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Advancing Interfaces: Level Set and Fast Marching Methods

J.A. Sethian

Dept. of Mathematics, University of California, Berkeley[1]

1 Introduction

In this article, we review past work on Level Set Methods, introduced by Osher and Sethian in [20], and Fast Marching Methods, introduced by Sethian in [25], for tracking propagating interfaces in two and three space dimensions. Both sets of techniques are based on a partial differential equations view of interface motion, and rely on the use of the theory of viscosity solutions, upwind finite difference schemes for hyperbolic conservation laws, and the theory of curve and surface evolution developed in [23]. Both sets of techniques require an adaptive methodology to obtain computational efficiency. We briefly review some of these methods, and show some examples of some applications.

2 Overview

Fast Marching Methods and Level Set Methods are computational techniques based on finite difference schemes for tracking propagating interfaces. They share the virtues of working in an arbitrary number of space dimensions with no change, handle topological merger and splitting with no special procedures, and accurately and efficiently compute the motion of fronts with sharp corners moving under speed laws which may include large variations in velocities.

Fast Marching Methods, introduced by Sethian in [25], approximate the solution of a boundary value partial differential equations view of propagating interface, while level set methods, introduced by Osher and Sethian in [20], approximate the solution of an initial value partial differential equation. At the core, both techniques rely on viscosity solutions for Hamilton-Jacobi equations, linking finite difference upwind schemes for hyperbolic conservation laws to propagating fronts, and aspects of the theory of curve and surface evolution. They have been used in a large variety of applications, including problems in fluid interface motion, combustion, dendritic solidification, etching and deposition in semi-conductor manufacturing, robotic navigation and path planning, image seg-

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mentation in medical imaging scans, computation of seismic travel times, and aspects of computational geometry and computer vision.

Both sets of techniques require an adaptive methodology to obtain computational efficiency. In the case of Fast Marching methods, this stems from a Dijkstra-like technique which exploits a causality relationship inherent in the chosen upwind finite difference formulation. In the case of level set methods, this leads to the Narrow Band Method given in [1].

This paper reviews some recent work in these areas. An overview and summary of current work on Fast Marching and level set methods is given in a recent cumulative introduction and resource book on level set and Fast Marching Methods [28]; an additional resource is the web site

http::/math.berkeley.edu/∼ sethian/level_set.html

3 Initial and Boundary Value Formulations of Front Propagation

3.1 A Boundary Value Formulation

Imagine a closed curve Γ in the plane propagating normal to itself with speed F. Furthermore, assume that $F > 0$, hence the front always moves "outwards". One way to characterize the position of this expanding front is to compute the arrival time $T(x, y)$ of the front as it crosses each point (x, y), as shown in Figure 1.

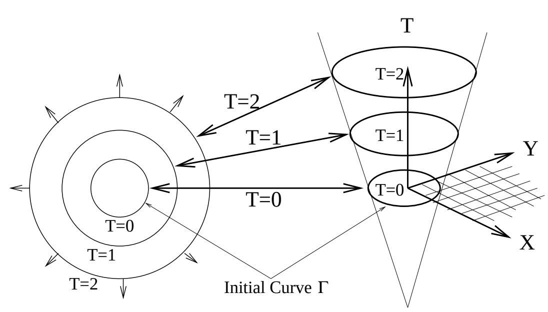

FIG. 1. Transformation of Front Motion into Boundary Value Problem

The equation that describes this arrival surface $T(x, y)$ is given by

$$|\nabla T|F = 1 \qquad\qquad T = 0 \text{ on } \Gamma. \tag{3.1}$$

This is a boundary value problem; if the speed F depends only on position, then the equation reduces to the familiar Eikonal equation.

3.2 Initial Value Formulation

Conversely, suppose we embed the initial position of the front as the zero level set of a higher dimension function ϕ, as was done in [20]. We can then identify the evolution of this function ϕ with the propagation of the front itself through a time-dependent initial value problem. At any time, the front is given by the zero level set of the time-dependent level set function ϕ, see Figure 2.

