SHAPOVALOV FORMS FOR POISSON LIE SUPERALGEBRAS

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1. Introduction

Poisson Lie superalgebras are the superalgebras of functions on a symplectic supermanifold. Subquotients of Poisson superalgebras, called superalgebras of Hamiltonian vector fields, appear in the list of simple finite-dimensional Lie superalgebras (see [K]). If the dimension of a supermanifold is even, then a Poisson superalgebra admits a non-degenerate invariant even symmetric form. In particular, there exists a Casimir operator. Poisson superalgebras also have root decomposition in the sense of [PS]. It was noticed in [GL] that in such situation it is possible to define a Shapovalov form using the approach suggested in [KK]. We give a precise formula for determinant of Shapovalov form for finite-dimensional Poisson superalgebra \( \mathfrak{po}(0|2n) \) with \( n \geq 2 \). The case \( n = 1 \) is well-known since \( \mathfrak{po}(0|2) \) is isomorphic to \( \mathfrak{gl}(1|1) \). We show that, contrary to the case of classical Lie superalgebras, the Jantzen filtration of a Verma module can be infinite.

One can use another approach to the problem of finding the Shapovalov form. It is well known that there is a deformation \( G_h \) of the Poisson superalgebra \( \mathfrak{po}(0|2n) \) such that the Lie superalgebra \( G_h \) is isomorphic to \( \mathfrak{gl}(2^{n-1}|2^{n-1}) \) for \( h \neq 0 \). The Shapovalov form for the latter superalgebra is known (see [KKK]). Since the deformation preserves a Cartan subalgebra and triangular decomposition, one can obtain the Shapovalov form for \( \mathfrak{po}(0|2n) \) isomorphic to \( G_0 \) by evaluating the Shapovalov form for \( G_h \) at \( h = 0 \). However this method seems more difficult. Indeed, several root subspaces are glued together when \( h = 0 \). The condition on weights of irreducible Verma modules also change dramatically. It seems that the direct approach using the Casimir operator works better. One can illustrate this on a simple example. Indeed, it is much easier to evaluate the Shapovalov form for the Heisenberg algebra than to consider its deformation to \( \mathfrak{sl}(2) \) and go back using the result for \( \mathfrak{sl}(2) \).

2. Preliminary

2.1. Poisson superalgebra \( \mathfrak{po}(0|n) \). Let \( \Lambda(n) \) be the Grassman superalgebra in \( \xi_1, \ldots, \xi_n \). The Poisson Lie superalgebra \( \mathfrak{po}(0|n) \) can be described as \( \Lambda(n) \) endowed with the bracket

\[
[f, g] = (-1)^{p(f)+1} \sum_i \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i}.
\]

It is easy to see that \( [g, g] = \sum_{i=0}^{n-1} \Lambda^i(n) \). Let \( \int : \mathfrak{g} \to \mathbb{C} \) be such a map that \( \ker \int = [g, g] \) and \( \int(\xi_1 \ldots \xi_n) = 1 \). For \( f, g \in \Lambda(n) \) define

\[
B(f, g) := \int fg.
\]

Clearly, \( B \) is a non-degenerate invariant bilinear form on \( \mathfrak{g} \). If \( n \) is even, \( B \) gives rise to the quadratic Casimir element.

In this text we consider the even case \( \mathfrak{g} := \mathfrak{po}(0|2n) \).
2.2. Triangular decompositions. A triangular decomposition of a Lie superalgebra $\mathfrak{g}$ can be constructed as follows (see [PS]). A Cartan subalgebra is a nilpotent subalgebra which coincides with its normalizer. It is proven in [PS] that any two Cartan subalgebras are conjugate by an inner automorphism. Fix a Cartan subalgebra $\mathfrak{h}$. Then $\mathfrak{g}$ has a generalized root decomposition

$$\mathfrak{g} := \mathfrak{h} + \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\Delta$ is a subset of $\mathfrak{h}^*$ and

$$\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} | (ad(h) - \alpha(h))^d \mathfrak{g}(x) = 0 \}.$$

