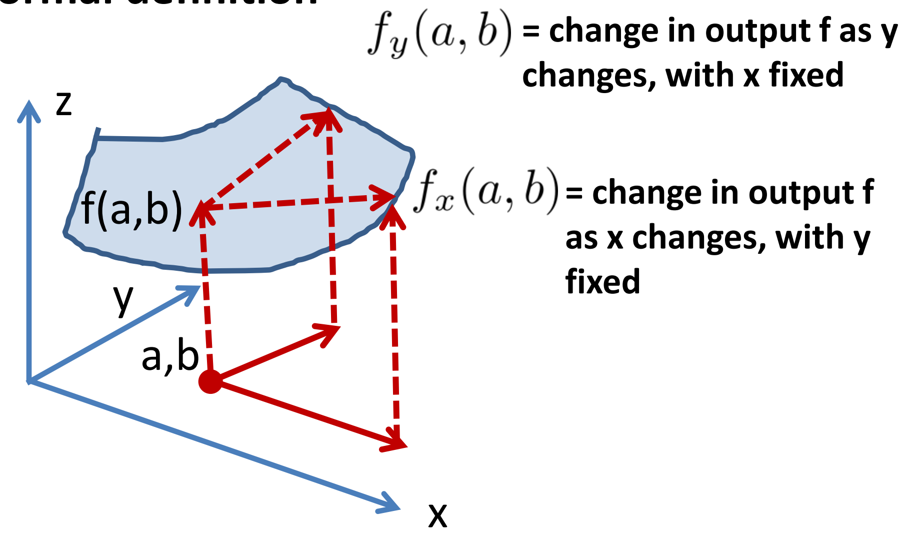
$$f_x = y\cos(xy)$$
$$f_y = x\cos(xy)$$

Section 14.3: The formal definition

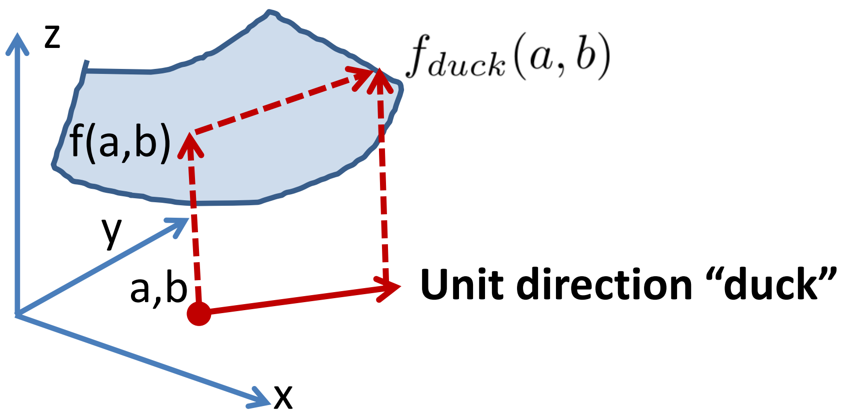
Let's define the partial derivatives formally:

$$f_x(a,b) \equiv \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$f_y(a,b) \equiv \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$



So, you should think of the partial derivative in the "duck" direction as how much the function changes as you move from (a,b) in the unit vector direction "duck"



Section 14.3: The formal definition

More examples: your turn $f(x,y,z) = x\,y^2 + x\,y\,z^3 + \cos(xyz)$

Find f_x, f_y, f_z

Solution:

$$f_x = y^2 + yz^3 - yz\sin xyz$$

$$f_y = 2xy + xz^3 - xz\sin xyz$$

$$f_z = 3xyz^2 - xy\sin xyz$$

It seems obvious we can keep going: $$f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} = \frac{\partial(\frac{\partial f}{\partial x})}{\partial x}$$

Find f_{xx}, f_{yy}, f_{zz}

$$f_{xx} = -(xy)(yz)\cos(xyz)$$

$$f_{yy} = 2x + -xz(xz)\cos(xyz)$$

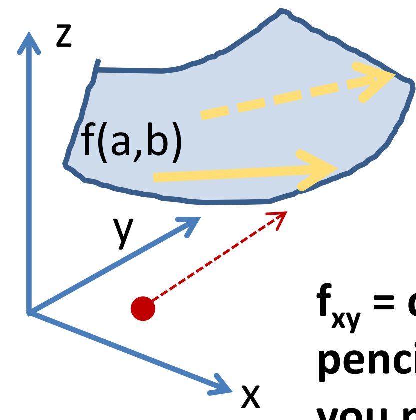
$$f_{zz} = 6xyz - xy(xy)\cos xyz$$

Section 14.3: We can also do "cross-derivatives"

$$f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial(\frac{\partial f}{\partial x})}{\partial y}$$