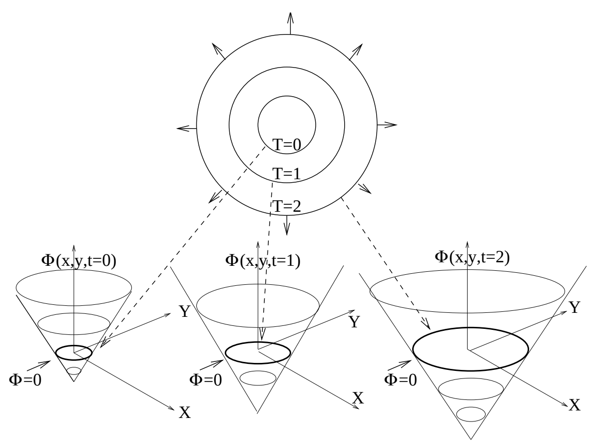

FIG. 2. Transformation of Front Motion into Initial Value Problem

In order to derive an equation of the motion for this level set function ϕ, we note that the stipulation that the zero level set of the evolving function ϕ always match the propagating hypersurface means that

$$\phi(x(t), t) = 0. \tag{3.2}$$

By the chain rule,

$$\phi_t + \nabla \phi(x(t), t) \cdot x'(t) = 0. \tag{3.3}$$

Since F supplies the speed in the outward normal direction, then $x'(t) \cdot n = F$ where $n = \nabla\phi / |\nabla\phi|$ and this yields an evolution equation for ϕ, namely,

$$\phi_t + F|\nabla\phi| = 0, \tag{3.4}$$

$$\text{given} \quad \phi(x, t = 0). \tag{3.5}$$

This is the level set equation introduced by Osher and Sethian [20]. For certain forms of the speed function F, one obtains a standard Hamilton–Jacobi equation.

This equation describes the time evolution of the level surface function ϕ in such a way that the zero level set of this evolving function is always identified with the propagating interface; see Figure 2.

Thus, we wish to solve

Initial Value Formulation	**Boundary Value Formulation**
$\phi_t + F\lvert\nabla\phi\rvert = 0$	$\lvert\nabla T\rvert F = 1$
Front= $\Gamma(t) = \{(x,y) \lvert \phi(x,y,t) = 0\}$	Front= $\Gamma(t) = \{(x,y) \lvert T(x,y) = t\}$
Applies for arbitrary F	Requires $F > 0$

(3.6)

3.3 Advantages of These Perspectives

There are certain advantages associated with these two perspectives on propagating interfaces. First, both are unchanged in higher dimensions; that is, for surfaces propagating in three dimensions and higher. Second, topological changes in the evolving front Γ are handled naturally; the position of the front at time t is given either by the zero level set $\phi(x, y, t) = 0$ of the evolving level set function or the contour $T = t$ of the boundary value solution. This set need not be connected, and can break and merge as t advances. $T(x, y)$ and the level set function ϕ remain single-valued. Both rely on viscosity solutions, and can be converted into computational schemes by exploiting adaptive schemes borrowed from hyperbolic conservation laws.

There are significant differences between the two approaches. The most obvious distinction between the two views is that the initial value level set formulation allows for both positive and negative speed functions F; the front may move forwards and backwards as it evolves. The boundary value perspective is restricted to fronts that always move in the same direction, because it requires a single crossing time T at each grid point, and hence a point cannot be revisited. While approximation of more complex speed functions F, such as those including curvature, are most naturally done in the initial value level set perspective, in contrast, speed functions F which depend on position and vary widely are best handled through the boundary value perspective approximated with Fast Marching Methods. This is because Fast Marching Methods employ no time step, and hence are not subject to CFL conditions, unlike level set methods.

3.4 Numerical Approximations

One of the main difficulties in solving the above equations is that the solution need not be differentiable, even with arbitrarily smooth boundary data. This non-differentiability is intimately connected to the notion of appropriate weak solutions; our goal is construct numerical techniques which naturally account for this non-differentiability in the construction of accurate and efficient approximation schemes, and admit physically correct non-smooth solutions.

In [23; 24], the equation for a curve propagating normal to itself with a given speed F, and which remains a graph as it moves, is studied. An entropy condition is given for this problem which is the limit of smoothed curvature-driven problems as the curvature term goes to zero; this motion is shown to be intimately connected to a hyperbolic conservation law with viscous right-hand-side. For example, consider a function $y = f(x)$ moving with speed $F = 1 - \epsilon\kappa$ in the normal direction; here κ is the curvature at the point x. In [24], this is shown to be related to a hyperbolic conservation law with viscosity, and the suggestion is made to borrow schemes from the numerical solution of gas dynamics to solve the resulting equation of motion. There is a close relationship between this motion and the idea of viscosity solutions for Hamilton-Jacobi equations; we refer the interested reader to [11] for more about this relationship.

