Math 53: Tu-Thurs, 8-9:30AM

J.A. Sethian, 725 Evans Hall sethian@math.berkeley.edu

Office Hours:

Right after class, outside

Course Website:

www.math.berkeley.edu/~sethian/course.html

Grade Calculation:

30% First midterm

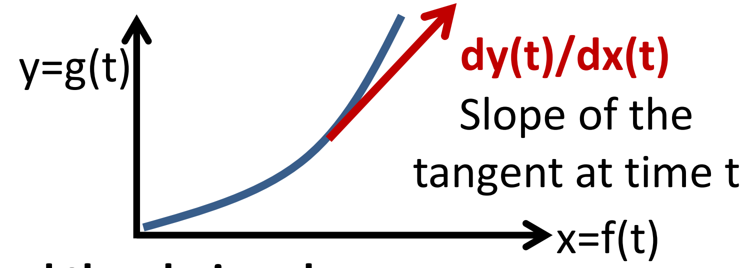
30% Second Midterm

30% Final Exam

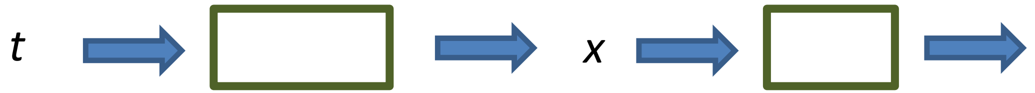
10% Homework and quizzes

Review of 10.1 and 10.2: (what we did last time)

We talked about parameterized curves



- **x = f(t) y = g(t)**
- **Example: x(t) = 3t ; y(t) = 8 t^3**
- **In order to write y as a function of x, we used the chain rule:**



Chain Rule: $$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \quad \Rightarrow \quad \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{dy}{dx}$$

(assume dx/dt ≠ 0)

So now we can compute the slope of tangent lines:

Example: Suppose $x(t) = \sqrt{t}$ and $y(t) = t^2 - 2t$ **Find the tangent line at t=4**

Solution: Step 1: dy/dx = [dy/dt]/[dx/dt] $= \frac{2t-2}{(1/2)t^{-1/2}} = 4(t-1)\sqrt{t}$ At $t = 4$ we get $\frac{dy}{dx} = 24$

Step 2: Find tangent point and tangent line

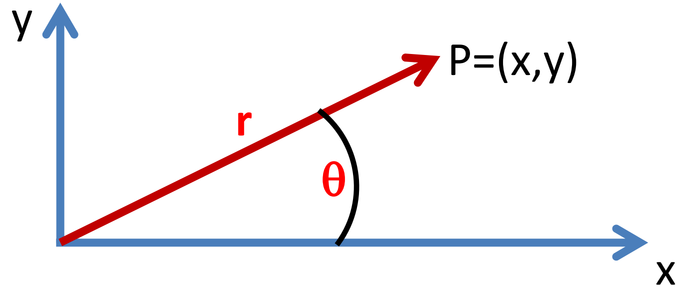
At t=4, point is (2,8), so tangent line is $(y-y_0)$ = slope * $(x-x_0)$

(y-8)=24*(x-2)

Section 10.3

Polar coordinates are a different way to characterize $\mathbb{R}^2$

[$\mathbb{R}^2$ is a fancy way to write 2D]
[$\mathbb{R}^1$ is the real line]
[$\mathbb{R}^3$ is 3D space]

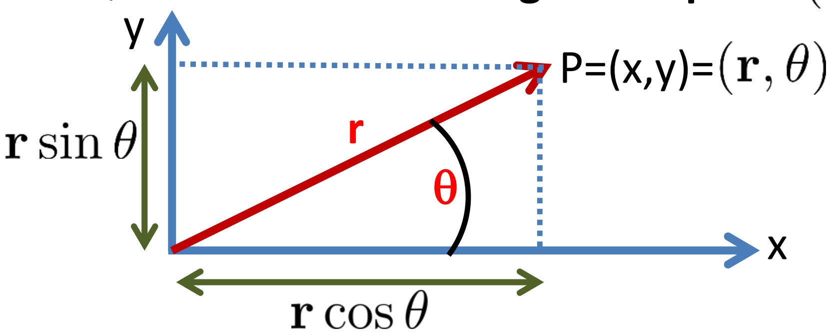


Examples of polar $(\mathbf{r}, \theta)$ representations of points:

$$(\mathbf{5}, \mathbf{0}), (\mathbf{5}, \mathbf{2}\pi), (\mathbf{5}, \mathbf{4}\pi), (-\mathbf{5}, \pi)$$

All the same point (just to make this less crazy, we'll rule out r<0)

Question: How do we go from polar $(\mathbf{r}, \theta)$ coordinates to Cartesian (x,y) coordinates?