In case considered in this paper a root space $\mathfrak{g}_\alpha$ is either odd or even. That allows one to define the parity on the set of roots $\Delta$. Denote by $\mathfrak{g}_\mathbb{P}$ (resp., $\mathfrak{g}_\mathbb{T}$) the even (resp., odd) component of $\mathfrak{g}$. Denote by $\Delta_0$ (resp. $\Delta_1$) the set of non-zero weights of $\mathfrak{g}_\mathbb{P}$ (resp., $\mathfrak{g}_\mathbb{T}$) with respect to $\mathfrak{h}$. Then $\Delta$ is a disjoint union of $\Delta_0$ and $\Delta_1$.

Now fix $h \in \mathfrak{h}_0^*$ satisfying $\alpha(h) \in \mathbb{R} \subset \{0\}$ for all $\alpha \in \Delta$. Set

$$\Delta^+ := \{ \alpha \in \Delta | \alpha(h) > 0 \},$$

$$n^+ := \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha$ is the weight space corresponding to $\alpha$.

Define $\Delta^-$ and $n^-$ similarly. Then $\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+$ is a triangular decomposition.

2.3. Notation. Denote by $\Delta^+_0$ (resp., $\Delta^+_1$) the set of even (resp., odd) positive roots. Let $Q \subset \mathfrak{h}^*$ be the root lattice that is the $\mathbb{Z}$-span of $\Delta^+$. Let $Q^+$ be the $\mathbb{Z}_{\geq 0}$-span of $\Delta^+$. Introduce the standard partial ordering on $\mathfrak{h}^*$ by setting $\mu \leq \nu$ if $\nu - \mu \in Q^+$.

Throughout the paper $\alpha$ and $\beta$ stand for positive roots.

For $\alpha \in \Delta^+$ denote by $D_\alpha$ the matrix of natural pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathfrak{h}$ given by $(x, y) \mapsto [x, y]$.

2.4. Verma modules. From now on suppose that $\mathfrak{h}$ is even and commutative. Set $\mathfrak{b} := \mathfrak{h} + n^+$. For each $\lambda \in \mathfrak{h}^*$ define $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda$ where $k_\lambda$ is a one-dimensional $\mathfrak{b}$-module which is trivial as $n^+$-module and corresponds to $\lambda$ as $\mathfrak{h}$-module. Each Verma module has a unique maximal submodule $\overline{M(\lambda)}$. The corresponding simple module $V(\lambda) := M(\lambda)/\overline{M(\lambda)}$ is called a highest weight simple module.

2.5. Shapovalov determinants. For finite dimensional semisimple Lie algebras N. Shapovalov ([Sh]) constructed a bilinear form $U(n^-) \otimes U(n^-) \to S(\mathfrak{h})$ whose kernel at a given point $\lambda \in \mathfrak{h}^*$ determines the maximal submodule $\overline{M(\lambda)}$ of a Verma module $M(\lambda)$. In particular, a Verma module $M(\lambda)$ is simple if and only if the kernel of Shapovalov form at $\lambda$ is equal to zero. The Shapovalov form can be realized as a direct sum of forms $S_\nu$; for each $S_\nu$ one can define its determinant (Shapovalov determinant). The zeroes of Shapovalov determinants determine when a Verma module is reducible.
2.5.1. A Shapovalov form for a Lie superalgebra \( \mathfrak{g} \) with an even commutative Cartan subalgebra \( \mathfrak{h} \) can be described as follows.

Identify \( U(\mathfrak{h}) \) with \( S(\mathfrak{h}) \). Let \( HC : U(\mathfrak{g}) \to S(\mathfrak{h}) \) be the Harish-Chandra projection i.e., a projection along the decomposition \( U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}^+) \). Define a form \( U^n(\mathfrak{n}^+) \otimes U(\mathfrak{n}^-) \to S(\mathfrak{h}) \) by setting \( S(x, y) := HC(xy) \). Using the natural identification of a Verma module \( M(\lambda) \) with \( U(\mathfrak{n}^-) \), one easily sees that \( \overline{M(\lambda)} \) coincides with the “right kernel” of the evaluated form \( S(\lambda) : U(n^+) \otimes U(n^-) \to k \) i.e., \( \overline{M(\lambda)} = \{ y \in U(n^-) \mid (x, y)(\lambda) = 0 \text{ for all } x \} \).

