Section 13.1

We have vectors $\vec{u} = (a_1, b_1, c_1, d_1, e_1)$ $\quad \vec{v} = (a_2, b_2, c_2, d_2, e_2)$

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta = \text{a number}$$

and new ways to combine them

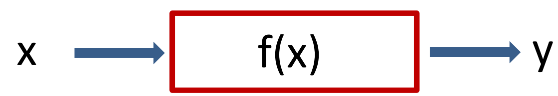
$$\vec{u} \times \vec{v} = \text{a funky expression} = \text{a vector}$$

$$\text{so far, only defined for 3D}$$

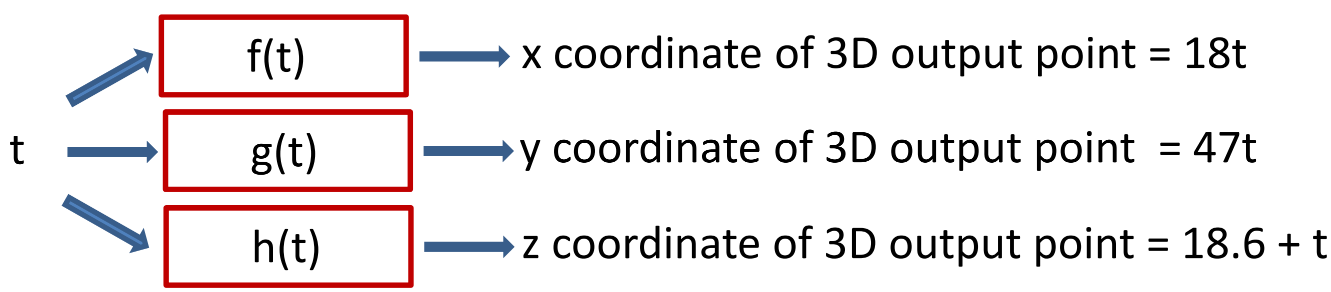
$$c\vec{u} = \text{a vector}$$

Now we want to build vector-valued functions

Example: $\mathbf{f}(\mathbf{x}) : \mathbb{R}^1 \to \mathbb{R}^1$ such as y = f(x) = 6x



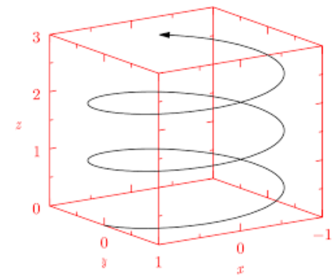
Example: $f(\vec{t}) : \mathbb{R}^1 \to \mathbb{R}^3$ such as (f(t), g(t), h(t)) = (18t, 47t, 18.6 + t)



Section 13.1

Example: **x= f(t) = cos (t)**
y=g(t) = sin (t)
z=h(t) = t

$$\vec{f}(x) : \mathbb{R}^1 \rightarrow \mathbb{R}^3$$



[image from artofproblemsolving.com]

Now, since we have functions, we want to talk about continuity and limits

Definition: the limit as $t \rightarrow a$ **of a vector function** $\vec{f}(t)$ **is the vector made up of the Individual, component-wise, limits:**

$$\lim_{t\to 8}(t, t^2, t^{1/3}) = \left(\lim_{t\to 8} t, \lim_{t\to 8} t^2, \lim_{t\to 8} t^{1/3}\right) = (8, 64, 2),$$

Realization: We can think of any vector-valued function

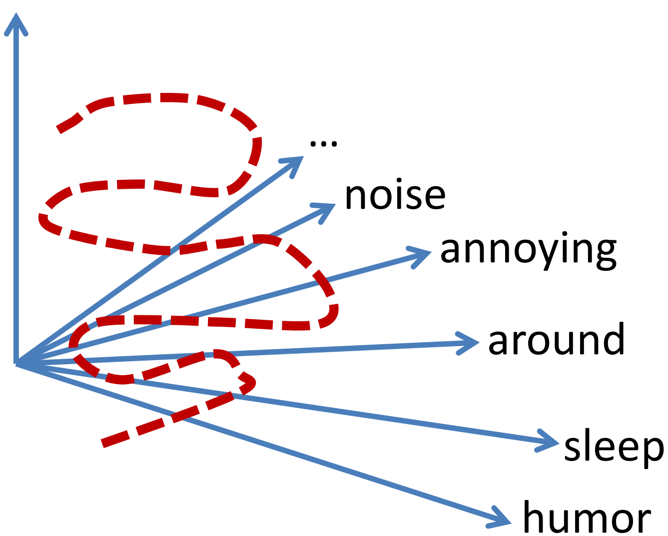
$$\vec{f}(x) : \mathbb{R}^1 \rightarrow \mathbb{R}^{82} = (f_1(x), f_2(x), f_3(x), ..., f_{82}(x))$$

as the trajectory of a mosquito in 82-dimensional space….