The rate of change as y changes of the rate of change of f as x changes

To see this visually, imagine a pencil tangent to a surface in a direction where y is constant. As that pencil is translated, always pointing in the xy plane, its change in slope is given by f_{xy}



f_{xy} = change in pencil slope as you move in the y direction

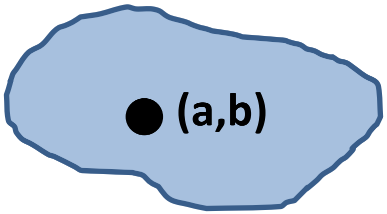
Example: $f(x,y) = x^2y^3 + e^{xy^4}$

Find f_x, f_y, f_{xy}, f_{yx}

$$f_x = 2xy^3 + y^4e^{xy^4}$$
$$f_{xy} = 6xy^2 + 4y^3e^{xy^4} + y^4(x4y^3)e^{xy^4}$$
$$f_y = x^2(3y^2) + (4xy^3)e^{xy^4}$$
$$f_{yx} = 2x(3y^2) + (4y^3)e^{xy^4} + (4xy^3)(y^4)e^{xy^4}$$

Wow! Why are these the same?

Section 14.3: Clairaut's Theorem



Suppose f(x,y) is defined on a disk around a point (a,b), and that f_{xy} and f_{yx} are both continuous in that disk. Then

$$f_{xy} = f_{yx} \qquad \text{Often referred to as the "Equality of Mixed Partials"}$$

Question: Suppose $f(x,y) = x^3 e^{x^3} \sin(x e^{xy} (x^y)^x)$

What is $f_{xy} - f_{yx}$?

Solution: By Clairaut's theorem, the answer is zero!

I won't prove this ---but try looking it up yourself to see how it is done—it basically depends on carefully analyzing the limits

$$f_{xy} = \lim_{h_2 \to 0} \frac{\lim_{h_1 \to 0} \frac{f(x+h_1, y+h_2) - f(x, y+h_2)}{h_1} - \lim_{h_1 \to 0} \frac{f(x+h_1, y) - f(x,y)}{h_1}}{h_2}$$

$$f_{yx} = \lim_{h_1 \to 0} \frac{\lim_{h_2 \to 0} \frac{f(x+h_1, y+h_2) - f(x, y+h_2)}{h_2} - \lim_{h_2 \to 0} \frac{f(x+h_1, y) - f(x,y)}{h_2}}{h_1}$$

And showing that they are the same (now you see why continuity in f_{xy} and f_{yx} is needed)

(I will come back to 14.4—but first): Section 14.5: Implicit Relationships

Suppose x + 2y = 6. Then x and y are not independent: x and y are forced to cooperate

This is called an "implicit relationship"

And we could ask "how does y change when x changes? This is what is meant by dy/dx

---

Question: Suppose $x^3 + y^3 + z^3 + 6xyz = 1$

This is an implicit relationship between x,y, and z. x,y, and z are linked So what is $\frac{\partial z}{\partial x}$ **?**

Commentary:

(1) What does this question even mean?

(2) It means, how does z change when x changes,
assuming y is held fixed (that's what a partial derivative means!) ?

Let's do it:

Step 1: Take the partial derivative of both sides with respect to x, remembering that y is held fixed, and z is now a function of x

$$\frac{\partial}{\partial x}(x^3 + y^3 + z^3 + 6xyz) = \frac{\partial}{\partial x}(1)$$