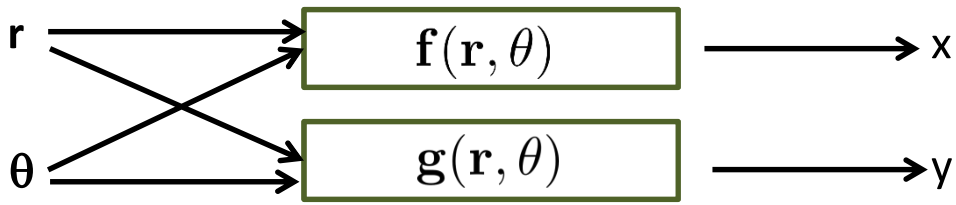


So:

$$\mathbf{x} = \mathbf{f}(\mathbf{r}, \theta) = \mathbf{r}\cos\theta$$

$$\mathbf{y} = \mathbf{g}(\mathbf{r}, \theta) = \mathbf{r}\sin\theta$$

It is useful to start thinking about function boxes

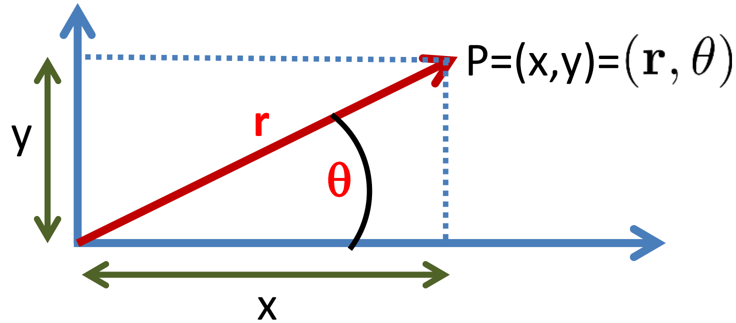


Start to think about mappings: this is a mapping from $\mathbb{R}^+ \times \mathbb{R}^1 \to \mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$

(actually, I should write [0,∞) instead of $\mathbb{R}^+$ but I'm too lazy to figure out how to do it…)

Section 10.3

Question: How do we go from Cartesian (x,y) coordinates to polar $(\mathbf{r}, \theta)$ coordinates?



So:

$$\mathbf{r} = \sqrt{\mathbf{x^2} + \mathbf{y^2}}$$

$$\tan\theta = \frac{\mathbf{y}}{\mathbf{x}}$$

Let's get fancy with our notation: this is a mapping $\mathbb{R}^2 \rightarrow \mathbb{R}^+ \times \mathbb{R}^1$

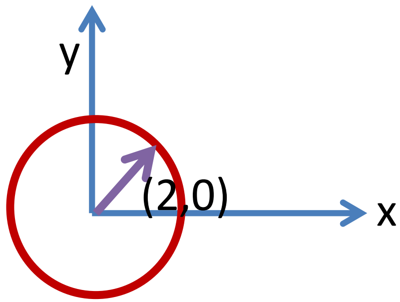
Graphing in polar coordinates:

(Cranky editorial comments:)

1. *This is non-intuitive.*
2. *It's a pain in the neck.*
3. *But you should see it once in your life.*
4. *And today's the day.*

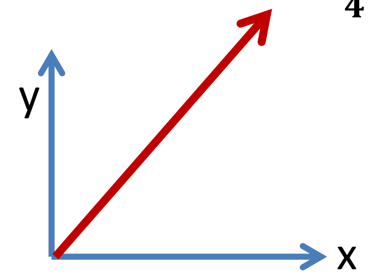
Question: What does the curve r=2 look like?

Circle of radius 2



Question: What does the curve $\theta = \frac{\pi}{4}$ look like?