Section 13.1

Example: How you currently feel about your roommate (or sister, brother, parents, etc.) is a vector-valued function of time t

Roommate (t) =
(sense of humor (t),
how much they sleep (t),
how often they are there (t),
how annoying their friends are (t),
how much noise they make (t),
…..)

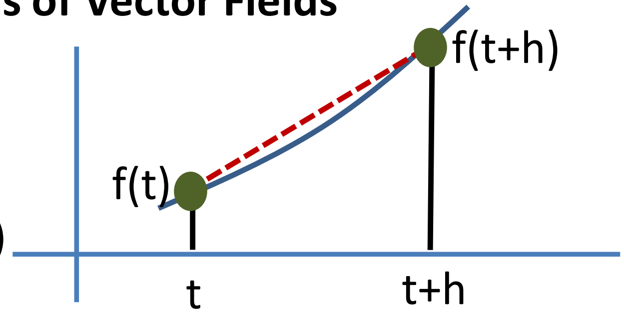


A trajectory through high-dimensional space

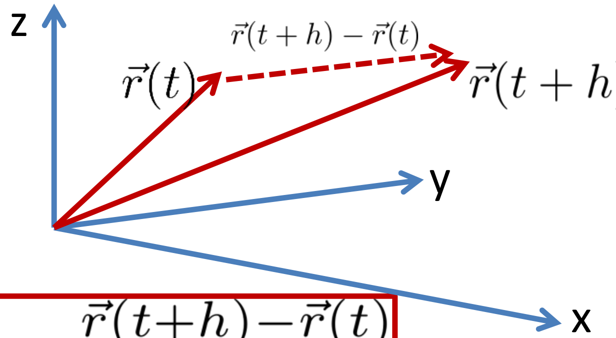
Section 13.2: Derivatives and Integrals of Vector Fields

Remember 1D: $f'(t) = \lim_{h \to 0} \frac{f(t+h)-f(t)}{h}$

= slope of straight line connecting (t,f(t)) with (t+h, f(t+h))



How can we think about this for vector-valued functions?



Definition of vector derivative

$$\frac{d\vec{r}(t)}{dt} = \text{Also written as } \vec{r}\,' = \lim_{h \to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

Please note: this "derivative" is a vector !

Fortunately, it is really easy to compute! $\vec{r}(t) = (f(t), g(t), h(t)) \to \vec{r}(t)' = (f'(t), g'(t), h'(t))$

Example: if $\vec{r}(t) = (t^2, t^3, t^5)$ **find** $\vec{r}\,'$ **at t=2:**

Solution: $\vec{r}\,' = (2t, 3t^2, 5t^4) = (4, 12, 80)$

Section 13.2: Derivatives and Integrals of Vector Fields

Okay—what about integration of a vector function?

If $\vec{r}(t) = (f(t), g(t), h(t))$ **then** $\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt\right)$

Example: Suppose $\vec{r}(t) = (f(t), g(t), h(t)) = (t, t^2, t^3)$ **Find** $\int_2^3 \vec{r}(t)dt$

Solution:

$$\int_2^3 \vec{r}(t)dt = \left(\int_2^3 tdt, \int_2^3 t^2dt, \int_2^3 t^3dt\right) = \left(\Big|_2^3 \frac{1}{2}t^2, \Big|_2^3 \frac{1}{3}t^3, \Big|_2^3 \frac{1}{4}t^4\right)$$

$$= \left(\frac{1}{2}(3^2 - 2^2), \frac{1}{3}(3^3 - 2^3), \frac{1}{4}(3^4 - 2^4)\right)$$

Yeah, but what does the vector-valued integral actually mean?

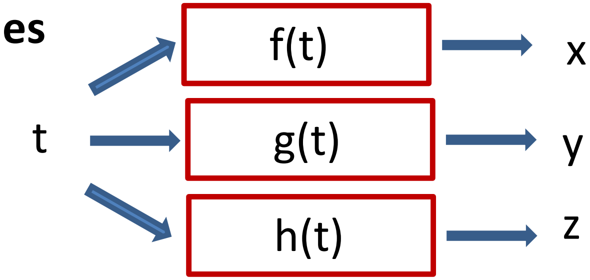
Answer: $\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt\right)$

is the total amount of :

f(t) between a < t < b, g(t) between a < t < b, h(t) between a <t <b

Section 14.1: Functions of Several Variables

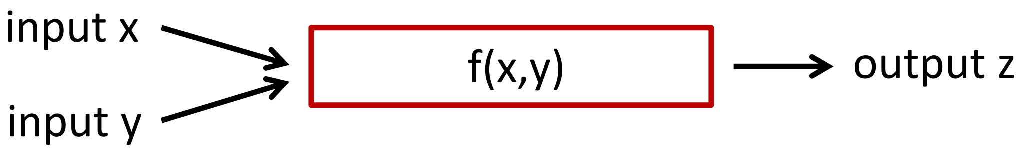
So far, we have talked about vector-valued functions of one input variable: $\vec{f}(t) : \mathbb{R}^1 \rightarrow \mathbb{R}^3$



