Section 12.3—Dot Products

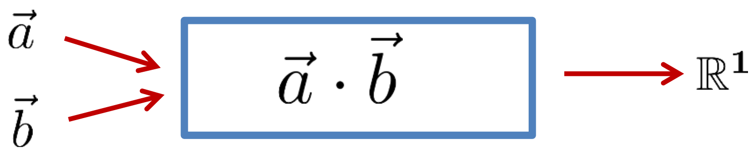
Given two vectors: $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3)$

We define the dot product $\vec{a} \cdot \vec{b}$ as follows:

$$\vec{a} \cdot \vec{b} \equiv a_1 b_1 + a_2 b_2 + a_3 b_3$$

Several things to observe:

(1) this takes two input vectors and returns a number



(2) That number can be positive, negative, or zero

(3) It makes sense regardless of the dimension of the vectors $\vec{a}$ and $\vec{b}$ $[\vec{a}, \vec{b} \in \mathbb{R}^{27}]$

(4) It does not make sense to take the dot product of a vectors of different dimensions:

$$\vec{a} = (a_1, a_2), \vec{b} = (b_1, b_2, b_3) \quad \vec{a} \cdot \vec{b} \text{ has no meaning at all}$$

$$\vec{b} = (4, 5) \text{ is not the same as } (4, 5, 0)$$

(5) This is a definition:

We could have defined it another way.

We have to discover if this definition is a good one.

Section 12.3—Dot Products

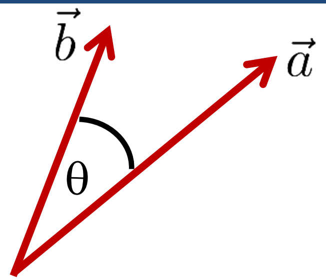
Example: $\vec{a} = (1,2,3), \vec{b} = (4,5,6)$ **What is** $\vec{a} \cdot \vec{b}$ **?**

Solution: $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = 1 \times 4 + 2 \times 5 + 3 \times 6 = 32$

It is easy to see from this definition that:

- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
- $\vec{a} \cdot \vec{0} = 0$
- $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ [since $\vec{a} \cdot \vec{a} = a_1 a_1 + a_2 a_2 + a_3 a_3$]

Section 12.3: Why is this a good definition?



Theorem: $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

Some things to note:

(1) this is *not* a definition. We have defined the dot product, and now we are claiming that something interesting is true.

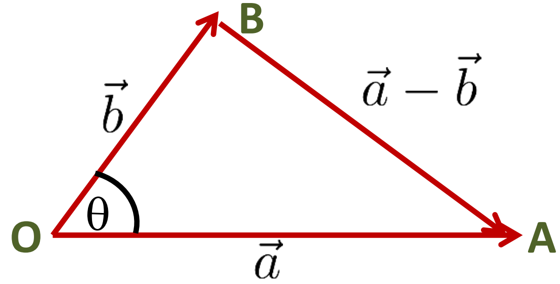
(2) We have to prove this.

(3) We are saying that the cosine of the angle between two vectors in 85-dimensional space is given by

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \cos\theta$$

That's quite a bold claim. Let's see if we can prove it.....

Let's prove that $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$



Step 1: Make a good drawing:

Step 2: Somehow, remember the law of cosines

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB|\cos\theta \quad [Eqn.*]$$

Step 3: Realize that we can convert these segment lengths into vector lengths:

$$|OA| = |\vec{a}| \qquad |OB| = |\vec{b}| \qquad |AB| = |\vec{a}-\vec{b}|$$

Step 4: So Eqn. * becomes

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\,|\vec{b}|\,\cos\theta \qquad [Eqn.**]$$

Step 5: Evaluate the left-hand-side of Eqn. **

$$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} = |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$$

Step 6: Set the left-hand-side of Eqn. ** equal to the right-hand-side:

$$|\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\;|\vec{b}|\cos\theta$$

$$\rightarrow 2\vec{a}\cdot\vec{b} = 2|\vec{a}|\;|\vec{b}|\cos\theta \rightarrow \boxed{\vec{a}\cdot\vec{b} = |\vec{a}|\;|\vec{b}|\cos\theta \quad \textit{Proved}}$$