Notice that \( S(x, y) = 0 \) if \( x \in U(n^+), y \in U(n^-) - \mu \) and \( \nu \neq \mu \). Thus \( S = \sum_{\nu \in Q^+} S_\nu \) where \( S_\nu \) is the restriction of \( S \) to \( U(n^+)_\nu \otimes U(n^-)_{-\nu} \). By the above, \( \dim V(\lambda)_{\lambda - \nu} = \text{codim ker}_r S_\nu(\lambda) \) where \( \ker_r \) stands for the “right kernel”.

2.5.2. Assume that \( \dim U(n^+)_\nu = \dim U(n^-)_{-\nu} < \infty \) for all \( \nu \in Q^+ \). Then \( \det S_\nu \) is an element of \( S(\mathfrak{h}) \) defined up to an invertible scalar. One obtains the following criterion of simplicity of a Verma module: \( M(\lambda) \) is simple iff \( \det S_\nu(\lambda) \neq 0 \) for all \( \nu \).

2.6. Case \( \mathfrak{g} := \mathfrak{po}(0|2n) \). The algebra \( \mathfrak{po}(0|2n) \) admits a \( \mathbb{Z} \)-grading

\[
\mathfrak{g} = \oplus_{i=-2}^{2n-2} \mathfrak{g}_i
\]

which is obtained from the natural grading on \( \Lambda(2n) \) by the shift by \(-2\).

2.6.1. We choose generators \( \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \) of \( \Lambda(2n) \) in such a way that

\[
[f, g] = (-1)^{p(f)+1} \sum_i \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i} \frac{\partial g}{\partial \xi_i}.
\]

2.6.2. Set \( I := \{1, \ldots, n\} \) and for each \( J \subset I \) define

\[
h_J := \prod_{i \in J} \xi_i \eta_i.
\]

The reader can check that the span of \( h_J \) is a Cartan subalgebra of \( \mathfrak{g} \) which we denote by \( \mathfrak{h} \). If \( h_J \in \mathfrak{h}_i \) for \( i \neq 0 \), \( \text{ad} h \) is nilpotent. Therefore the set of roots \( \Delta \) can be realized as a subset of \( \mathfrak{h}_0^* \) where the embedding \( \mathfrak{h}_0^* \subset \mathfrak{h}_0^* \) comes from the decomposition \( \mathfrak{h}_0^* = \mathfrak{h}_{-2} \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_{2n-2} \).

Using the standard notation for \( \mathfrak{g}_0 = \mathfrak{so}(n) \), one obtains

\[
\Delta = \{ \pm \varepsilon_{i_1} \pm \cdots \pm \varepsilon_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n \}.
\]

Clearly, \( \mathfrak{h}_{-2} \) is spanned by \( h_0 \in \Lambda^0(2n) \) and coincides with the centre of \( \mathfrak{g} \). Define triangular decompositions as in 2.2.
2.6.3. Example. Take \( h := 2^n - 1 \varepsilon_1^* + 2^{n-2} \varepsilon_2^* + \ldots + \varepsilon_n^* \). Then
\[
\Delta^+ = \{ \varepsilon_{i_1} \pm \varepsilon_{i_2} \pm \ldots \pm \varepsilon_{i_k} : i_1 < i_2 < \ldots < i_k \}.
\]
Simple roots are
\[
\pi := \{ \varepsilon_1 - \varepsilon_2 - \ldots - \varepsilon_n, \varepsilon_2 - \ldots - \varepsilon_n, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n \}.
\]

2.6.4. Example. For \( n = 3 \) take \( h := 4 \varepsilon_1^* + 3 \varepsilon_2^* + 2 \varepsilon_3^* \). Then
\[
\pi := \{ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, -\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \}.
\]

2.6.5. The group of signed permutations of \( \{1, \ldots, n\} \) is a subgroup of Aut \( g \): a non-signed permutation acts as permutation of indexes and the permutation \( i \mapsto -i \) corresponds to the interchange \( \xi_i \leftrightarrow \eta_i \). As a consequence, any root is simple with respect to a suitable triangular decomposition (such a decomposition can be obtained from one in the first example by the action of a signed permutation).