Let's have more fun, and talk about functions of several input variables: z = f(x,y)

Definition: A function of two variables is a mapping with two inputs and one output

Example: z= f(x,y)

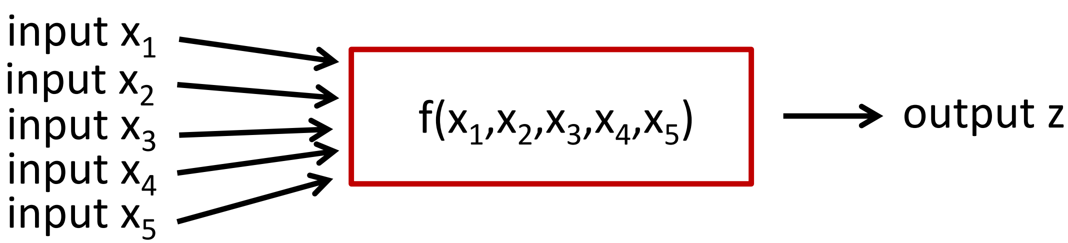


Example:
$z=f(x,y)= x^2 + 6y$

The notation here is precise, but tricky: I should write: $f(\vec{x}) \equiv f(x,y) = z : \mathbb{R}^2 \rightarrow \mathbb{R}^1$

(the input vector $\vec{x}$ has two components, namely x and y)

Another example: a function of 5 variables $z = f(\vec{x}) \equiv f(x_1, x_2, x_3, x_4, x_5) : \mathbb{R}^5 \rightarrow \mathbb{R}^1$



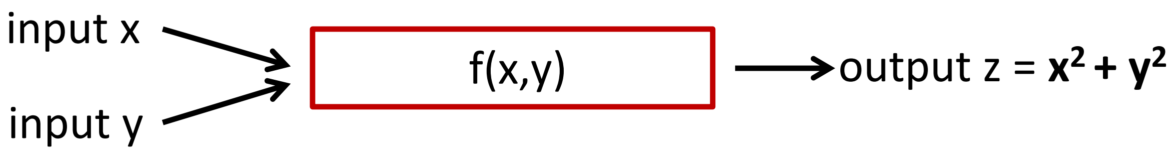
Section 14.1: Functions of Several Variables

And now comes a moment of slipperiness

Question: What is meant by $f(x,y)=x^2 + y^2$?

Important point: it is giving the output for a function of two input variables x and y

It is *not* a circle.

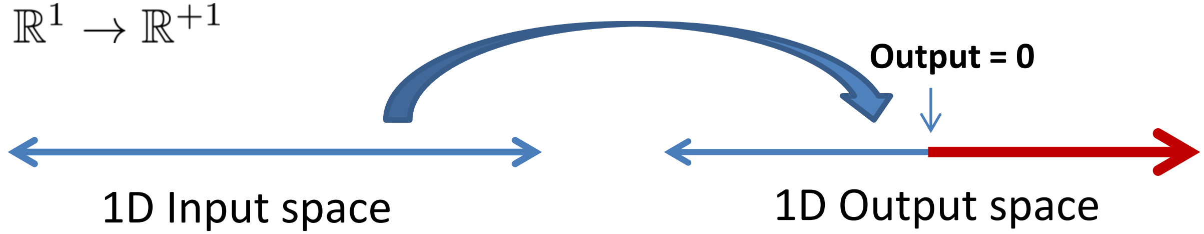


Let's really dig into this:

Question: Back to 1D: What is $f(x) = x^2$?

