Math 53: Tu-Thurs, 8-9:30AM

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Office Hours:

I will be on-line from 9:30 to 10:30 after class, Tu-Thurs.

Course Website:

www.math.berkeley.edu/~sethian/course.html

Grade Calculation:

30% First midterm

30% Second Midterm

30% Final Exam

10% Homework and quizzes

Course Logistics:

- **These lecture notes will be posted on the course web page within 48 hours of the lecture.**
- **I will post homework assignments on the web page**
- **Homeworks are due in your section: first meeting of the week they are due.**
- **Midterms are during class time (8-9:30, Tues/Thurs. Dates are:**
  First Midterm = Thursday, Oct. 2nd—in class.
  Second Midterm = Thursday, Nov 6th----in class.
  Final = Wednesday, Dec 17th, 3-6PM
- **Quizzes are given in Section**
- **I cannot move the final. Please check that you do not have a conflict.**

Tips for doing very well in this course:

- **Show up/log in on time.**
- **Come to class. I discuss and emphasize things that I think are important.**
- **I assign lots of homework:**
- **Doing lots of homework increases your ability to understand, recognize how to do a problem, and work quickly on exams.**
- **I'm lazy: I take at least 80% of my exam questions from the homework.**

If you don't do any of the homework, it's only 5% of your grade. But….(see above)

I like to call on people during class—this is so I get a sense of who is understanding what.

Don't worry—You can always say "pass"—and then I will move on.

Now, normally, at this point I would say

Turn your cell phones off.

So let me say something different. What's the point of getting up at 8AM if you don't pay attention?

Questions?

How should you prepare yourself to do well in this class?

Ask questions. Anything (about math) is okay.

Optimally, read the section before I talk about it.

You may not understand it, but it will make my lectures make more sense.

You should expect to spend 10-15 hours a week on this class, not including class time.

Treat it like a job. Focus and do your homework during the day. So that you can get exercise, read books, relax with friends, etc. at night.

Do not start the homework at 9PM the night before it's due.

Treat this like a job.

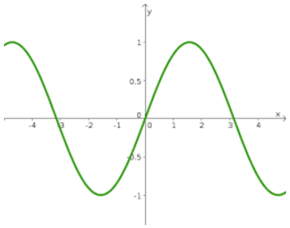
Okay. Enough preliminaries. What is this course about? (the non-math explanation).

You have many examples of functions of one variable:

One input determines one output:

(Ex.: your height as a function of time [trust me, at some point in life, you start to get shorter])

Often: want to find maxima and minima



Examples of functions of many variables:

Multiple inputs determine one output:

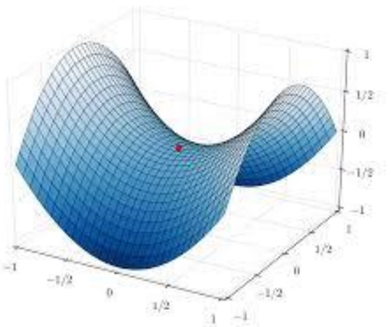
Example #1: The price of a car (depends on make, mileage, condition, color, ….)

Example #2: Desirability of a roommate (Messy? Clean? Goes to bed early? Late?...)

Just like 1D: we want to optimize:

Find maxima and minima

Add constraints:



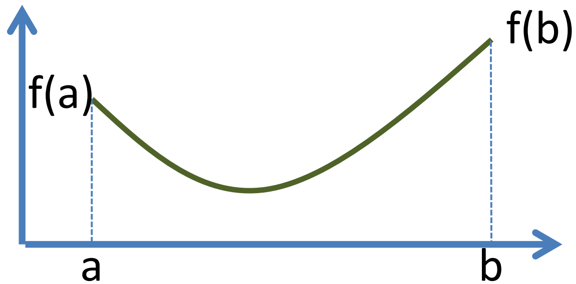
Just like 1D: we also want to integrate:

Find area (volume) under a function

What is this course about? (the math explanation).

