SAMPLE MIDTERMS SOLUTIONS
MATH 55

Sample 1.
1. \( \exists x, y \in \mathbb{R} \ (x < y) \land (f(x) \geq f(y)) \).

2. (a) \( \mathbb{Z} \times \mathbb{Z} \) is countable. In class we constructed the bijections \( f: \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) and \( g: \mathbb{Z} \rightarrow \mathbb{Z}^+ \). Now we define a bijection \( h: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}^+ \) by the formula \( h(m, n) = f(g(m), g(n)) \) for all \((m, n) \in \mathbb{Z} \times \mathbb{Z}\).

(b) \( \mathbb{R} - \mathbb{Z} \) is not countable, it contains an open interval \((0, 1)\) as a subset. The cardinality of \((0, 1)\) is continuum and \(|(0, 1)| \leq |\mathbb{R} - \mathbb{Z}|\).

(c) the set of all irrational real numbers is not countable. To prove it assume that this set is countable, denote it by \( I \). Then \( \mathbb{R} = \mathbb{Q} \cup I \). We have proved that the union of two countable sets is countable and that \( \mathbb{Q} \) is countable, but \( \mathbb{R} \) is not. Thus, we obtain contradiction with our assumption that \( I \) is countable.

3. Find the binary and hexadecimal expansions for 500. For binary we get
\[
500 = 250 \times 2 + 0, \quad 250 = 125 \times 2 + 0, \quad 125 = 62 \times 2 + 1, \quad 62 = 31 \times 2 + 0, \quad 31 = 15 \times 2 + 1, \\
15 = 7 \times 2 + 1, \quad 7 = 3 \times 2 + 1, \quad 3 = 1 \times 2 + 1, \quad 1 = 2 \times 0 + 1.
\]
So we obtain \( 500 = (111110100)_2 \). For hexadecimal expansion take the binary expansion and translate 4-digit groups into hexadecimal digits \( 500 = (1F4)_{16} \).

4. Just follow the algorithm from the textbook. We have \( m = 315 \), \( M_1 = 63 \), \( M_2 = 45 \), \( M_3 = 35 \).
\[
63y_1 \equiv 1 \pmod{5}, \quad y_1 \equiv 2 \pmod{5}, \quad 45y_2 \equiv 1 \pmod{7}, \quad y_2 \equiv 5 \pmod{7}, \quad 35y_3 \equiv 1 \pmod{9}, \quad y_1 \equiv -1 \pmod{9}.
\]
\[
x \equiv 3 \times 63 \times 2 + 4 \times 45 \times 5 + 2 \times 35 \times (-1) = 1208 \equiv 0 \pmod{315}
\]
or \( x \equiv 263 \pmod{315} \).

5. Basic step:
\[
1(1 + 1) = 2 = \frac{1(1 + 1)(1 + 2)}{3}.
\]
Inductive step:
\[
\sum_{i=1}^{k+1} i(i + 1) = \sum_{i=1}^{k} i(i + 1) + (k + 1)(k + 2) = \frac{k(k + 1)(k + 2)}{3} + (k + 1)(k + 2) = \\
\frac{k(k + 1)(k + 2) + 3(k + 1)(k + 2)}{3} = \frac{(k + 1)(k + 2)(k + 3)}{3}.
\]
Sample 2.
1. Since $|A| = |B|$ and $|B| = |C|$, there exist bijections $f : A \rightarrow B$ and $g : B \rightarrow C$. The composition $g \circ f : A \rightarrow C$ is a bijection. Therefore $|A| = |C|$.

2. (a) $1, 2, 6, 42, 1806$.
(b) We have $a_{n+1} - a_n = a_n^2 + a_n - a_n = a_n^2 > 0$. Hence $a_{n+1} > a_n$.

3. 

$$292 = 2012 \mod 344, \quad 52 = 344 \mod 292, \quad 32 = 292 \mod 52, \quad 20 = 52 \mod 32, \quad 12 = 32 \mod 20, \quad 8 = 20 \mod 12, \quad 4 = 12 \mod 8, \quad 0 = 8 \mod 4.$$ 

Thus, $\gcd(2012, 344) = 4$.

4. If $p = 3$ then $p^2 + 14 = 23$ is prime. If $p \neq 3$, then $3$ does not divide $p$ and hence $p$ is congruent to $1$ or $2$ modulo $3$. Then $p^2 \equiv 1 \pmod{3}$ and $p^2 + 14 \equiv 0 \pmod{3}$. Hence $3$ divides $p^2 + 14$ and therefore $p^2 + 14$ is composite.

5. $x \in \bigcap_{n=1}^{\infty} A_n$ iff $x$ is divisible by any positive integer. Therefore $\bigcap_{n=1}^{\infty} A_n = \{0\}$.

Sample 3

1. Let $f : A \rightarrow B$ be a surjective function. To prove that $|B| \leq |A|$ we have to show the existence of an injective function $g : B \rightarrow A$. For every $b \in B$ there exists at least one $a \in A$ such that $f(a) = b$. Set $g(b) = a$. If $g(b_1) = g(b_2) = a$, then $f(a) = b_1 = b_2$. Therefore $g$ is one-to-one. The statement is proven.

2. Since $10 \equiv 1 \pmod{9}$ we have $10^k \equiv 1 \pmod{9}$ for any $k \in \mathbb{N}$. Hence $a_n10^n + \cdots + a_0 \equiv a_n + \cdots + a_0 \pmod{9}$.

3. Basic step: $n = 1$ is the first Fibonacci number. Inductive step: assume that the statement is true for all positive $k < n$. We want to prove it for $n$. If $n$ is a Fibonacci number $n = f_m$, then the statement is clear. Otherwise let $f_m$ be the largest Fibonacci number less than $n$ and let $k = n - f_m$. Then $k < n$. Note also $n < f_{m+1}$ implies $k < f_{m+1} - f_m = f_{m-1}$. Since $k$ is a sum of distinct Fibonacci numbers $f_{i_1} + \cdots + f_{i_s}$, we obtain $n = f_{i_1} + \cdots + f_{i_s} + f_m$. Inequality $k < f_{m-1}$ ensures that $f_m$ is greater than any of $f_{i_j}$. Hence $n$ is a sum of distinct Fibonacci numbers.

4. Let $P(n)$ denote the predicate $n^2 + n + 41$ is prime.

(a) The truth values of $P(1), P(2), P(5)$ is $T$, because $43, 47$ and $71$ are prime numbers.

(b) For $n = 41$ $P(n)$ is false. Indeed $41^2 + 41 + 41$ is divisible by $41$ and therefore not prime.

5. Procedure root ($p > 2$ prime integer)

\[ \text{for } r = 2 \text{ to } p - 1 \]
\[ a := r, \ x := 1 \]
\[ \text{while } a \neq 1 \ a := ar \mod p, \ x := x + 1 \]
\[ \text{if } x = p - 1 \text{ return } r \]