Example: Find the angle between $\vec{a} = (2, 2, 1)$ $\vec{b} = (5, -3, 2)$

Solution:

Step 1: Need to evaluate $\vec{a} \cdot \vec{b} = |\vec{a}| \; |\vec{b}| \cos\theta$

$$|\vec{a}| = \sqrt{2^2 + 2^2 + (1)^2} = 3 \qquad |\vec{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

$$\vec{a} \cdot \vec{b} = 2 \times 5 + 2 \times (-3) + (1) \times (2) = 6$$

Step 2: So that means that:

$$6 = 3\sqrt{38}\cos\theta$$

$$\cos\theta = \frac{2}{\sqrt{38}}$$

Orthogonality: So now we know the dot product tells about the angle between two vectors:

Definition: We say that two vectors are **orthogonal** if the angle between them is $\boxed{\pi/2}$

Claim: Two vectors are orthogonal iff $\vec{a} \cdot \vec{b} = 0$

Note: ("iff" = if and only if) means that both directions are true:

(1) Proof in $\Longrightarrow$ direction:

(a) Assume that $\vec{a}$ **and** $\vec{b}$ **are orthogonal**

(b) Then $\vec{a} \cdot \vec{b} = |\vec{a}| \; |\vec{b}| \cos\theta = |\vec{a}| \; |\vec{b}| \cos\frac{\pi}{2} = 0$

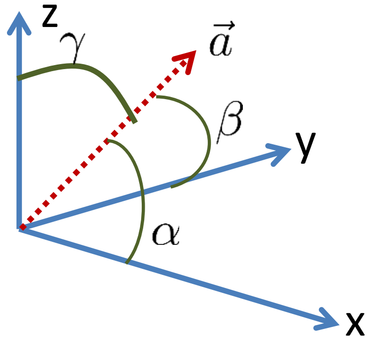
(2) Proof in $\Longleftarrow$ direction:

(a) Assume that $\vec{a} \cdot \vec{b} = 0$

(b) Then $0 = \vec{a} \cdot \vec{b} = |\vec{a}| \; |\vec{b}| \cos\theta \rightarrow \theta = \frac{\pi}{2} \rightarrow \text{are orthogonal}$

Direction angles and cosines:

The direction angles of a non-zero vector $\vec{a}$



α **= angle with x axis**

β **= angle with y axis**

γ **= angle with z axis**

We can use our dot product definition to find nice tidy expressions for direction angles:

Recall that

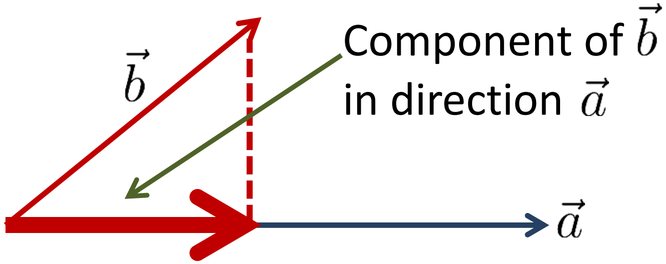
$$\vec{i} = (1,0,0) = \text{ x axis} \qquad \vec{j} = (0,1,0) = \text{ y axis} \qquad \vec{k} = (0,0,1) = \text{ z axis}$$

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}|\,|\vec{i}|} = \frac{a_1}{|\vec{a}|} \qquad \cos\beta = \frac{\vec{a}\cdot\vec{j}}{|\vec{a}|\,|\vec{j}|} = \frac{a_2}{|\vec{a}|}$$

$$\cos\gamma = \frac{\vec{a}\cdot\vec{k}}{|\vec{a}|\,|\vec{k}|} = \frac{a_3}{|\vec{a}|}$$

Projections:

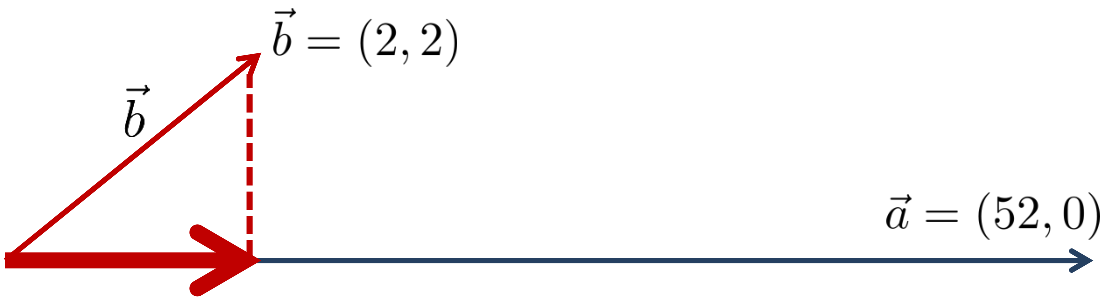
Suppose we want to find the "component" of vector $\vec{b}$ pointing in direction $\vec{a}$



Definition: **The vector projection of $\vec{b}$ onto $\vec{a}$ is the actual thick dark red vector above.**

Definition: **The component (or scalar projection) is the magnitude of the vector projection**

Before producing formulae to calculate these ideas, here's a silly example:

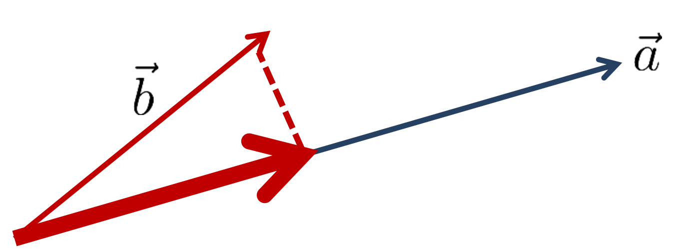


Let $\vec{b} = (2,2)$ and $\vec{a} = (52,0)$

Then the vector projection of $\vec{b}$ onto $\vec{a}$ is the vector (2,0), and the scalar projection is 2

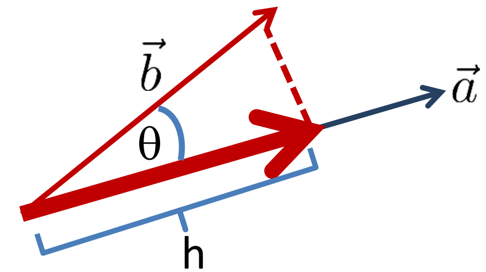
Definition: The **vector projection** of $\vec{b}$ onto $\vec{a}$ is the dark red vector.

Definition: The **component (or scalar projection)** is the magnitude of the vector projection



How can we find formulae for these things?

IDEA!: (back to the drawing)



Step 1: The unit vector in the $\vec{a}$ direction is $\dfrac{\vec{a}}{|\vec{a}|}$

Step 2: The part of $\vec{b}$ lying on $\vec{a}$ has length $h = |\vec{b}|\cos\theta$

So the scalar projection is $h = |\vec{b}|\cos\theta$

Step 3: So the vector projection is h times the unit vector in the direction

So the vector projection is (h) (unit vector in direction $\vec{a}$) = $|\vec{b}|\cos\theta \ \dfrac{\vec{a}}{|\vec{a}|}$

The book likes to write these using dot products:

$$\text{Vector Projection:} = |\vec{b}|\cos\theta\,\frac{\vec{a}}{|\vec{a}|} = \left(|\vec{a}||\vec{b}|\cos\theta\right)\frac{\vec{a}}{|\vec{a}|^2} = \left[\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right]\vec{a}$$

Example: Find the vector projection of $(1,1,2)=\vec{b}$ onto $(-2,3,1)=\vec{a}$

$$\text{Vector Projection:} = |\vec{b}|\cos\theta\frac{\vec{a}}{|\vec{a}|} = \left(|\vec{a}||\vec{b}|\cos\theta\right)\frac{\vec{a}}{|\vec{a}|^2} = \left[\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right]\vec{a}$$