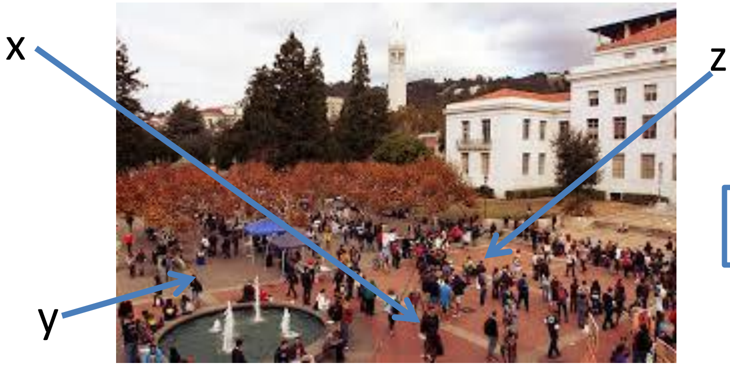
$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Step 2: Solve for $\frac{\partial z}{\partial x}$

$$3x^2 + 6yz + (3z^2 + 6xy)\frac{\partial z}{\partial x} = 0 \rightarrow \frac{\partial z}{\partial x} = -\frac{3x^2+6yz}{3z^2+6xy}$$

Section 14.5: Implicit Relationships

Let's use our function boxes to make sure we know what we are talking about:



We start with no relation between x, y and z

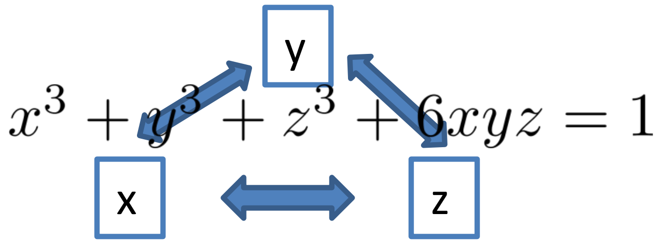
Image from busy.org

(2) We discover there is an equation

$$x^3 + y^3 + z^3 + 6xyz = 1$$

that links them all together:

Image from netclipart.com



$$x^3 + y^3 + z^3 + 6xyz = 1$$

If we choose to make z the output

x, y → **Outputs z given x and y** → z

$\frac{\partial z}{\partial x}$ Change in output z as input x changes, with other input y held fixed

---

And we could have chosen any variable as the output

x, z → **Outputs y given x and z** → y

$\frac{\partial y}{\partial z}$ Change in output y as input z changes, with other input x held fixed

Section 14.3: Implicit Relationships

Your turn: $x^2 + 2y^2 + 3z^2 = 1 \quad \text{Find} \quad \frac{\partial z}{\partial x}$

Solution: Step 1: take the partial derivative of both sides with respect to x (holding y fixed)

$$\frac{\partial}{\partial x}(x^2 + 2y^2 + 3z^2) = \frac{\partial}{\partial x}(1)$$

$$2x + 0 + 6z\frac{\partial z}{\partial x} = 0$$

Step 2: Solve for $\frac{\partial z}{\partial x} \rightarrow \frac{\partial z}{\partial x} = (-2x)/(6z)$

Your turn: $e^z = xyz \quad \text{Find} \quad \frac{\partial y}{\partial x}$

Solution: Step 1: take the partial derivative of both sides with respect to x (holding z fixed)

$$\frac{\partial}{\partial x}(e^z) = \frac{\partial}{\partial x}(xyz)$$

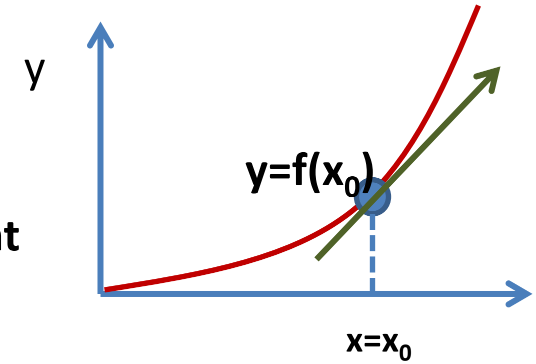
(Remember: y = y(x) and z is fixed—use product rule)

$$0 = yz + x\frac{\partial y}{\partial x}z$$

Step 2: Solve for $\frac{\partial y}{\partial x} \rightarrow \frac{\partial y}{\partial x} = (-yz)/(xz) = -y/x$

Section 14.4: Tangent Planes and linear approximations

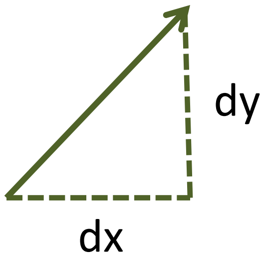
Let's recall 1D Calculus: y=f(x)



Tangent line with slope f'(x_0) going through the point (x_0,f(x_0))

Tangent line at (x_0,f(x_0)) touches the graph y=f(x0) at only one point in a near (x_0,f(x_0))