Straight line

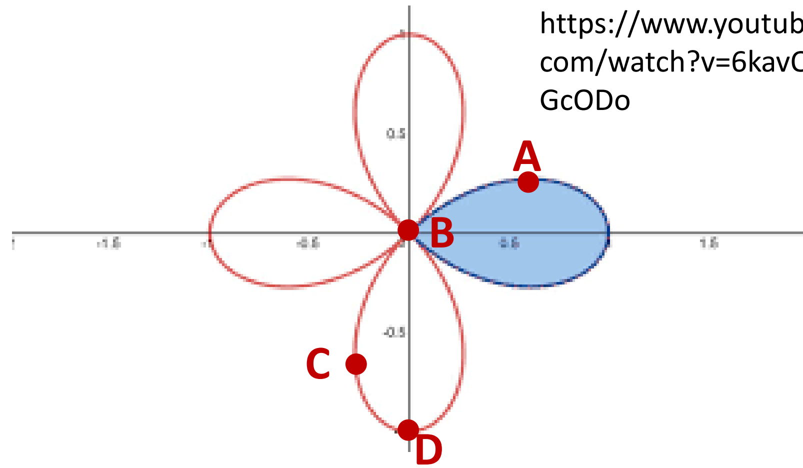


Section 10.3

Question: What does the curve r= cos (2θ) look like?

Solution: First, try plotting some points:

Point	θ	r	x	y
A	$\frac{\pi}{8}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}\cos\frac{\pi}{8}$	$\frac{\sqrt{2}}{2}\sin\frac{\pi}{8}$
B	$\frac{\pi}{4}$	0	0	0
C	$\frac{3\pi}{8}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}\cos\frac{3\pi}{8}$	$-\frac{\sqrt{2}}{2}\sin\frac{3\pi}{8}$
D	$\frac{\pi}{2}$	-1	0	-1



From https://www.youtube.com/watch?v=6kavQyGcODo

Suppose we are given a polar curve r = r (θ). How do we find tangents to polar curves?

Step 1: We have

$$\mathbf{x} = \mathbf{r}\cos\theta = \mathbf{r}(\theta)\cos\theta$$

$$\mathbf{y} = \mathbf{r}\sin\theta = \mathbf{r}(\theta)\sin\theta$$

So we are viewing θ as parameterizing the curve

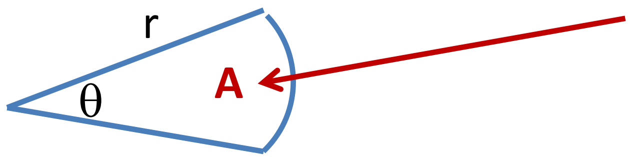
Step 2: So now we can find dy/dx:

$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{\frac{\mathbf{dy}}{\mathbf{d}\theta}}{\frac{\mathbf{dx}}{\mathbf{d}\theta}} = \frac{\frac{\mathbf{dr}}{\mathbf{d}\theta}\sin\theta + \mathbf{r}(\theta)\cos\theta}{\frac{\mathbf{dr}}{\mathbf{d}\theta}\cos\theta - \mathbf{r}(\theta)\sin\theta}$$

Section 10.4

Question: How can we find areas in polar coordinates?

Recall that:



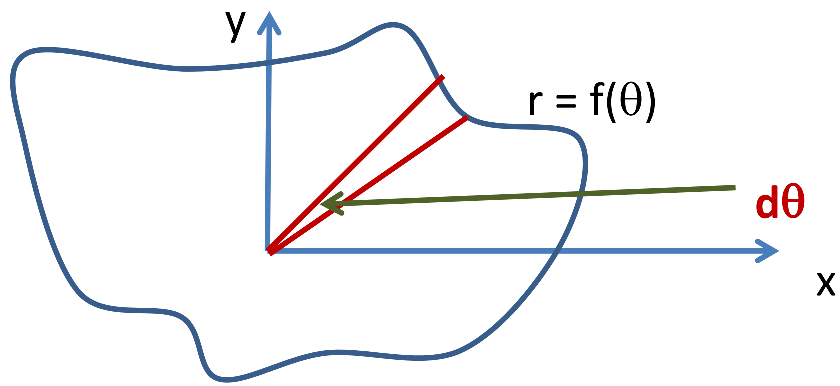
Area = (1/2) $r^2\,\theta$

Why?

