SAMPLE MIDTERMS SOLUTIONS
MATH 55

Sample 1.

1.

$$\exists x, y \in \mathbb{R} \quad (x < y) \wedge (f(x) \geq f(y)).$$

2.

(a) $\mathbb{Z} \times \mathbb{Z}$ is countable. In class we constructed the bijections $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ and $g : \mathbb{Z} \to \mathbb{Z}^+$. Now we define a bijection $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}^+$ by the formula $h(m, n) = f(g(m), g(n))$ for all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$.

(b) $\mathbb{R} - \mathbb{Z}$ is not countable, it contains an open interval $(0, 1)$ as a subset. The cardinality of $(0, 1)$ is continuum and $|(0, 1)| \leq |\mathbb{R} - \mathbb{Z}|$.

(c) the set of all irrational real numbers is not countable. To prove it assume that this set is countable, denote it by I. Then $\mathbb{R} = \mathbb{Q} \cup I$. We have proved that the union of two countable sets is countable and that $\mathbb{Q}$ is countable, but $\mathbb{R}$ is not. Thus, we obtains contradiction with our assumption that I is countable.

3. Find the binary and hexadecimal expansions for 500. For binary we get

$$500 = 250 \times 2 + 0,\ 250 = 125 \times 2 + 0,\ 125 = 62 \times 2 + 1,\ 62 = 31 \times 2 + 0, 31 = 15 \times 2 + 1,$$

$$15 = 7 \times 2 + 1,\ 7 = 3 \times 2 + 1,\ 3 = 1 \times 2 + 1, \quad 1 = 2 \times 0 + 1.$$

So we obtain $500 = (111110100)_2$. For hexadecimal expansion take the binary expansion and translate 4-digit groups into hexadecimal digits $500 = (1F4)_{16}$.

4. Just follow the algorithm from the textbook. We have $m = 315$ $M_1 = 63, M_2 = 45, M_3 = 35$.

$$63y_1 \equiv 1(\mod 5), \quad y_1 \equiv 2(\mod 5),$$
$$45y_2 \equiv 1(\mod 7), \quad y_2 \equiv 5(\mod 7),$$
$$35y_3 \equiv 1(\mod 9), \quad y_1 \equiv -1(\mod 9).$$
$$x \equiv 3 \times 63 \times 2 + 4 \times 45 \times 5 + 2 \times 35 \times (-1) = 1208(\mod 315)$$

or $x \equiv 263(\mod 315)$.

5. Basic step:

$$1(1 + 1) = 2 = \frac{1(1 + 1)(1 + 2)}{3}.$$

Inductive step:

$$\sum_{i=1}^{k+1} i(i + 1) = \sum_{i=1}^{k} i(i + 1) + (k + 1)(k + 2) = \frac{k(k + 1)(k + 2)}{3} + (k + 1)(k + 2) =$$

$$\frac{k(k + 1)(k + 2) + 3(k + 1)(k + 2)}{3} = \frac{(k + 1)(k + 2)(k + 3)}{3}.$$

Sample 2.

1. Since $|A| = |B|$ and $|B| = |C|$, there exist bijections $f : A \to B$ and $g : B \to C$. The composition $g \circ f : A \to C$ is a bijection. Therefore $|A| = |C|$.

2.

(a) $1, 2, 6, 42, 1806$.

(b) We have $a_{n+1} - a_n = a_n^2 + a_n - a_n = a_n^2 > 0$. Hence $a_{n+1} > a_n$.

3.

$$292 = 2012 \mod 344, \quad 52 = 344 \mod 292, \quad 32 = 292 \mod 52,$$
$$20 = 52 \mod 32, \quad 12 = 32 \mod 20, \quad 8 = 20 \mod 12,$$
$$4 = 12 \mod 8, \quad 0 = 8 \mod 4.$$

Thus, $gcd(2012, 344) = 4$.

4. If $p = 3$ then $p^2 + 14 = 23$ is prime. If $p \neq 3$, then 3 does not divide p and hence p is congruent to 1 or 2 modulo 3. Then $p^2 \equiv 1(\mod 3)$ and $p^2 + 14 \equiv 0(\mod 3)$. Hence 3 divides $p^2 + 14$ and therefore $p^2 + 14$ is composite.

5. $x \in \bigcap_{n=1}^{\infty} A_n$ iff x is divisible by any positive integer. Therefore $\bigcap_{n=1}^{\infty} A_n = \{0\}$.

Sample 3

1. Let $f : A \to B$ be a surjective function. To prove that $|B| \leq |A|$ we have to show the existence of an injective function $g : B \to A$. For every $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$. Set $g(b) = a$. If $g(b_1) = g(b_2) = a$, then $f(a) = b_1 = b_2$. Therefore g is one-to-one. The statement is proven.

2. Since $10 \equiv 1(\mod 9)$ we have $10^k \equiv 1(\mod 9)$ for any $k \in \mathbb{N}$. Hence

$$a_n 10^n + \cdots + a_0 \equiv a_n + \cdots + a_0(\mod 9).$$

3. Basic step: $n = 1$ is the first Fibonacci number. Inductive step: assume that the statement is true for all positive $k < n$. We want to prove it for n. If n is a Fibonacci number $n = f_m$, then the statement is clear. Otherwise let f_m be the largest Fibonacci number less that n and let $k = n - f_m$. Then $k < n$. Note also $n < f_{m+1}$ implies $k < f_{m+1} - f_m = f_{m-1}$. Since k is a sum of distinct Fibonacci numbers $f_{i_1} + \cdots + f_{i_s}$, we obtain $n = f_{i_1} + \cdots + f_{i_s} + f_m$. Inequality $k < f_{m-1}$ ensures that f_m is greater than any of f_{i_j}, Hence n is a sum of distinct Fibonacci numbers.

4. Let $P(n)$ denote the predicate $n^2 + n + 41$ is prime.

(a) The truth values of $P(1)$, $P(2)$, $P(5)$ is T, because 43, 47 and 71 are prime numbers.

(b) For $n = 41$ $P(n)$ is false. Indeed $41^2 + 41 + 41$ is divisible by 41 and therefore not prime.

5. Procedure root ($p > 2$ prime integer)

$$\text{for } r = 2 \text{ to } p - 1$$
$$a := r,\ x := 1$$
$$\text{while } a \neq 1\ a := ar \mod p,\ x := x + 1$$
$$\text{if } x = p - 1 \text{ return } r$$