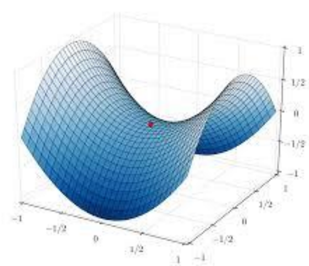
One Dimensional Calculus: $y = f(x)$ ⇐ **Fundamental Theorem of Calculus:** ⇒ $f(b) - f(a) = \int_a^b \frac{df}{dx} \; dx$

Which says that : **"the difference between a function at two ends of an interval, namely f(b)-f(a), is the total amount of 'derivative' on the inside"**



Multiple Variable Calculus: $z = f(x, y)$ ⇐ **(some version of the) Fundamental Theorem of Calculus:** ⇒ $\int_C \mathbf{F} \cdot \mathbf{r} = \int \int_\mathbf{S} \nabla \times \mathbf{F} \cdot \mathbf{dS}$ **???**

Which says that : **"the difference between the amount on the boundary has something to do with a derivative on the inside"**



Don't worry. This will all make sense at the end of this course!

$$\int_C \mathbf{F} \cdot \mathbf{r} = \int \int_{\mathbf{S}} \nabla \times \mathbf{F} \cdot \mathbf{dS}$$

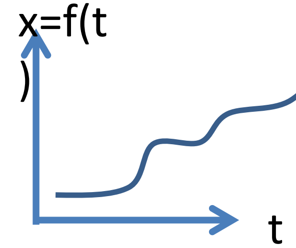
In fact, by the end of this semester, I will tell you where to get your own multivariable calculus mug

So, let's go!!!

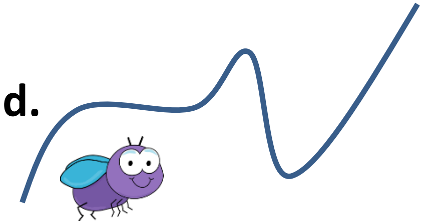
Section 10.1

Suppose x =f(t), and f is a given function.
We usually plot f against t:

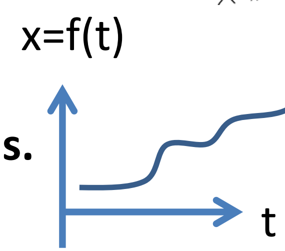
In other words, at any time t we get the position x(t).



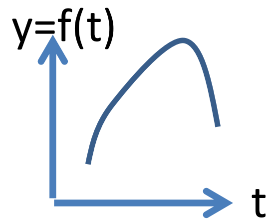
Let's think of what this might mean in 2D
Suppose we have a bug crawling the ground.



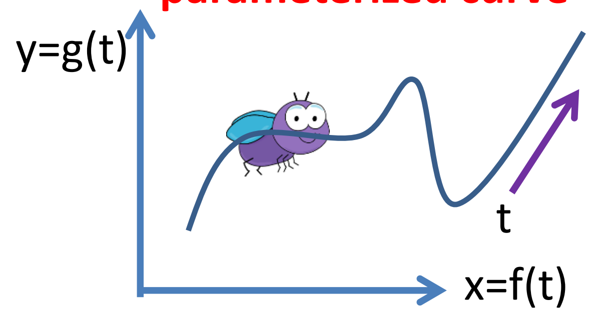
We could let x =f(t) be the
x-coordinate of the bug as it crawls.



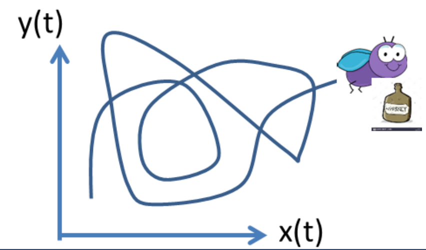
We could let y =g(t) be the
y-coordinate of the bug as it
crawls.