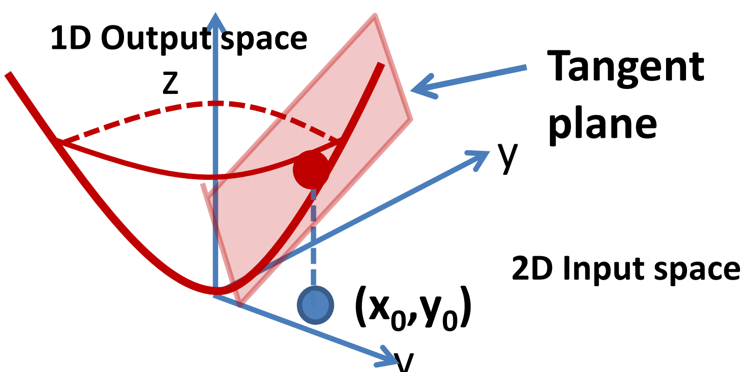
The tangent line is given by $y - f(x_0) = slope(x - x_0) = \left.\frac{df}{dx}\right|_{x_0} (x - x_0)$



Tangent vector = (dx,dy) = (1, dy/dx)= (1, f'(a))

We want to construct a similar idea for functions of two (or more variables):

The Tangent Plane

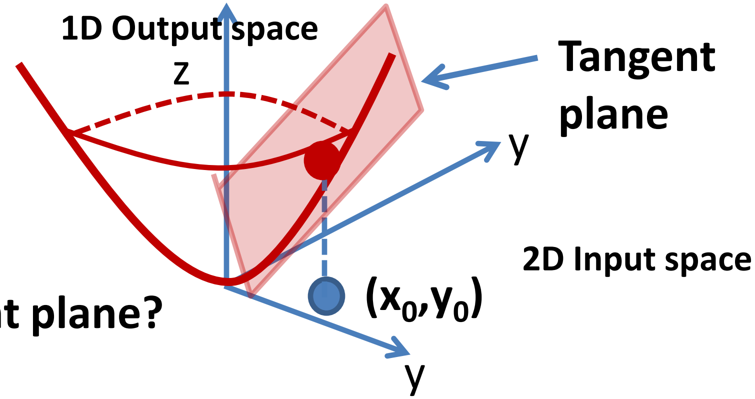


Tangent plane at (x_0,y_0,f(x_0,y_0)) touches the graph z=f(x_0,y_0) at only one point in a near (x_0,y_0,f(x_0,y_0))

Section 14.4: Tangent Planes and linear approximations

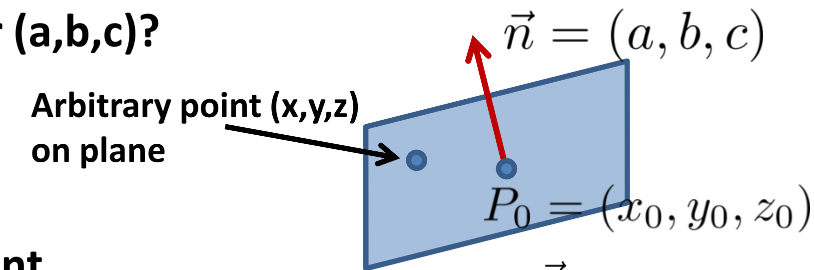
Tangent plane at $(x_0,y_0,f(x_0,y_0))$ touches the graph $z=f(x_0,y_0)$ at only one point in a near $(x_0,y_0,f(x_0,y_0))$

How are we going to find an equation for the tangent plane?



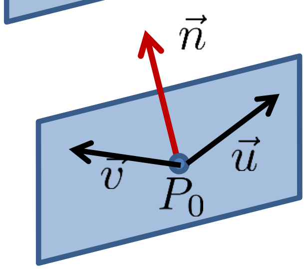
Idea #1! Do you remember we had a formula for a plane going through the point (x_0,y_0,z_0) with normal vector (a,b,c)?