So we integrate all the slivers and find an area integral:

Consider a curve r = f(θ)

Area of little sliver = (1/2) $[f(\theta)]^2\,d\theta$
= (1/2) $r^2\,d\theta$



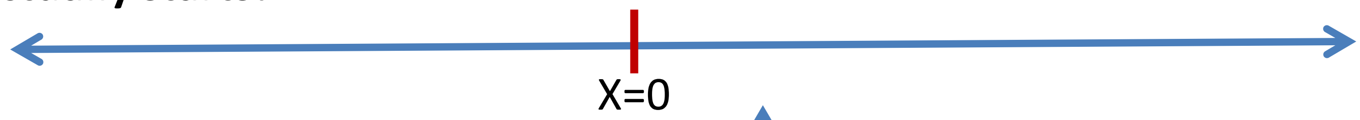
So it's now calculus—add them all up:

$$\mathbf{Area} = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2} r^2\, d\theta$$

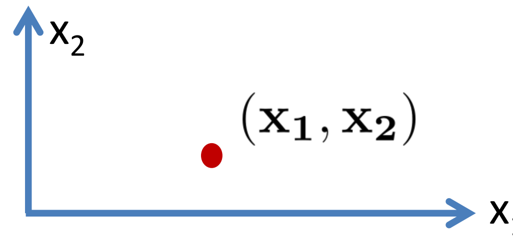
Section 12.1

And now the course actually starts!

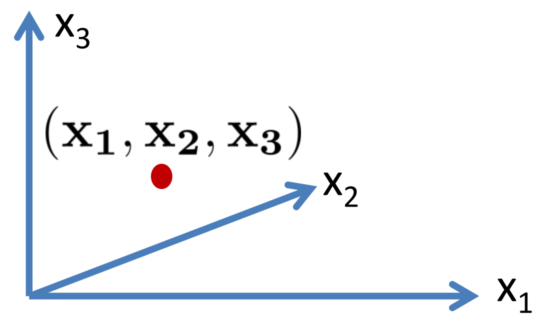
$\mathbb{R}^1$ **The real line:**



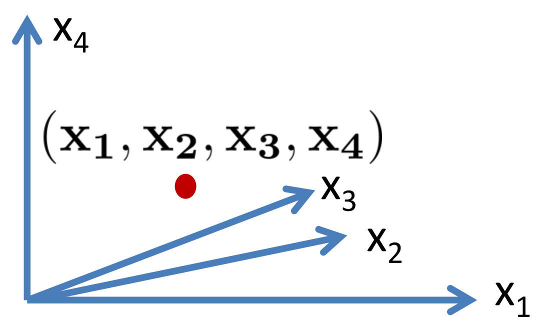
$\mathbb{R}^2$ **Two-dimensional space = the plane**



$\mathbb{R}^3$ **Three-dimensional space**



Why stop there? We have $\mathbb{R}^4$



Just because I can't draw it doesn't mean it's not there!

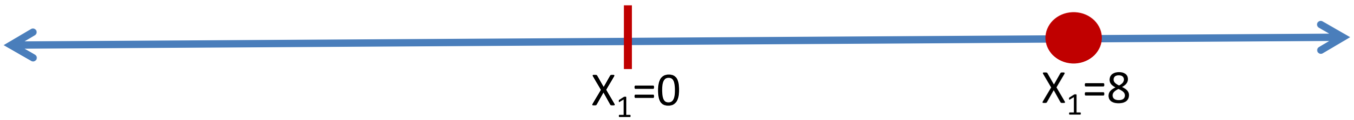
$\mathbb{R}^5$? (sure!) $\mathbb{R}^6$? (why not?)

Section 12.1

Some examples:

(1) Question: what is the object in $\mathbb{R}^1$ represented by x_1 =8?