The above discussion is limited to curves which remain graphs. The level set method, introduced by Osher and Sethian [20], takes the perspective of embedding the front in one higher dimension. In addition, that work developed a particular set of multi-dimensional schemes to approximate the relevant gradients; details on these schemes may be found in [20].

4 Approximation Schemes

4.1 Stationary Formulation

For the boundary value formulation, we wish to solve

$$|\nabla u(x,y,z)| = f(x,y,z). \tag{4.1}$$

Using the multi-dimensional approximations from [20] we have

$$\begin{aligned} |\nabla u| \quad &\approx \quad (\max(D^{-x}_{ijk}u,0)^2 + \min(D^{+x}_{ijk}T,0)^2 \\ &\qquad + \max(D^{-y}_{ijk}u,0)^2 + \min(D^{+y}_{ijk}T,0)^2 \\ &\qquad + \max(D^{-z}_{ijk}u,0)^2 + \min(D^{+z}_{ijk}T,0)^2)^{1/2} \\ &= \quad f_{ijk}. \end{aligned} \tag{4.2}$$

The forward and backwards operators D^{-y}, D^{+y}, D^{-z}, and D^{+z} in the other coordinate directions are similar to the one defined earlier for the x direction.

A slightly different upwind scheme, given in [21], which will turn out to be more convenient, is given by

$$\begin{bmatrix} \max(D^{-x}_{ijk}u, -D^{+x}_{ijk}u, 0)^2 + \\ \max(D^{-y}_{ijk}u, -D^{+y}_{ijk}u, 0)^2 + \\ \max(D^{-z}_{ijk}u, -D^{+z}_{ijk}u, 0)^2 \end{bmatrix}^{1/2} = f_{ijk}, \tag{4.3}$$

where we use the same forward and backward operators D^- and D^+ and f_{ijk} is the slowness at the gridpoint ijk.

4.2 Level Set Initial Value Formulation

For the Initial Value Formulation, an entropy-satisfying viscosity scheme for initial value formulation was introduced in [20], leading to the numerical method known as the level set method, namely

$$\phi^{n+1}_{ijk} = \phi^n_{ijk} - \Delta t[\max(F_{ijk},0)\nabla^+ + \min(F_{ijk},0)\nabla^-], \tag{4.4}$$

where

$$\nabla^+ = \begin{bmatrix} \max(D^{-x}_{ijk},0)^2 + \min(D^{+x}_{ijk},0)^2 + \\ \max(D^{-y}_{ijk},0)^2 + \min(D^{+y}_{ijk},0)^2 + \\ \max(D^{-z}_{ijk},0)^2 + \min(D^{+z}_{ijk},0)^2 \end{bmatrix}^{1/2}$$

$$\nabla^- = \begin{bmatrix} \max(D^{+x}_{ijk},0)^2 + \min(D^{-x}_{ijk},0)^2 + \\ \max(D^{+y}_{ijk},0)^2 + \min(D^{-y}_{ijk},0)^2 + \\ \max(D^{+z}_{ijk},0)^2 + \min(D^{-z}_{ijk},0)^2 \end{bmatrix}^{1/2}$$

Higher order schemes are available, some based on the ENO formulations introduced in [14]. For details, see [20].

5 Adaptivity:

The schemes given for both the boundary value and initial value formulations are computationally inefficient. In this section, we make both schemes optimal through the use of variations on adaptivity and causality.

5.1 The Narrow Band Level Set Scheme

Equation 4.4 is an explicit scheme, and hence can be solved directly. The time step requirement depends on the nature of the speed function F; for an F that depends only on position, the time step behaves like $\frac{\Delta t}{\Delta x}F \leq 1$. In the case when the speed function F depends on curvature terms (for example, $F = -\kappa$). the equation has a parabolic component, and hence the time step requirement resembles that of a non-linear heat equation; the time step depends roughly on $\frac{\Delta y}{\Delta x^2}$.

In the level set formulation, both the level set function and the speed are embedded into a higher dimension. This then implies computational labor through the entire grid, which is inefficient. A rough operation count of this technique assumes N grid points in each space dimension of a three-dimensional problem. For a simple problem of straightforward propagation with speed $F = 1$; assuming that it takes roughly N time steps for the front to propagate through the domain (here, the CFL condition is taken almost equal to unity), this produces an $O(N^4)$ method.