2.7. Casimir element \( C \). Let \( C \) be the quadratic Casimir element corresponding to the non-degenerate bilinear form \( B \) defined in 2.1. Clearly, \( C \) has degree \( 2n - 4 \) with respect to the \( \mathbb{Z} \)-grading defined in 2.6. One easily sees that
\[
\text{HC}(C) = \sum_{J \subset I} h_J h_{I \setminus J} + h_C \quad \text{for some} \ h_C \in \mathfrak{h}_{2n-4}.
\]

3. Shapovalov determinants for \( \mathfrak{po}(0|2n) \), \( n > 2 \).

Recall that any root is of the form \( \alpha = \sum s_i \varepsilon_i \) where \( s_i \in \{-1, 0, 1\} \); for \( \alpha \in \Delta^+ \) set
\[
h_\alpha := \sum_{j \in I} h_j(\alpha) h_{I \setminus \{j\}} = \sum_{j \in I} s_j h_{I \setminus \{j\}}.
\]
Notice that \( h_\alpha \in \mathfrak{h}_{2n-4}^* \).

In this section we will prove the following formula.

3.1. Theorem.
\[
\det S_\nu = \prod_{\alpha \in \Delta_0^+} h_\alpha^{\dim \mathfrak{g}_\alpha} \sum_{m=1}^{\infty} \tau(\nu - ma) \prod_{\alpha \in \Delta_1^+} h_\alpha^{\dim \mathfrak{g}_\alpha} \sum_{m=1}^{\infty} (-1)^{m+1} \tau(\nu - ma).
\]
Proof. Fix \( \nu \in \mathbb{Q}^+ \). Recall that \( \det S_\nu(\lambda) = 0 \) iff \( M(\lambda) \) has a primitive vector of weight \( \lambda - \mu \) for some \( 0 < \mu \leq \nu \). Therefore
\[
\det S_\nu(\lambda) = 0 \implies \text{HC}(C)(\lambda) = \text{HC}(C)(\lambda - \mu) \quad \text{for some } 0 < \mu \leq \nu.
\]
Combining the formula (1) and the fact that \( h(\mu) = 0 \) if \( h \in \mathfrak{h} \) has a non-zero degree and \( \mu \in \mathbb{Q}^+ \), we obtain
\[
\text{HC}(C)(\lambda) - \text{HC}(C)(\lambda - \mu) = 2 \sum_{J \subset I} h_J(\mu) h_{I \setminus J}(\lambda) - \sum_{J \subset I} h_J(\mu) h_{I \setminus J}(\mu) = 2 \sum_{j \in I} h_j(\mu) h_{I \setminus \{j\}}(\lambda).
\]
Therefore
\[
\det S_\nu(\lambda) = 0 \implies h_\mu(\lambda) = 0 \quad \text{for some } 0 < \mu \leq \nu
\]
where \( h_\mu := \sum_{j \in I} h_j(\mu) h_{I \setminus \{j\}} \). In other words, all zeros of the polynomial \( \det S_\nu \) lie in the union of hyperplanes \( h_\mu = 0 \). Hence \( \det S_\nu = \prod_{0 < \mu \leq \nu} h_\mu^{d_\mu(\nu)} \) for some \( d_\mu(\nu) \geq 0 \). In particular, \( \det S_\nu \) is homogeneous and thus coincides with its leading term. Now Theorem 4.2 reduces the assertion to the formula \( \det D_\alpha = h_\alpha^{\dim g_\alpha} \).