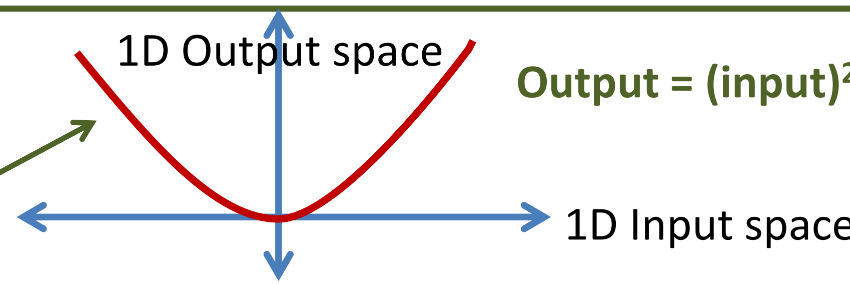
Answer: It is a function: input is any real number and output is non-negative real number

It is a mapping from $\mathbb{R}^1 \rightarrow \mathbb{R}^{+1}$



On the other hand, if I ask you to graph the output of this function vs. the input

Then the graph is a parabola

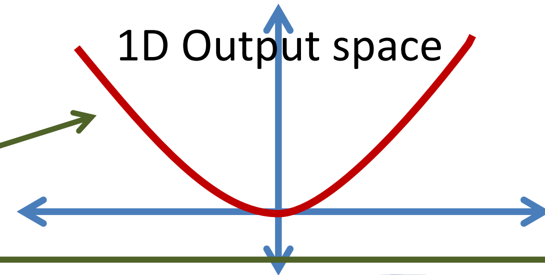


Section 14.1: Functions of Several Variables

I really want to push this issue

If we plot output against input

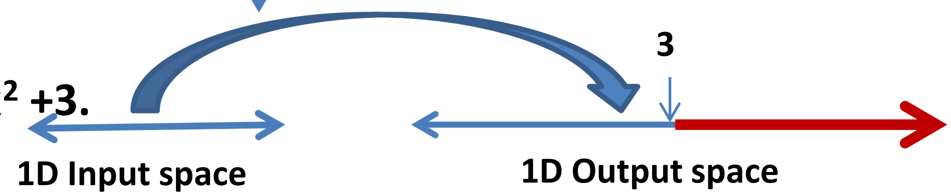
then the graph is a parabola



$f(x) = x^2$

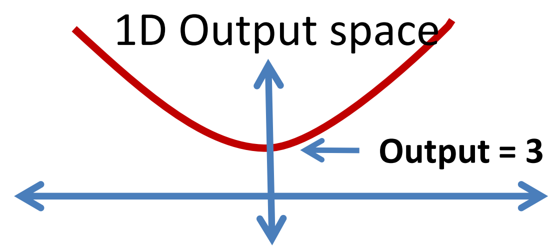
Now suppose I change the function to $f(x) = x^2 + 3$.

Still a mapping from $\mathbb{R}^1 \to \mathbb{R}^{+1}$



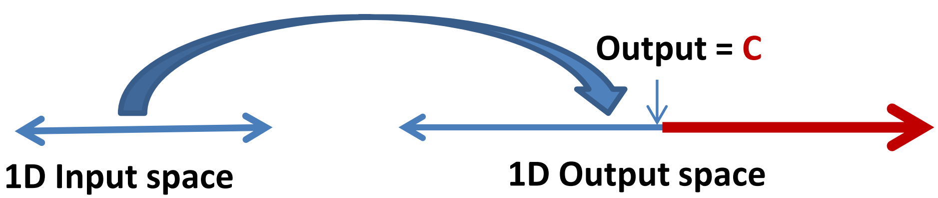
If we plot output against input

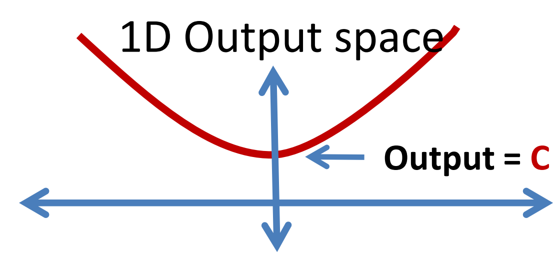
then the graph is a different parabola



$f(x) = x^2 + 3$

We can write something more general: $f(x) = x^2 + C$ (C is a constant)