Or, we could plot them together: called a parameterized curve



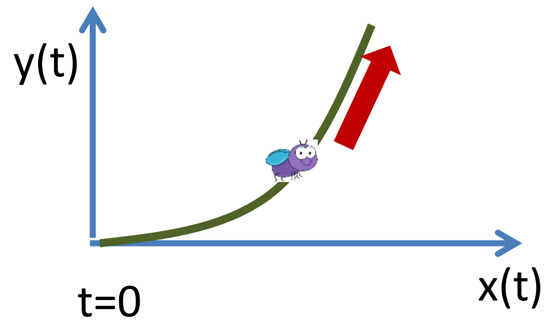
Advantage: Goes beyond simple functions:
You can track the path of a drunk bug:



Section 10.1

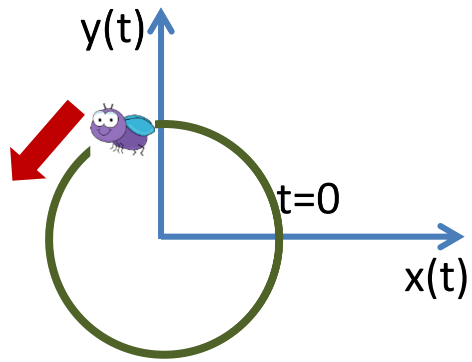
Example #1: $x = f(t) = t$
$y = g(t) = t^2$

The bug crawls on along a parabola



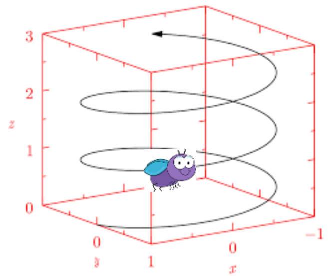
Example #2:
$x = f(t) = 2\cos(t)$
$y = g(t) = 2\sin(t)$

The bug crawls on along a circle of radius 2



Example #3:
$x = f(t) = 2\cos(t)$
$y = g(t) = 2\sin(t)$
$z = h(t) = t$

Now the bug is a mosquito tracing out a helix in 3D



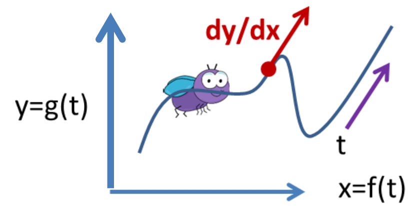
[image from artofproblemsolving.com]

Section 10.2

Well, now that we have parameterized curves, we want tangents and derivatives to curve....

Suppose we have *x=f(t)* and *y=g(t)*:

- **So *x* depends on *t*, and *y* depends on *t***
- **which means that *t* can be written as a function of *x***
- ***and that means that we can write y as a function of x.***



Graphically, what does all this mean?

link #1: $t \rightarrow$ **$f(t)$** $\rightarrow x$

link #3: $t \rightarrow$ **$g(t)$** $\rightarrow y$

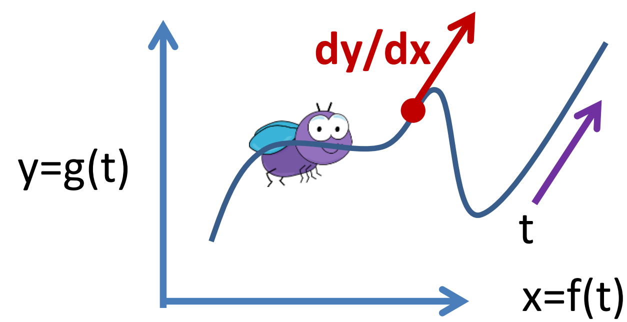
So we can invert this:

link #2: $x \rightarrow$ **$f^{-1}(x)$** $\rightarrow t$

So we can chain them together:

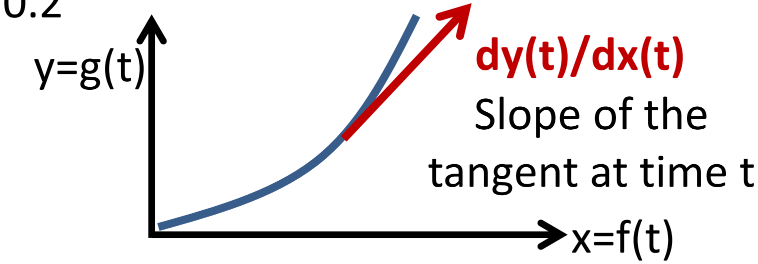
$x \rightarrow$ **$f^{-1}(x)$** (link #2) $\rightarrow t \rightarrow$ **$g(t)$** (link #3) $\rightarrow y$

So we get to think of y as a function of x.
Which means we can ask about dy/dx:
which is the tangent slope to the curve

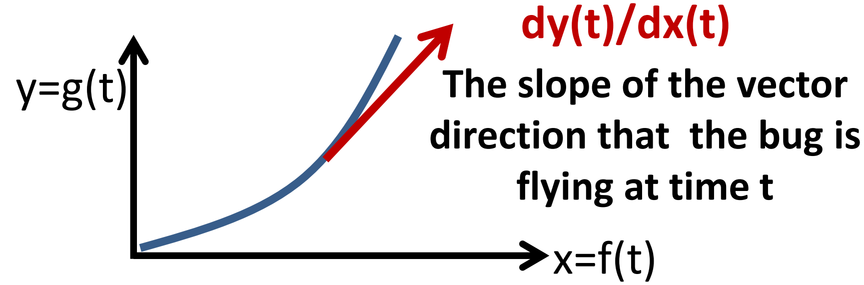


Section 10.2

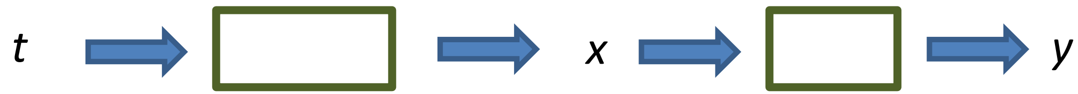
(1) **Before finding** **dy/dx, let's be clear about what it is geometrically:**



(2) **What is dy/dx "mosquito-wise"?**



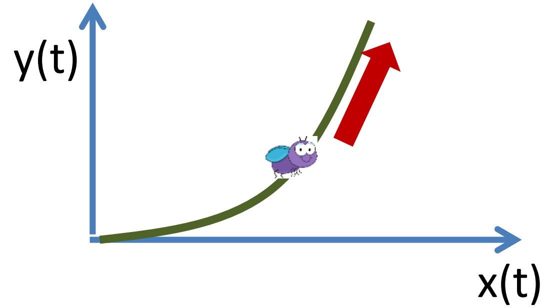
(3) **What is dy/dx analytically?** **y is a function of x which is a function of t**



Chain Rule: $$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt} \quad \Longrightarrow \quad \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{dy}{dx}$$

(assume dx/dt ≠ 0)

Section 10.2



[Eqn. #1] [Eqn. #2]

Example 1: Suppose $x=f(t) = t$ **and** $y=g(t)=t^2$

What is dy/dx at t = 2 ?
(same question as: what is the slope of the parameterized curve at time t=2)?

The old way you would have done this:

$$y = g(t) = t^2$$ [this is Eqn. #1]

$$x = f(t) = t$$ [this is Eqn. #2]

$$y = t^2 = x^2$$ [substituting Eqn. #1 into Eqn. #2]

$$\frac{dy}{dx} = 2x$$ [take the derivative]

At t=2, we have that x=2, so dy/dx = 4

The new way: $$\frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{dy}{dx}$$

Taking derivatives we have

dy/dt =2t
dx/dt = 1

So: dy/dx = [dy/dt]/[dx/dt]= [2t]/1 = 2t which at t=2 gives dy/dx=4

Section 10.2

Question: Why not always do it the old way?
Answer: Because you are not often going to have such nice functions!