To prove the last formula, recall that \( D_\alpha \) is a matrix of the natural pairing \( g_\alpha \times g_{-\alpha} \to \mathfrak{h} \). This matrix does not depend on a triangular decomposition. Since any root is simple with respect to a certain triangular decomposition, we can assume that \( \alpha \) is simple that is \( D_\alpha = S_\alpha \). Then the above reasoning gives \( \det D_\alpha = h_\alpha^{d_\alpha(\alpha)} \). Observe that the entries of \( D_\alpha \) lie in \( \mathfrak{h} \) and so the degree of \( \det D_\alpha \) is \( \dim g_\alpha \); hence \( d_\alpha(\alpha) = \dim g_\alpha \) as required. \( \square \)

3.2. Corollary.

(i) A Verma module \( M(\lambda) \) is simple if and only if \( h_\alpha(\lambda) \neq 0 \) for all \( \alpha \in \Delta^+ \).
(ii) A Verma module \( M(\lambda) \) contains a primitive vector of weight \( \lambda - \alpha \) if \( h_\alpha(\lambda) = 0 \).

4. The leading term of a Shapovalov determinant.

Let \( \mathfrak{g} \) be a Lie superalgebra with a fixed triangular decomposition such that

(i) the Cartan subalgebra is even and commutative;
(ii) \( \dim \mathcal{U}(\mathfrak{n}^+)_\nu = \dim \mathcal{U}(\mathfrak{n}^-)_{-\nu} < \infty \) for all \( \nu \in \mathbb{Q}^+ \).

Define Shapovalov determinants as in 2.5.2. In this section we compute the leading term of Shapovalov determinants for such algebras.

4.1. Retain notation of 2.2,2.3. The Kostant partition function \( \tau : Q \to \mathbb{Z}_{\geq 0} \) is defined by the formula
\[
\text{ch} \mathcal{U}(\mathfrak{n}^-) = \prod_{\alpha \in \Delta^+_1} (1 + e^{-\alpha}) \prod_{\alpha \in \Delta^+_0} (1 - e^{-\alpha})^{-1} =: \sum_{\eta \in Q} \tau(\eta) e^{-\eta}.
\]
Note that \( \tau(Q \setminus \mathbb{Q}^+) = 0 \).
4.2. Theorem. The leading term of $\det S_\nu$ is equal to
\[ \prod_{\alpha \in \Delta_0^+} (\det D_\alpha)^{\sum_{m=1}^\infty \tau(\nu - m\alpha)} \prod_{\alpha \in \Delta_1^+} (\det D_\alpha)^{\sum_{m=1}^\infty (-1)^{m+1}\tau(\nu - m\alpha)} \]
up to a non-zero scalar.

4.3. Proof. Denote by $\tilde{\Delta}_0^+, \tilde{\Delta}_1^+$ the corresponding multisets of roots (the multiplicity of $\alpha$ is equal to $\dim g_\alpha$). Set $\Delta^+ := \tilde{\Delta}_0^+ \cup \tilde{\Delta}_1^+$.

**Definition.** A vector $k = \{k_\alpha\}_{\alpha \in \tilde{\Delta}^+}$ is called a partition of $\nu \in Q^+$ if
\[ \nu = \sum_{\alpha \in \tilde{\Delta}^+} k_\alpha \alpha; \quad k_\alpha \in \mathbb{Z}_{\geq 0} \text{ for } \alpha \in \tilde{\Delta}_0^+ \text{ and } k_\alpha \in \{0, 1\} \text{ for } \alpha \in \tilde{\Delta}_1^+. \]
Denote by $P(\nu)$ the set of all partitions of $\nu$. Clearly, $|P(\nu)| = \tau(\nu)$.

4.3.1. Set $|k| := \sum_{\alpha \in \tilde{\Delta}^+} k_\alpha$. Take $\alpha \in \Delta^+$ and let $\alpha^{(i)} : i = 1, \ldots, \dim g_\alpha$ be the corresponding elements of the multiset $\tilde{\Delta}^+$. Denote by $k_\alpha$ the subpartition $k_\alpha := (k^{(i)}_\alpha : i = 1, \ldots, \dim g_\alpha)$.