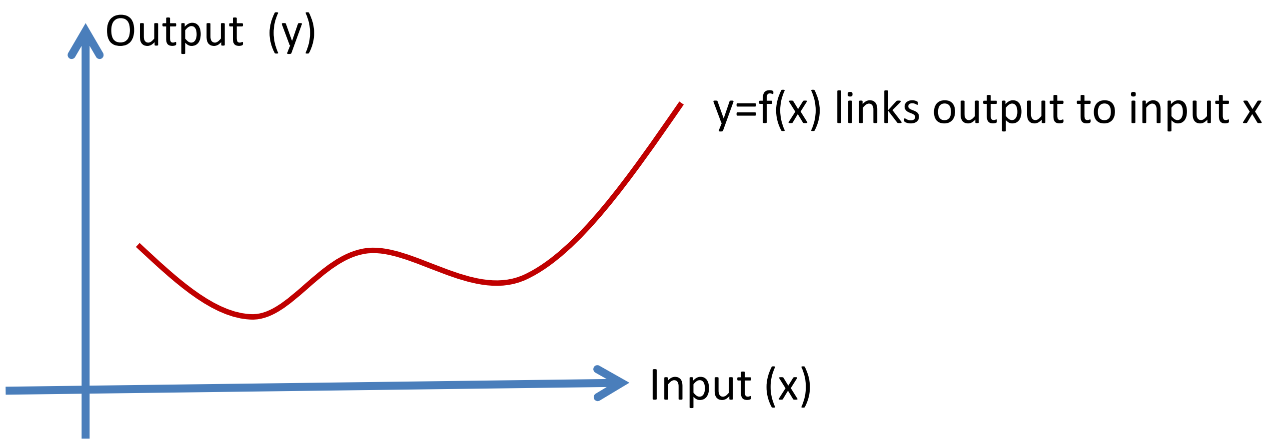


Section 14.1: Functions of Several Variables

The key (very important) point: What do mean when we write y = f(x) ?

Realization: f(x) is a function which maps $\mathbb{R}^1 \rightarrow \mathbb{R}^1$

By setting y = f(x), then we linking a y variable to an input variable x



Again: **Input (x)** → **$f(x) = x^2$** → **Output (y)**

I know, I know. You think I am crazy for making such a big deal out of this.
Hang on.

Section 14.1: Functions of Several Variables

Again: **Input (x)** → **$f(x) = x^2$** → **Output**

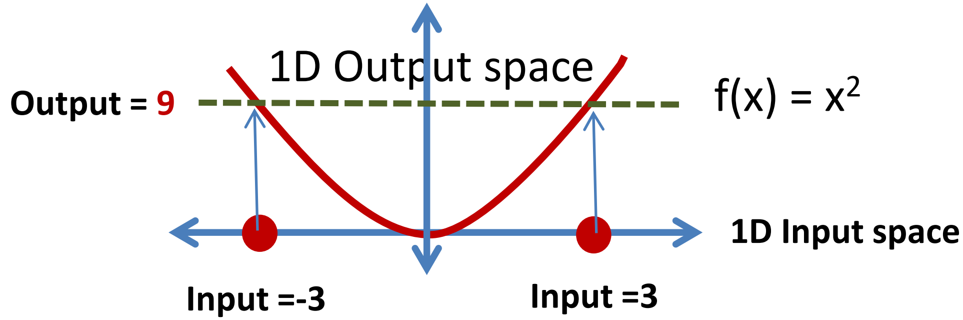
Now, suppose I ask:

what is the set of all inputs that get sent by $f(x) = x^2$ to the output 9 ?

Answer: the input values x= -3 and x = 3

Definition: **Given f(x), we call the "k" level set to be the set of all inputs that get sent to the output value of k.**

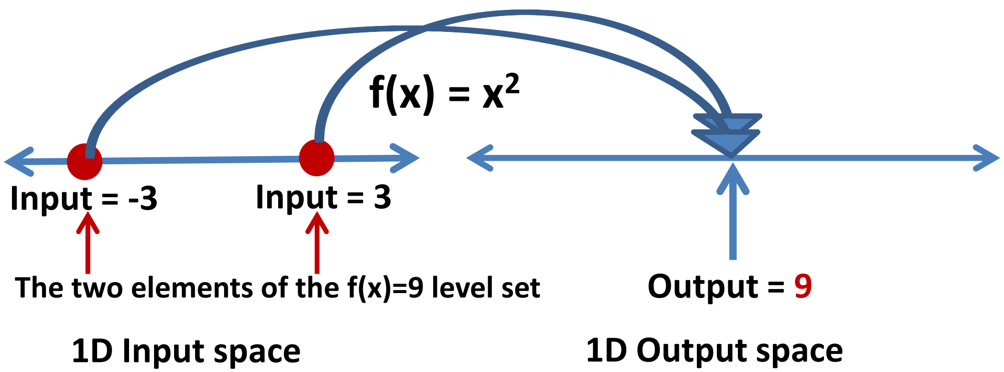
So, for the above example, the "k=9" level set is the set of all inputs that get sent to output y =9, so the "k= 9" level set is the two inputs x=-3 and x=3