Example 2: Let C be the curve parameterized by $x=f(t) = t^2$ and $y=g(t)=t^3 - 3t$

Find dy/dx at t=4

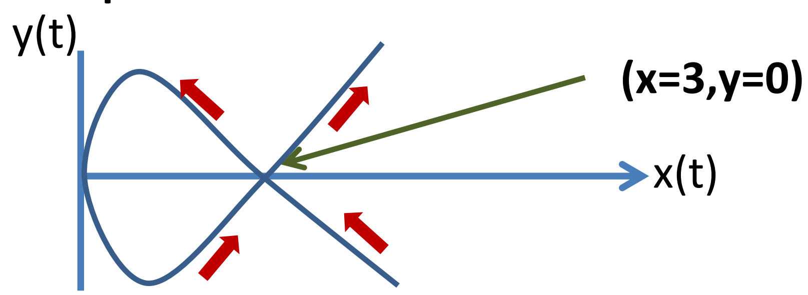
The new way: $$\frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{dy}{dx}$$ **Taking derivatives we have**

$dy/dt = 3t^2 - 3$
$dx/dt = 2t$

So: dy/dx = [dy/dt]/[dx/dt]= $[3t^2 - 3]/2t$ which at t=4 gives dy/dx=[48-3]/[2*4] =45/8

Example 3: Using the same equations, show that the parameterized curve C has two tangents at the point (x=3,y=0), and find their equations

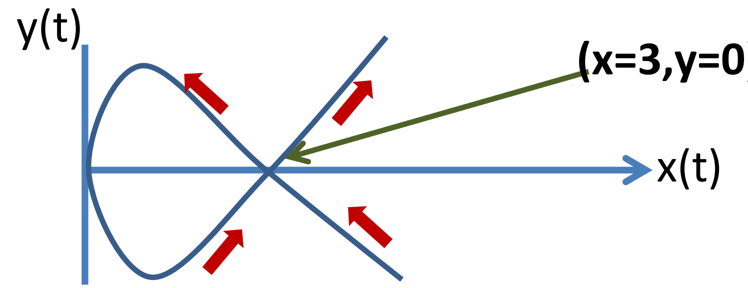
First, we graph it:



[Check this yourself!]

Section 10.2

Example 3: Using the same equations, show that the parameterized curve C has two tangents at the point (x=3,y=0), and find their equations



Step 1: **What are values of t corresponding to when the curve intersects itself at (3,0)?**

Solution: Solve for t such that x=3 and y= 0

$3=x=f(t) = t^2$ ⇒ $t = \pm\sqrt{3}$

$0=y=g(t)=t^3 - 3t$ ⇒ $t = 0;\ t = \pm\sqrt{3}$

So, bug goes through (3,0) at $t = \sqrt{3}$ and $t = -\sqrt{3}$

Step 2: Find dy/dx at both of these times: **dy/dx = [dy/dt]/[dx/dt]= $[3t^2 - 3]/2t$**

Solution:
at $t = \sqrt{3}:\quad \mathbf{dy/dx = 3/2 * (\sqrt{3} - 1/\sqrt{3}) = \sqrt{3}}$
at $t = -\sqrt{3}: \mathbf{dy/dx = 3/2 * (-\sqrt{3} + 1/\sqrt{3}) = -\sqrt{3}}$

So both tangent lines go through (3,0): one with slope $\sqrt{3}$ **the other with slope** $-\sqrt{3}$

Step 3: Write down equations of tangent lines: $\mathbf{(y - y_0) = slope * (x - x_0)}$