Considerable computational speedup in the level set method comes from the use of the "Narrow Band Level Set Method", introduced by Adalsteinsson and Sethian in [1]. It is clear that performing calculations over the entire computational domain is wasteful. Instead, an efficient modification is to perform work only in a neighborhood of the zero level set; this is known as the *Narrow Band Approach*. This drops the operation count in three dimensions drops to $O(kN^3)$, where k is the number of cells in the narrow band, a significant cost reduction; it also means that extension velocities need only be done to points lying in the narrow band, as opposed to all points in the computational domain.

Use of narrow bands leads to level set front advancement algorithms that are computationally equivalent in terms of complexity to traditional marker methods and cell techniques, while maintaining the advantages of topological merger, accuracy, and easy extension to multi-dimensions. Typically, the speed associated with the narrow band method is about ten times faster on a 160×160 grid than the full matrix method. Such a speed-up is substantial; in three-dimensional simulations, it can make the difference between computationally intensive problems and those that can be done with relative ease. Details on the accuracy, typical tube sizes, and number of times a tube must be rebuilt may be found in Adalsteinsson and Sethian [1].

5.2 The Fast Marching Method

One approach to solving the finite difference scheme given for the stationary formulation (Eqn. 4.3), given by Rouy in [21], is through iteration, which leads to an $O(N^4)$ algorithm in three dimensions, where N is the number of points in each direction. Instead, we take a different approach.

The Fast Marching method [25] is connected to Huyghen's principle, which is a construction involving expanding wavefronts, and Dijkstra's method, which is depth-first search algorithm on network paths. The viscosity solution to the Eikonal equation $|\nabla u(x)| = F(x)$ can be interpreted through Huyghen's principle in the following way; circular wavefronts are drawn at each point on the boundary, with the radius proportional to $F(x)$. The envelope of these wavefronts is then used to contruct a new set of points, and the process is repeated; in the limit the Eikonal solution is obtained. The Fast Marching Method mimics this construction; a computational grid is used to carry the solution u, and an upwind, viscosity-satisfying finite difference scheme is used to approximate this wavefront.

The order in which the grid values produced through these finite difference approximations are obtained is reminiscent of Dijkstra's method, which is a depth-search technique for computing shortest paths on a network. In that technique, the algorithm keeps track of the speed of propagation along the network links, and fans out along the network links to touch all the grid points. The Fast Marching Method exploits this idea in the context of a continuous finite difference approximation to the underlying partial differential equation, rather than the discrete network links.

The Fast Marching Method evolved in part from examining the limit of the narrow band method as the band was reduced to one grid cell. Fast Marching Methods, by taking the perspective of the large body of work on higher order upwind, finite difference approximants from hyperbolic conservation laws, allow for higher order versions on both structured and unstructured meshes. Several other Dijkstra-like algorithms for solving the Eikonal equation are available. One which is not an upwind finite difference scheme but also respects the viscosity solutions of the underlying partial differential equation was given earlier by Tsitsiklis in [34]. He obtains a control-theoretic discretization of the Eikonal equation, which then leads to a causality relationship based on the optimality criterion rather than on the upwind finite difference operators employed in Fast Marching Methods. In the particular special case of a first order upwind finite difference for the Fast Marching Method on a square mesh, the resulting update equation at each grid point can be seen to be the same quadratic equation obtained through Tsitsikilis's control theoretic approach. We refer the reader to Tsitsiklis [34] for further details on his approach.

In more detail, the Fast Marching Method is as follows. Suppose at some time the Eikonal solution is known at a set of points (denoted *Accepted* points). For every not-yet accepted grid point such that it has an accepted neighbor, we compute a trial solution to the above quadratic Eqn. 4.3, using the given values for u at accepted points, and values of ∞ at all other points. We now observe that the smallest of these trial solutions must be correct, since it depends only on accepted values which are themselves smaller. This " causality" relationship can be exploited to efficiently and systematically compute the solution as follows:

First, tag points in the initial conditions as *Accepted*. Then tag as $Considered$ all points one grid point away and compute values at those points by solving Eqn. 4.3. Finally, tag as Far all other grid points. Then the loop is :

(1) Begin Loop: Let $Trial$ be the $Considered$ point with smallest value of u.
(2) Tag as $Considered$ all neighbors of $Trial$ that are not *Accepted*. If the neighbor is in Far, remove it from that set and add it to the set $Considered$.
(3) Recompute the values of u at all $Considered$ neighbors of $Trial$ by solving the piecewise quadratic equation according to Eqn. 4.3.
(4) Add point $Trial$ to *Accepted*; remove from $Considered$
(5) Return to top until the $Considered$ set is empty.