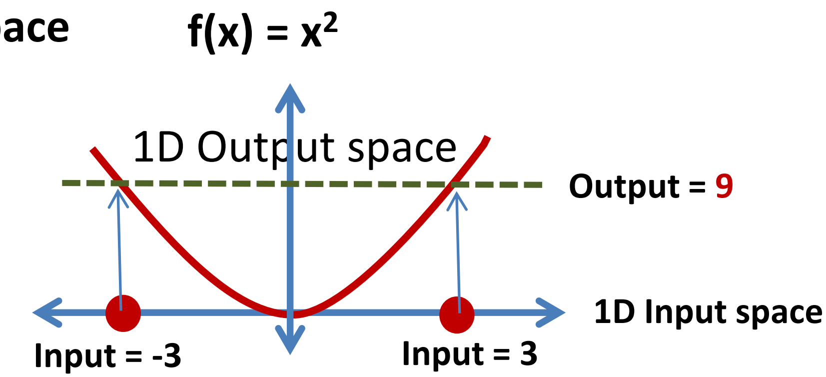


Section 14.1: Functions of Several Variables

Definition (again!): **Given f(x), we define the "f(x)=k" level set to be the set of all inputs that get sent to the output value of k.**

I want to stress that the level set lies in input space





The level set lies in *input* space

The level set lies in *input* space

The level set lies in *input* space

The level set lies in *input* space

The level set lies in *input* space

The level set lies in *input* space

Section 14.1: Functions of Several Variables

Example: f(x) = sin x What is the level set for k=1?

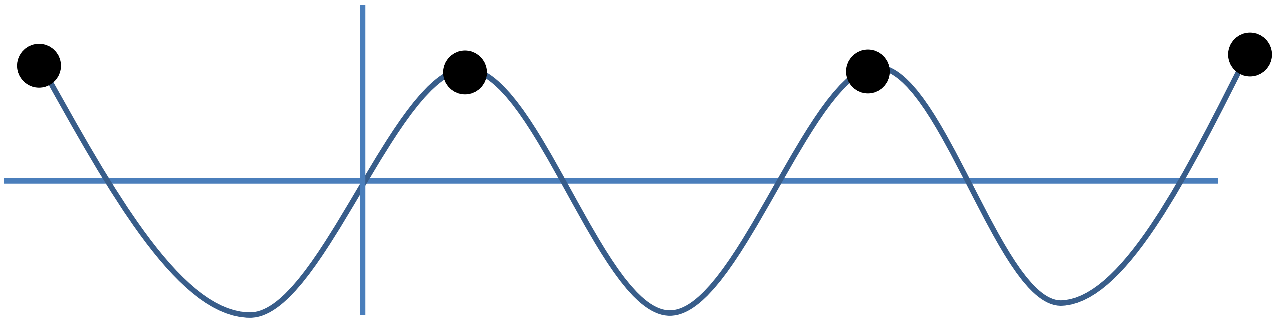
Answer: Step 1: We are trying to find the set of all inputs x such that the output f(x) is =1

Step 2: So we need to find all x that satisfy 1 = f(x) = sin x

Step 3: So the answer is all input values x that are of the form

$$x = \frac{4n+1}{2}\pi \rightarrow \pi/2, 5\pi/2, \ldots$$

where n is an integer (positive, negative, or zero)



f(x) = sin x

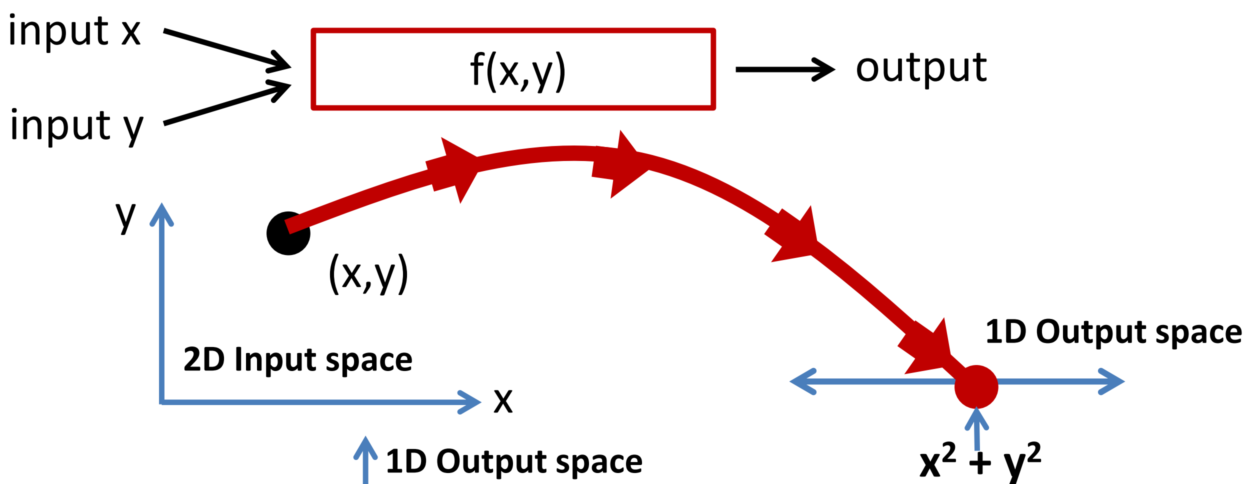
Section 14.1: Functions of Several Variables