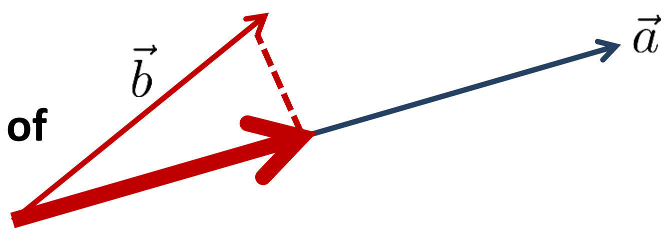
Solution: We compute the pieces in the last expression:

$$\vec{a}\cdot\vec{b} = -2\times 1 + 3\times 1 + 2\times 1 = 3$$

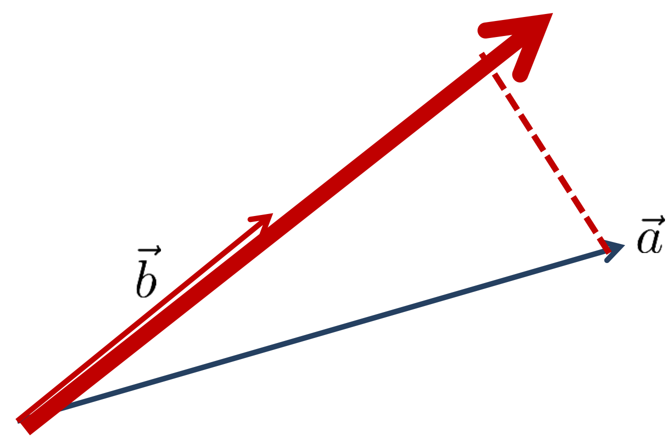
$$|\vec{a}|^2 = (2^2+3^2+1^2) = 14$$

So the answer is $\frac{3}{14}(-2,3,1)$

Be careful: the vector projection of $\vec{b}$ onto $\vec{a}$



Is not the same as the projection of $\vec{a}$ onto $\vec{b}$



Section 12.4— Cross Products

Definition: If $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3)$ **then we define the cross product** $\vec{a} \times \vec{b}$

$$\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1) \qquad [***]$$

Please note: the cross product $\vec{a} \times \vec{b}$ **is a vector**

The complicated expression in [***] is actually quite natural, though hard to remember. It is related to the determinant. Please read that section in the book—though I will return to it at the right time.

Example: $\vec{a} = (1, 3, 4), \qquad \vec{b} = (2, 7, -5)$

Find $\vec{a} \times \vec{b}$

Solution:

$$\vec{a} \times \vec{b} = ((3)(-5) - (4)(7), (4)(2) - (1)(-5), (1)(7) - (3)(2)) = (-43, 13, 1)$$

Section 12.4— Cross Products

Let's prove some interesting things about the cross-product

(1) Interesting thing #1: $\vec{a} \times \vec{b}$ **is orthogonal to both** $\vec{a}$ **and** $\vec{b}$

How do we prove this?

Step 1: We recall that two vectors are orthogonal if their dot product is zero

Step 2: Here goes: we will check that $\left[\vec{a} \times \vec{b}\right] \cdot \vec{a} = 0$

Step 3: $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

$$\text{So} \qquad (\vec{a} \times \vec{b}) \cdot \vec{a} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot \vec{a}$$

$$= (a_2b_3 - a_3b_2)a_1 + (a_3b_1 - a_1b_3)a_2 + (a_1b_2 - a_2b_1)a_3$$

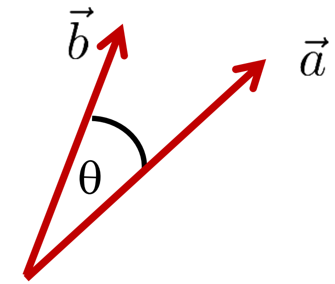
$$= a_1a_2b_3 - a_1a_3b_2 + a_2a_3b_1 - a_1a_2b_3 + a_1a_3b_2 - a_2a_3b_1$$

$$= 0$$

Section 12.4— Cross Products

(2) Interesting thing #2: If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}| \quad |\vec{b}| \sin\theta$$



How do we prove this? (just a bunch of arithmetic)

$$|\vec{a} \times \vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$$

$$= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 \quad \leftarrow \text{Definition of dot product}$$