Define an equivalence relation on $P(\nu)$ by setting $k \approx m$ if $|k_\alpha| = |m_\alpha|$ for all $\alpha$. Thus the equivalence classes are indexed by vectors $\kappa = (\kappa_\alpha : \alpha \in \Delta^+)$ where $k \in \kappa$ iff $|k_\alpha| = \kappa_\alpha$ for all $\alpha$. Set
\[ \text{supp } \kappa = \{ \alpha \in \Delta^+ : \kappa_\alpha \neq 0 \} \]
and define $\text{supp } k$ similarly. Set
\[ P(r, \alpha) := \{ k | \text{ supp } k = \{ \alpha \}, |k| = r \} \]
and denote by $p(r, \alpha)$ the cardinality of $P(r, \alpha)$.

4.3.2. Fix a total ordering on $\tilde{\Delta}^+$ compatible with the standard partial ordering on $h^*$. Fix bases $\{e_\alpha : \alpha \in \tilde{\Delta}^+\}$ of $n^+$ and $\{f_\alpha : \alpha \in \tilde{\Delta}^+\}$ of $n^-$ where $e_\alpha$ (resp., $f_\alpha$) has weight $\alpha$ (resp., $-\alpha$). For every $k \in P(\nu)$ set
\[ f_k := \prod_{\alpha \in \tilde{\Delta}^+} f_\alpha^{k_\alpha} \]
where the factors are arranged with respect to the total ordering: the first factor corresponds to the minimal root. Define $e_k$ by the similar formula but with factors arranging in the reverse order. The sets $\{f_k : k \in P(\nu)\}$ and $\{e_k : k \in P(\nu)\}$ form PBW bases of $U(n^-)_\nu$ and $U(n^+)_\nu$ respectively. Let $S_\nu$ be the matrix of Shapovalov form written in these bases: its columns and rows are indexed by the partitions $k \in P(\nu)$ and the $(k, m)$th entry is $\text{HC}(e_k f_m)$.
4.4. Let $A, B$ be two square matrices. One can naturally define $A \otimes B$ as the matrix of the corresponding linear operator.

On the other hand, view $B$ as a matrix of bilinear form on $V$ and define

$$\tilde{S}^k(B)(v_1 \otimes \ldots \otimes v_k; v'_1 \otimes \ldots \otimes v'_1) := \sum_{\sigma \in S_k} \prod_{i=1}^k B(v_i, v'_{\sigma(i)}),$$

$$\tilde{\Lambda}^k(B)(v_1 \otimes \ldots \otimes v_k; v'_1 \otimes \ldots \otimes v'_1) := \sum_{\sigma \in S_k} (-1)^{\text{sgn}\sigma} \prod_{i=1}^k B(v_i, v'_{\sigma(i)}).$$

Now define $S^k(B)$ and $\Lambda^k(B)$ as the restrictions of $\tilde{S}^k(B)$ and $\tilde{\Lambda}^k(B)$ to $S^k(V)$ and $\Lambda^k(V)$ respectively.

4.4.1. Let $C$ be an $m \times m$ matrix with entries in $S(\mathfrak{h})$. For each $\sigma \in S_m$ let $\text{deg}(C, \sigma)$ be the degree of $\prod_{i=1}^m c_{i\sigma(i)}$; put $\text{deg}(C) := \max_{\sigma} \text{deg}(C, \sigma)$ and denote by $\det' C$ the term of degree $\text{deg}(C)$ in the polynomial $\det C$. Thus $\det' C$ is either zero or equal to the leading term of $\det C$.

4.5. Fix $\alpha \in \Delta^+$. Let $D_{ma}$ be the submatrix of the Shapovalov matrix $S_{ma}$ formed by the entries whose both coordinates lie in $P(m, \alpha)$. For $m = 1$ this definition gives the same matrix as was defined in 2.3. Observe that $D_{ma} = S_{ma}$ if $\alpha$ is simple. Recall that all entries of $D_{\alpha}$ has degree one and so $\det' D_{\alpha} = \det D_{\alpha}$.