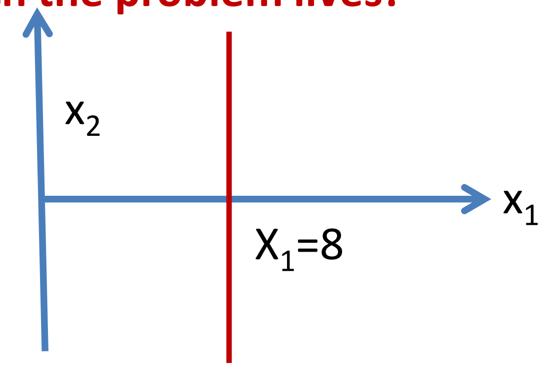
Answer:



It's a point located at X_1=8 that divides one-dimensional space into two parts

Note that the answer depends on the dimension in which the problem lives!

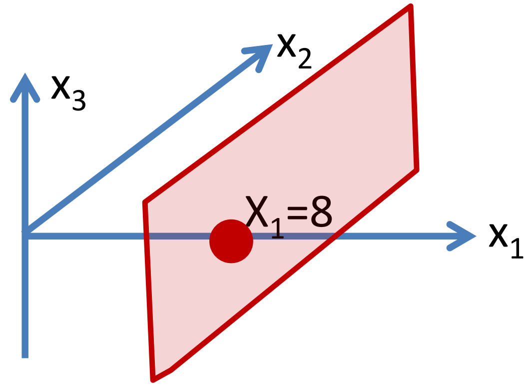
(2) Question: what is the object in $\mathbb{R}^2$ represented by x_1 =8?



Answer: *It's the line located at X_1=8 that divides two-dimensional space into two parts*

Note that the answer depends on the dimension in which the problem lives!

(3) Question: what is the object in $\mathbb{R}^3$ represented by x_1 =8?



Answer: *It's the plane located at X_1=8 that divides three-dimensional space into two parts*

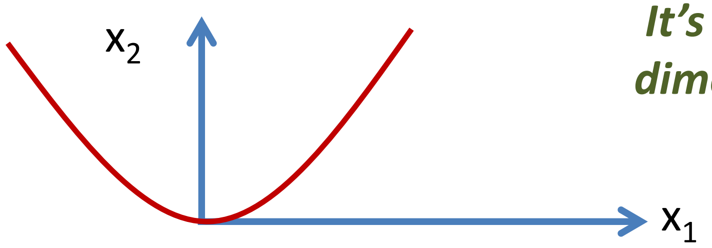
Note that the answer depends on the dimension in which the problem lives!

Section 12.1

More examples:

(3) Question: what is the object in $\mathbb{R}^2$ represented by $\mathbf{x_2 = x_1{}^2}$?

Answer:



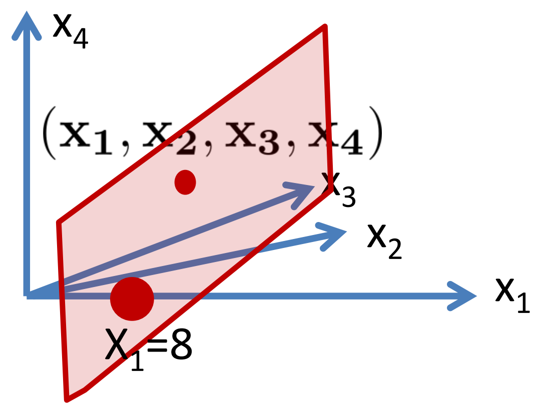
It's a parabola that divides two-dimensional space into two parts

(4) Question: what is the object in $\mathbb{R}^4$ represented by $x_1 = 8$?

Answer: Well, I can't draw it---but the answer is all "4-tuples"

$$(\mathbf{x_1, x_2, x_3, x_4})$$

in 4D space such that $x_1 = 8$

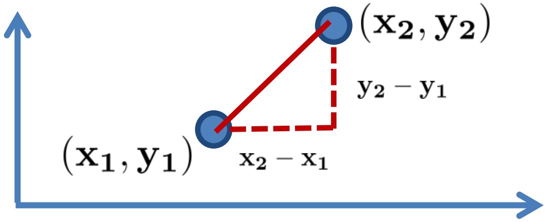


It divides 4D space into 2 parts— each of which is a 4D space

(you gotta open up your mind to see this!)

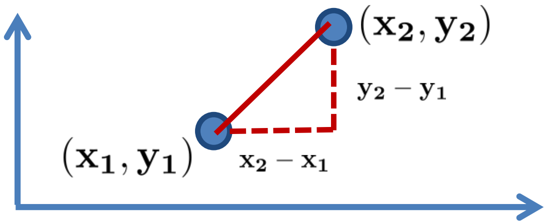
Section 12.1

What is the distance between two points $(x_1, y_1), (x_2, y_2)$ in $\mathbb{R}^2$?