This is the Fast Marching Method given in [25]. Helmsen compares a similar algorithm with a volume-of-fluid approach for photolithography simulations in [15]; Malladi and Sethian apply the Fast Marching Method to image segmentation in [19].

The key to an efficient implementation of the above technique lies in a fast way of locating the grid point in the narrow band with the smallest value for u. An efficient scheme for doing so, discussed in detail in [28], can be devised using a min-heap structure, similar to what is done in Dijkstra's method. Given N elements in the heap, this allows us to change any element in the heap and re-order the heap in $O(\log N)$ steps. Thus, the computational efficiency of the total Fast Marching Method for the mesh with N points is $O(N \log N)$: N steps to touch each mesh point, where each step is $O(\log N)$ since the heap has to be re-ordered each time the values are changed.

5.3 Flow Chart of Methods

These techniques have been used to tackle a wide collection of front propagation problems, including problems in geometry and singularity formation [8; 9], robotic navigation and path planning, computing first arrivals in seismic travel times [30], computing geodesics [16], the construction of extension velocity fields [3], two-phase flow [33; 6], etching and deposition in semiconductor manufacturing [2], surface diffusion [10], variational level set methods for multiple interface in [35], crystal growth and dendritic solidification [31; 7], applications to photolithography development [27; 15], and application to medical imaging and image segmentation [5; 17].

In Figure 3, we give a perspective on how some of these topics are related. There are many other contributors to the evolution of these ideas; the chart is meant to give perspective on how the theory, algorithms, and applications have evolved. The text and bibliography of [28] gives a more complete sense of the literature and the range of work underway.