By Lemma 4.6.1 the leading terms of the entries of $D_{ma}$ form the matrix $S^m(D_{\alpha})$ if $\alpha$ is even and the matrix $\Lambda^m(D_{\alpha})$ if $\alpha$ is odd. Consequently,

$$\det' D_{ma} = \begin{cases} \det S^m(D_{\alpha}) & \text{if } \alpha \text{ is even}, \\ \det \Lambda^m(D_{\alpha}) & \text{if } \alpha \text{ is odd}. \end{cases}$$

Notice that for any square matrix $A$ one has $\det S^m(A) = c(\det A) \frac{m! (s^m(A))}{s(A)}$ where $c \in \mathbb{Z}_{>0}$ and $s(B)$ stands for the size of a matrix $B$; $\det \Lambda^m(A)$ has the similar formula. Hence, up to a non-zero constant, one has

$$(2) \quad \det' D_{ma} = (\det D_{\alpha})^{\frac{\dim P(\alpha)}{\dim \mathfrak{g}_{\alpha}}}. \quad \text{(2)}$$

4.6. By 4.6.2 the degrees of the entries of $k$th row (resp., column) of a Shapovalov matrix $S_{\nu}$ is not greater than $|k|$. Moreover, if $|k| = |m|$ the degree of $(k, m)$th entry is less than $|k|$ if $k \not\approx m$. Finally, if $k \approx m$ then the leading term of $(k, m)$th entry coincides with the leading term of $c_{k,m} := \prod_{\alpha \in \Delta^+} \text{HC}(e^{k,m}\mathfrak{g}_{\alpha})$; note that $\text{HC}(e^{k,m}\mathfrak{g}_{\alpha})$ is an entry of the matrix $D_{|k,m|\alpha}$.

As a consequence, $\deg(S_{\nu}) = \sum_{k \in P(\nu)} |k|$ and $\det' S_{\nu} = \det' C_{\nu}$ where $C_{\nu} = (c_{k,m})_{k,m \in P(\nu)}$ and $c_{k,m}$ is given by the above formula for $k \approx m$, $c_{k,m} = 0$ for $k \not\approx m$. Thus $C_{\nu}$ is a block matrix with the blocks indexed by the equivalence classes of partitions; the block indexed by $\kappa = (\kappa_{\alpha})$ is the tensor product of the matrices $D_{\kappa_{\alpha},\alpha}$ for all $\alpha \in \Delta^+$. 
Observe that \( \det(A \otimes B) = (\det A)^{s(B)}(\det B)^{s(A)} \). Using the formula (2) we get
\[
\det' S_\nu = \prod_\kappa \prod_{\alpha \in \Delta^+} (\det' D_{\kappa \alpha})^{p(\kappa \beta, \beta)} = \prod_{\alpha \in \Delta^+} (\det D_{\alpha})^{d(\alpha)}
\]
where
\[
d(\alpha) = \sum_\kappa \frac{\kappa \prod_{\beta} p(\kappa \beta, \beta)}{\dim g_\alpha} = \frac{1}{\dim g_\alpha} \sum_{k \in \mathcal{P}(\nu)} |k_\alpha|
\]
since \( \prod_{\beta} p(\kappa \beta, \beta) \) is equal to the cardinality of \( \kappa \). Now Lemma 4.6.3 completes the proof of Theorem 4.2. \( \square \)

4.6.1. Lemma. The leading terms of the entries of \( D_{ma} \) form the matrix \( S_m(D_\alpha) \) if \( \alpha \) is even and the matrix \( \Lambda_m(D_\alpha) \) if \( \alpha \) is odd.