Pythagorean Theorem = Distance = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

What is the distance between two points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ in $\mathbb{R}^3$?



Pythagorean Theorem = Distance = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

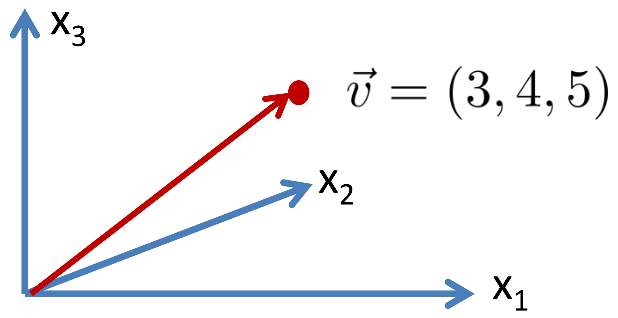
How can we use this to find equation for a sphere of radius R in $\mathbb{R}^3$ centered around (j,k,l)?

Answer: Want to write an expression for all (x,y,z) that a distance R from the point (j,k,l)

$$R = \sqrt{(x - j)^2 + (y - k)^2 + (z - l)^2}$$

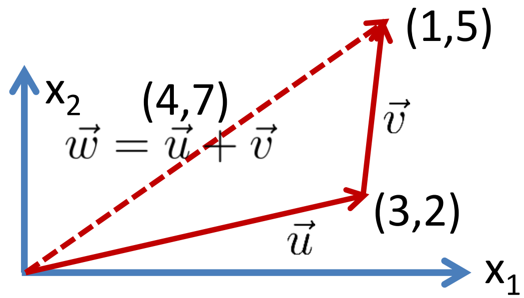
Section 12.2

Vectors:



Vector addition: $\vec{u} = (3, 2)$ $\vec{v} = (1, 5)$ $\rightarrow$ $\vec{u} + \vec{v} = (3 + 1, 2 + 5) = (4, 7)$

Geometrically:

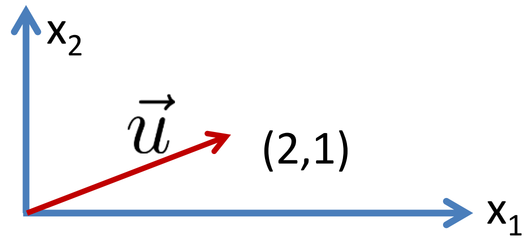


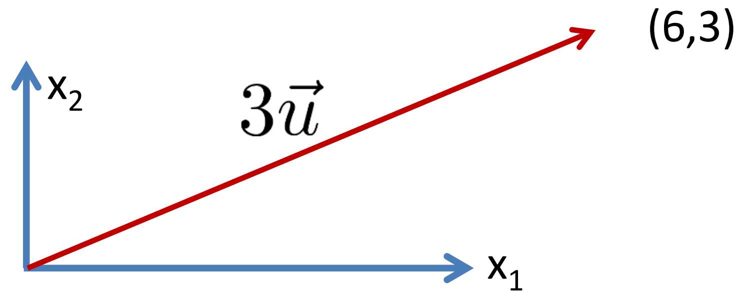
Section 12.2

We can do lots of things with vectors….

Thing 1: Definition: Scalar Multiplication

If c is a scalar, and $\vec{v}$ is a vector, then the scalar multiple $c\vec{v}$ is defined as a new vector that points in the same direction as $\vec{v}$ with length c times the original length





Section 12.2

We can do lots of things with vectors....

Thing 2: Definition: Length of a vector

$\vec{v} = (v_1, v_2, v_3)$ then the length of $\vec{v}$ is given by $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Thing 3: Properties of Vectors:

- $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- $\vec{a} + \vec{0} = \vec{a}$
- $c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}$
- $cd(\vec{a}) = c(d\vec{a})$
- $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
- $\vec{a} + (-\vec{a}) = \vec{0}$
- $(c + d)\vec{a} = c\vec{a} + d\vec{a}$
- $1\vec{a} = \vec{a}$