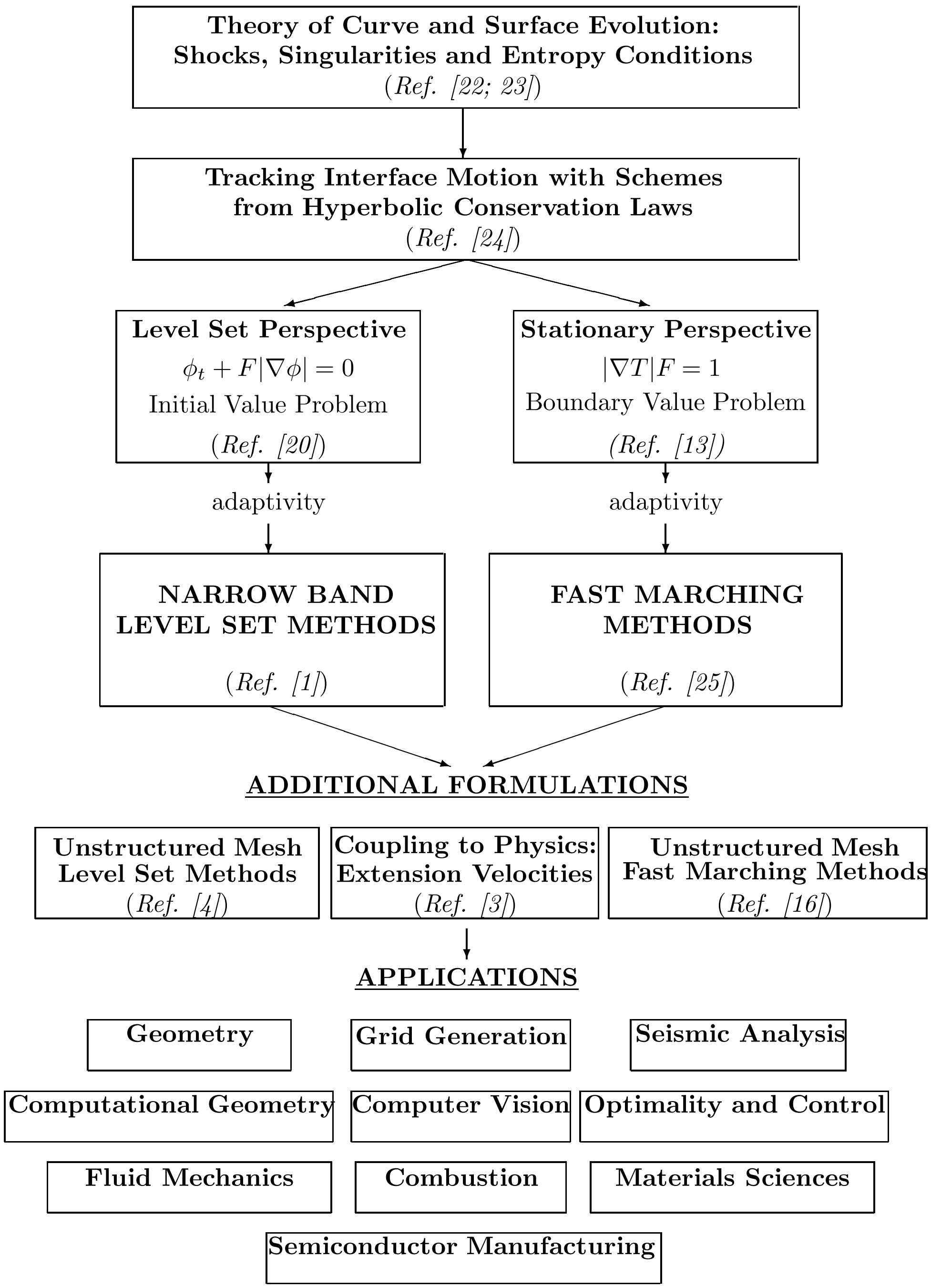

FIG. 3. Algorithms and Applications for Interface Propagation

6 Applications

Finally, we present a few results. To begin, these interface techniques can be used to segment images; the central idea is grow a seed inside a region with a propagation velocity which depends on the image gradient and hence stops when the boundary is reached. This strategy was proposed by Malladi, Sethian, and Vemuri in [17] and Caselles, Catte, Coll, and Dibos in (see [5]). Malladi makes his approach fast by coupling it to Narrow Band Methods in [18], extends it to three dimensions, couples his approach to Fast Marching Methods in [19]. Figure 4 shows a 3D reconstruction of the liver (the z axis from head to toe is not to scale). Figure 4a shows a 2D slice and the 2D contour; Figure 4b shows the full 3D reconstructed shape displayed embedded in the same 2D slice.

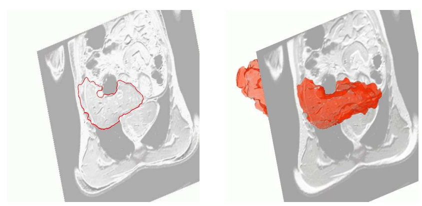

Fig. 4. Reconstruction of three-dimensional liver.

In [16], Kimmel and Sethian give a technique for computing geodesics on manifolds using the Fast Marching Method. A second order version of the Fast Marching Method on triangulated domains to compute shape offsets on machine parts is given by Sethian and Vladimirsky in [32]. Figure 5 shows offsets equidistant from the bounding box on a manifold which represents a complex machine part; the triangulation used is obtained by mapping a regular triangular mesh in the xy plane onto the surface; this creates a large number of obtuse and non-nice triangles, including some with angles bigger than 160^o.

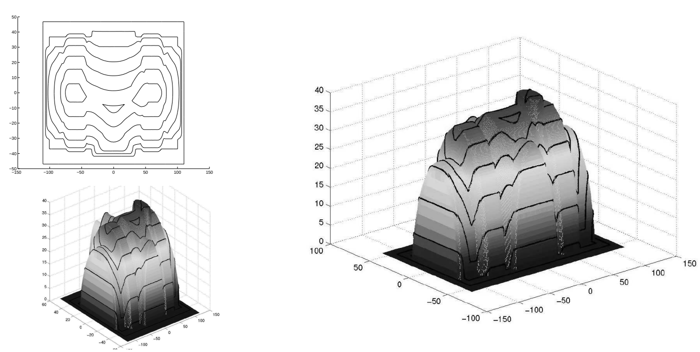

Fig. 5. Obtuse triangulated Fast Marching Method

Further information may be found at http::/math.berkeley.edu/∼ sethian/level_set.html.

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