Solution: $(y - 0) = \sqrt{3}(x - 3)$ $\qquad (y - 0) = -\sqrt{3}(x - 3)$

Section 10.2

New topic: Now that we know how to find slope dy/dx of a parameterized curve, what about

the second derivative: $\left(\frac{d^2y}{dx^2}\right)$

We have: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ **So, replacing y by** $\frac{dy}{dx}$ **we then have:**

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

(you have to think about this a minute:)



$$\frac{d[\ \ ]}{dx} = \frac{\frac{d[\ \ ]}{dt}}{\frac{dx}{dt}} \text{ and let } [\textbf{duck}\ \ ] = \frac{dy}{dx}$$

Section 10.2

Let's use it: back to our example: *$x=f(t) = t^2$* **and** *$y=g(t)=t^3 - 3t$*

What is: $\left(\frac{d^2y}{dx^2}\right)$

Solution: We have that $$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

From before, we found out that: $dy/dx = [dy/dt]/[dx/dt]= [3t^2 - 3]/2t$

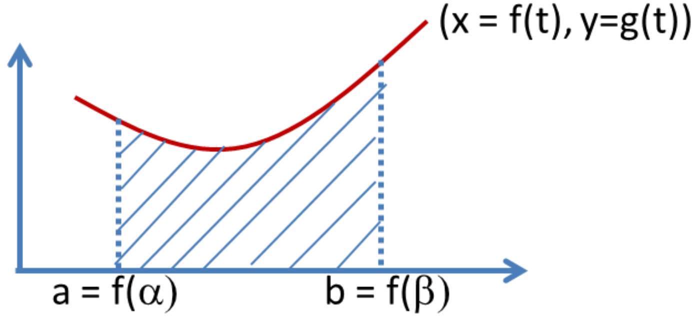
First, let's simplify a bit: $[3t^2 - 3]/2t = (3/2) * (t – 1/t)$

So,

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(3/2*(t-1/t))}{\frac{dx}{dt}} = \frac{3/2*(1+\frac{1}{t^2})}{2t}$$

Section 10.2

Onwards:
How do we find the area under a parameterized curve?



Solution: $\text{Area} = \int_a^b y\,dx$ **Then x=f(t), y=g(t) and dx =f '(t) dt**

Suppose $\mathbf{a = f(t = \alpha)}$ and $\mathbf{b = f(t = \beta)}$

So we can change variables from x to t in the integral, and get

$$\text{Area} = \int_a^b y\,dx = \int_\alpha^\beta g(t) f'(t) dt$$

Example: Find the area under x=f(t)=t^2 ; y=g(t)=t^4 from x= 1 to x=4

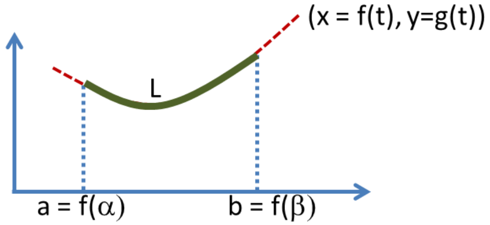
Solution: Step 1: Find limits of integration for "t": $x = 1 \rightarrow \alpha = 1 \quad x = 4 \rightarrow \beta = 2$

Step 2: Evaluate integral:

$$\text{Area} = \int_\alpha^\beta g(t) f'(t) dt = \int_1^2 t^4 2t\,dt = \int_1^2 2t^5 dt = \Big|_1^2 \frac{2t^6}{6} = \frac{1}{3}[2^6 - 1] = 21$$

Section 10.2

Onwards: Arc-length:
How do we find the length of section of parameterized curve?



Solution: Recall that the length of the curve y=f(x) from X=a to X=b is

$$\mathbf{Length\,L} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

What should this be in parameterized form? (x=f(t), y=g(t))

Using our formula that $\frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{dy}{dx}$ **, and remembering that dx = (dx/dt) dt:**

$$\mathbf{Length\,L} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\,\frac{dx}{dt}\,dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$