Now, let's go to a function of two variables and see what all this means:

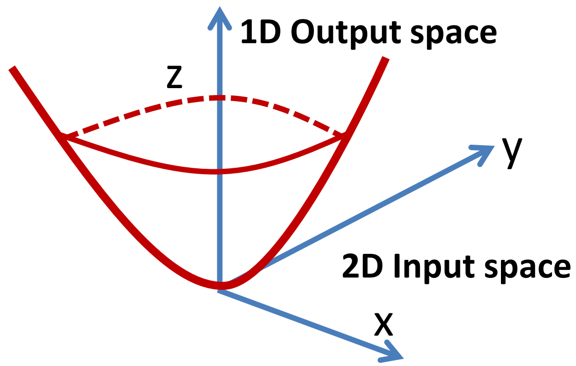
Suppose $f(x,y) = x^2 + y^2$

$f(\vec{x}) \equiv f(x,y) : \mathbb{R}^2 \to \mathbb{R}^1$

Then input space is 2D, and output space is 1D



We could choose to plot output space against input space:

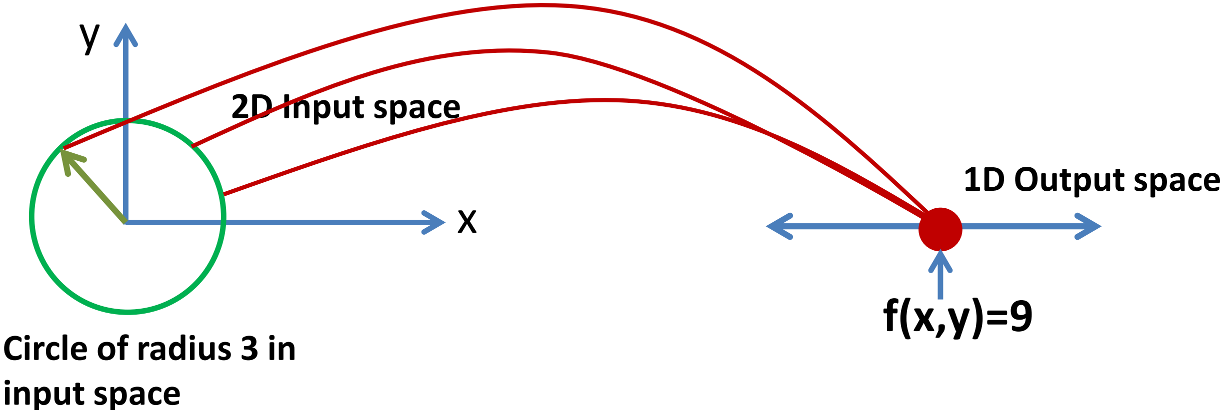


And now I could ask: what is the level set $f(x,y) = 9$?

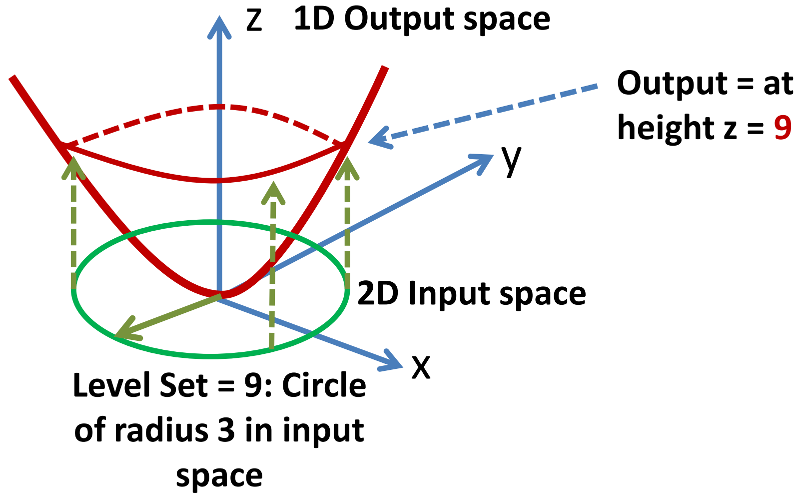
Answer: it is the set of all input points that get sent to the output 9

Section 14.1: Functions of Several Variables

$f(x,y) = x^2 + y^2$ What is the level set $f(x,y) = 9$?