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$



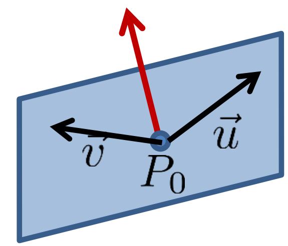
Realization 1: we know the point P_0 where the tangent plane touches the surface: $P_0=(x_0,y_0,z_0)=(x_0,y_0,f(x_0,y_0))$

Realization 2: if we had two vectors $\vec{u}$ and $\vec{v}$ in the tangent plane, we could take their cross product to find the normal $\vec{n}=\vec{u}\times\vec{v}$



Section 14.4: Tangent Planes and linear approximations

Formula for a plane going through the point (x_0,y_0,z_0) with normal vector (a,b,c)



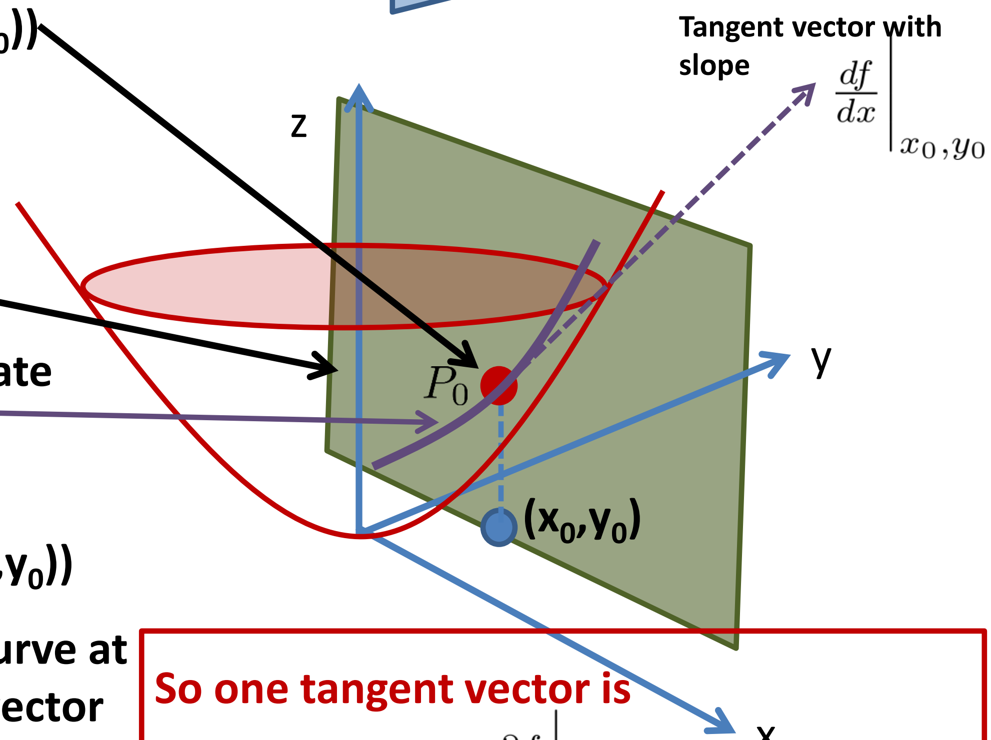
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Tangent point is $P_0 = (x_0,y_0,z_0)=(x_0,y_0,f(x_0,y_0))$

How can we find two vectors in the tangent plane?

Idea #2: We can slice the graph of $z=f(x,y)$ with a plane $y=y_0$

This gives a purple curve whose y coordinate never changes and lies on the surface.



So this purple curve is the graph of $(x,y_0,f(x,y_0))$

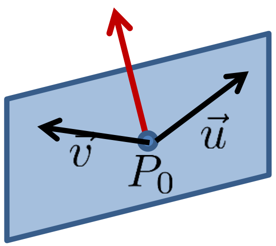
And the x partial derivative of this purple curve at (x_0,y_0) gives the slope $\left.\frac{df}{dx}\right|_{x_0,y_0}$ of the tangent vector at P_0 lying in the slicing plane