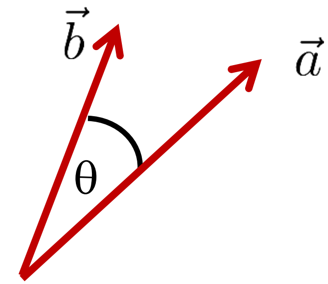
$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2 \cos^2\theta \quad \leftarrow \text{Theorem we just proved}$$

$$= |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta)$$

$$= |\vec{a}|^2|\vec{b}|^2(\sin^2\theta)$$

So taking square root of both sides: $\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|(\sin\theta)$ **Done**

Section 12.4— Cross Products



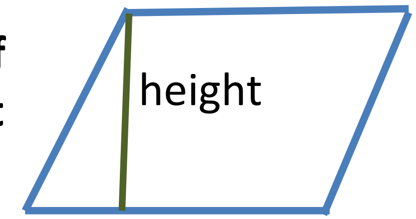
(3) Interesting thing #3: We know that the dot product has a interesting geometric meaning:

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

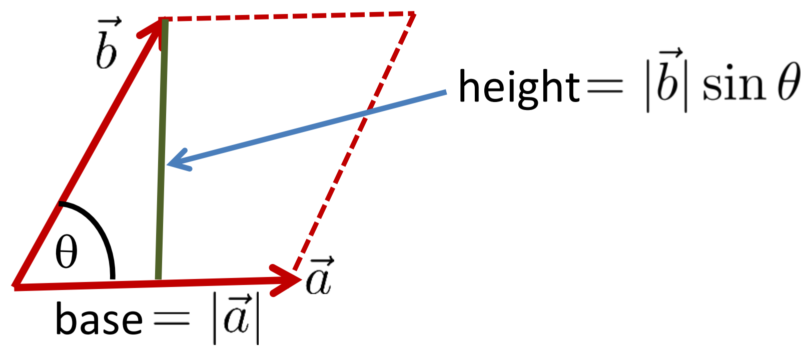
measures the angle between the two vectors.

Question: Does the cross product have a geometric meaning?

First remember that the area of a parallelogram is base x height



And now look at our drawing:



So, Area = base x height = $|\vec{a}| \quad |\vec{b}| \sin\theta = |\vec{a} \times \vec{b}|$

So the cross product measures the size of the parallelogram staked out by the vectors

Section 12.5— Equations of lines and planes

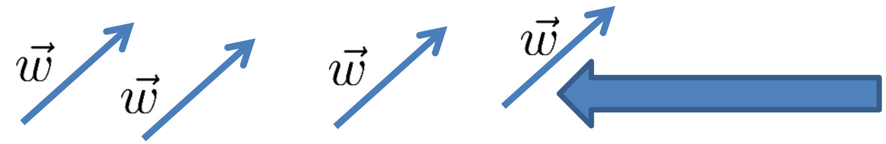
Now that we have these nifty new things like dot products and cross products, we can use them to come up with snappy formulae for lines and planes

The notation here is a little confusing: the vector $\vec{w}$ can mean both

(a) the point where a vector starting at the origin O in the direction $\vec{w}$ ends up

or

(b) the vector $\vec{w}$ which describes a length and direction, but can be anchored anywhere

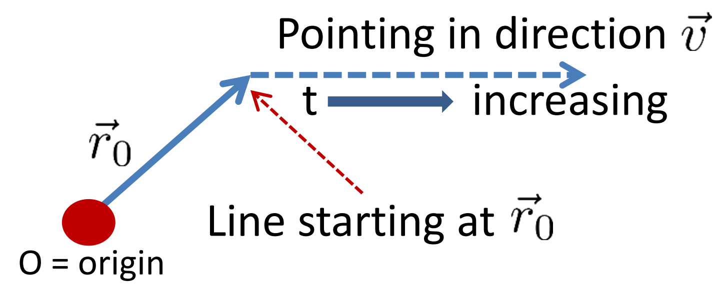


These are all the same vector $\vec{w}$

So let's find a new way to write the equation of a line

Called "vector eqn. of line" $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

- **Parameter**: t
- **Starting point**: $\vec{r}_0$
- **Direction**: $\vec{v}$



Example:

$\vec{r}_0 = (x_0, y_0, z_0)$

$\vec{v} = (a, b, c)$

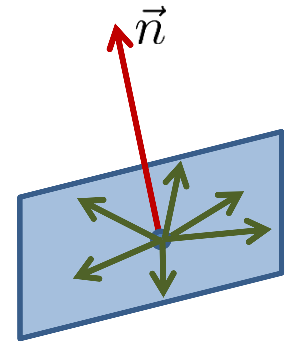
t →

$$x(t) = x_0 + ta$$ → x coor. of line

$$y(t) = y_0 + tb$$ → y coor. of line

$$z(t) = z_0 + tc$$ → z coor. of line

Section 12.5— Equations of lines and planes

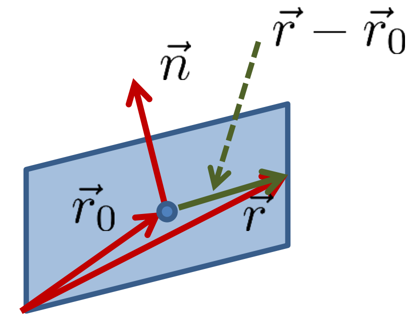


What about a fancy way to describe a plane?

Idea: Use the normal vector to write an equation for a plane

Let the vector $\vec{n}$ be normal to the plane: that means that every vector lying in the plane is orthogonal to $\vec{n}$

$$\vec{n} \cdot \textbf{every green vector} = 0$$

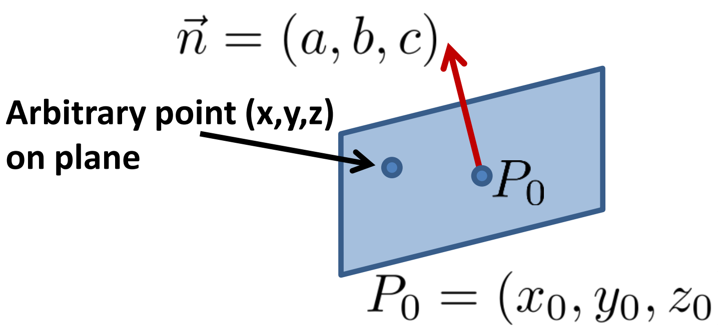


(1) So, let $\vec{r}$ and $\vec{r}_0$ be points in the plane

(2) Then $\vec{r} - \vec{r}_0$ is a vector lying in the plane

(3) So that means that $\vec{r} - \vec{r}_0$ must be orthogonal to the normal $\vec{n}$

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$



What about a scalar form of this?
Then any point (x,y,z) on plane must satisfy

$$(a, b, c) \cdot ((x, y, z) - (x_0, y_0, z_0)) = 0$$

Expanding this out, we get

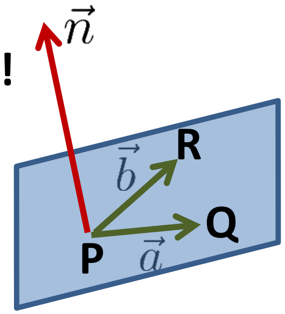
$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Called " scalar eqn. for plane"

Section 12.5

Example: Find the equation of the plane passing through the points:
P = (1,3,2) Q = (3,-1,6) R = (5,2,0)

Solution:



Step 0: We realize that three points determine a plane, so that's good!

Step 1: We need to find the normal to this plane:

idea! If we had two vectors $\vec{a}$ and $\vec{b}$ in the plane then the only vector orthogonal to both of them is the normal $\vec{n}$

Step 2: But we do have two vectors in the plane

$$\vec{a} = Q - P = (3-1, -1-3, 6-2) = (2, -4, 4)$$
$$\vec{b} = R - P = (5-1, 2-3, 0-2) = (4, -1, -2)$$

Step 3: And given these two vectors, we know that $\vec{a} \times \vec{b}$ is orthogonal to both

$$\vec{n} = \vec{a} \times \vec{b} = (12, 20, 14) \qquad \text{please check this}$$

Step 4: So given our formula for a plane given a normal (a,b,c) and a point (x_0, y_0, z_0)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

we now have

12 (x-1) + 20 (y-3) + 14 (z-2) = 0