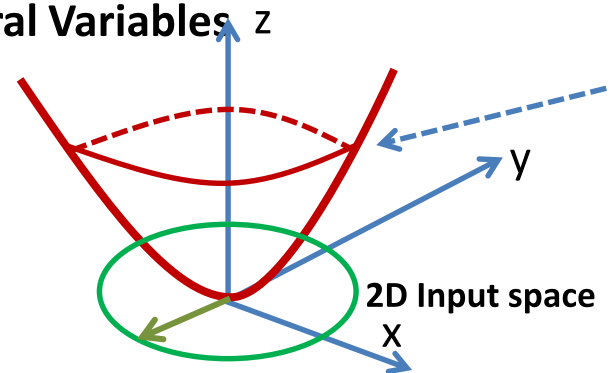
We could choose to plot output space against input space:



Answer: it is the set of all input points that get sent to the output 9

Section 14.1: Functions of Several Variables

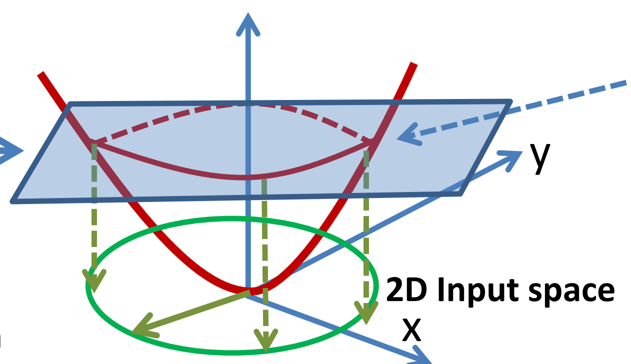
$f(x,y) = x^2 + y^2$ What is the level set $f(x,y) = 9$?



Analytic answer:

it is the set of all input points that get sent to the output 9

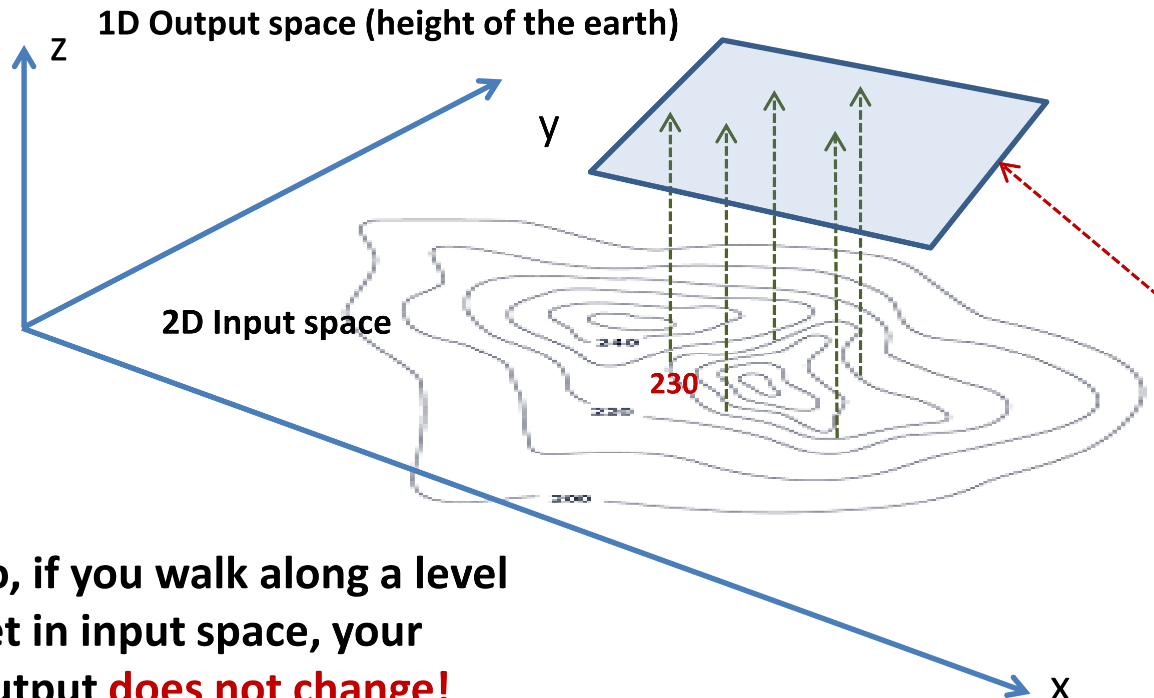
Geometric answer: if you slice the graph with a plane at height k, and project the result down to the input plane, the points you get are the k level set.



Section 14.1: Functions of Several Variables

And that's what a topographic map is!

A topographic gives you different k-level sets in the input space, showing what the height of the earth would be. Contour lines show points of the same elevation.



All sent to same height z = 230

So, if you walk along a level set in input space, your output does not change!

Topo map from https://datavizproject.com/data-type/topographic-map/