So one tangent vector is

$$\vec{v} = (1, 0, \left.\frac{\partial f}{\partial x}\right|_{x_0,y_0})$$

Section 14.4: Tangent Planes and linear approximations

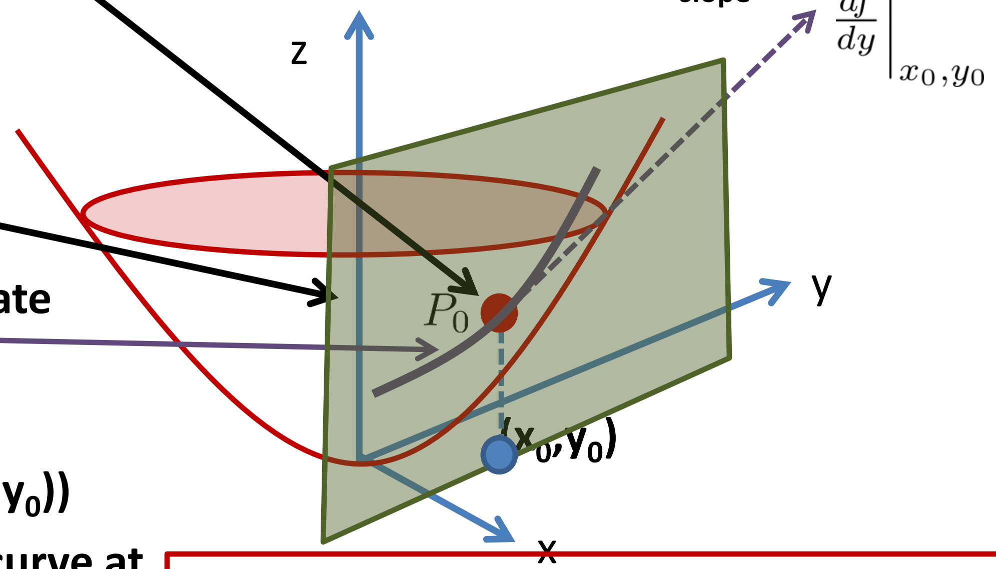
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



Tangent point is $P_0 = (x_0,y_0,z_0)=(x_0,y_0,f(x_0,y_0))$

Obtain another tangent vector by slicing the graph of $z=f(x,y)$ with a plane $x=x_0$

This gives a purple curve whose x coordinate never changes and lies on the surface.



So this purple curve is the graph of $(x_0,y,f(x,y_0))$

And the y partial derivative of this purple curve at (x_0,y_0) gives the slope $\left.\frac{df}{dy}\right|_{x_0,y_0}$ of the tangent vector at P_0 lying in the slicing plane

So one tangent vector is

$$\vec{v} = \left(0, 1, \left.\frac{\partial f}{\partial y}\right|_{x_0,y_0}\right)$$

Section 14.4: Tangent Planes and linear approximations

One tangent vector is $\vec{u} = (1, 0, \frac{\partial f}{\partial x}\Big|_{x_0,y_0})$ **One tangent vector is** $\vec{v} = (0, 1, \frac{\partial f}{\partial y}\Big|_{x_0,y_0})$

$$\vec{n} = (a,b,c) = \vec{u} \times \vec{v} = (1, 0, \frac{\partial f}{\partial x}) \times (0, 1, \frac{\partial f}{\partial y}) = (-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1) \leftarrow \text{check this!}$$

Back to our formula for a plane:

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

(I temporarily stopped writing $\Big|_{x_0,y_0}$

Substitute everybody in:

$$-\frac{\partial f}{\partial x}(x - x_0) + -\frac{\partial f}{\partial y}(y - y_0) + 1(z - f(x_0, y_0)) = 0$$

Solve for z (remembering that $z_0 = f(x_0,y_0)$):

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}\Big|_{x_0,y_0}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{x_0,y_0}(y - y_0)$$

Formula for the tangent plane

Section 14.4: Tangent Planes and linear approximations

Formula for the tangent plane

$$z - f(x_0, y_0) = \frac{\partial f}{\partial x}\bigg|_{x_0,y_0} (x - x_0) + \frac{\partial f}{\partial y}\bigg|_{x_0,y_0} (y - y_0)$$

Notice how much it looks like our formula for the slope of a tangent line:

$$y - f(x_0) = slope(x - x_0) = \frac{df}{dx}\bigg|_{x_0} (x - x_0)$$

Example: Find the equation for the plane tangent to $z= 2x^2 + y^2$ at the input point $x_0=1$ $y_0=1$

Solution: Step 1: find the point on the surface at the input $x_0=1$ $y_0=1$

$z_0 = f(x_0,y_0)=f(1,1) = 2 (1)^2 + 1^2 = 3$

Step 2: find the partial derivatives at the input point:

$f_x = 4x$ so at input $x_0=1$ $y_0=1$ $f_x=4$
$f_y = 2x$ so at input $x_0=1$ $y_0=1$ $f_y=2$

Step 3: put them into your equation for the tangent plane: $z-3 = 4 (x-1) + 2 (y-1)$