4.6.2. Lemma. Take \( \nu \in Q^+ \) and \( k, m \in \mathcal{P}(\nu) \). Then
\[(i) \text{ deg } HC(e^k f^m) \leq \min(|k|, |m|). \]
\[(ii) \text{ Assume that } \text{deg } HC(e^k f^m) = |k| = |m|. Then } \]
\[k \approx m \]
and the leading term of \( HC(e^k f^m) \) is equal to the leading term of
\[
\prod_{\alpha \in \Delta^+} HC(e^{k_\alpha} f^{m_\alpha}).
\]
Proof is by induction on \( \nu \in Q^+ \).

4.6.3. Lemma.
\[(i) \text{ For any } \alpha \in \Delta_0^+ \text{ one has } \]
\[\sum_{k \in \mathcal{P}(\nu)} |k_\alpha| = \dim g_\alpha \sum_{m=1}^{\infty} \tau(\nu - m\alpha). \]
\[(ii) \text{ For any } \alpha \in \Delta_1^+ \text{ one has } \]
\[\sum_{k \in \mathcal{P}(\nu)} |k_\alpha| = \dim g_\alpha \sum_{m=1}^{\infty} (-1)^{m+1} \tau(\nu - m\alpha). \]

Proof. Recall that \( |k_\alpha| = \sum_{i=1}^{\dim g_\alpha} k^{(i)}_{\alpha} \). For each \( i \) the formula
\[\sum_{k \in \mathcal{P}(\nu)} k^{(i)}_{\alpha} = \sum_{m=1}^{\infty} \tau(\nu - m\alpha)\]
for \( \alpha \in \Delta_0^+ \) and a similar formula for \( \alpha \in \Delta_1^+ \) can be obtained by a standard reasonings (see, for instance [G2], 3.3.1). \( \square \)
5. The case \( \mathfrak{po}(0|4) \).

5.1. For the Lie superalgebra \( \mathfrak{g} := \mathfrak{po}(0|4) \) all triangular decompositions are conjugated. We fix a triangular decomposition with the following positive roots: \( \varepsilon_1 \pm \varepsilon_2, \varepsilon_1, \varepsilon_2 \). One easily sees that \( \mathrm{HC}(C) = 2(h_\emptyset h_{1,2} + h_1 h_2 - h_1) \) where \( h_1 \) stands for \( h_{\{1\}} \) and other notations are similar.

5.2. The even roots \( \varepsilon_1 \pm \varepsilon_2 \) have multiplicity one and \( D_{\varepsilon_1 \pm \varepsilon_2} = \pm h_1 + h_2 \). The odd roots \( \varepsilon_1, \varepsilon_2 \) have multiplicity two. To compute \( D_{\varepsilon_2} \) notice that the weight space \( \mathfrak{g}_{\varepsilon_2} \) (resp., \( \mathfrak{g}_{-\varepsilon_2} \)) has a basis \( \{ \xi_2, \xi_1 \eta_1 \xi_2 \} \) (resp., \( \{ \eta_2, \xi_1 \eta_1 \eta_2 \} \)). The matrix \( D_{\varepsilon_2} \) written in these bases takes form

\[
D_{\varepsilon_2} = \begin{pmatrix}
    h_\emptyset & h_1 \\
    -c_2(\nu) & 0
\end{pmatrix}
\]

and so \( \det D_{\varepsilon_2} = -h_1^2 \); similarly \( \det D_{\varepsilon_1} = -h_2^2 \). By Theorem 4.2, the leading term of \( \det S_\nu \) is, up to a non-zero scalar, equal to

\[
(h_1 - h_2)^{d(\nu)} h_1 c_2(\nu) h_2^{c_1(\nu)}
\]

where \( d(\nu) := \sum_{m=1}^\infty \tau(\nu - m(\varepsilon_1 - \varepsilon_2)) \), \( d'(\nu) := \sum_{m=1}^\infty \tau(\nu - m(\varepsilon_1 + \varepsilon_2)) \) and \( c_i := 2 \sum_{m=1}^\infty (-1)^{m+1} \tau(\nu - m \varepsilon_i) \).

5.3. Arguing as in 3.1, we conclude that all Shapovalov determinants admit linear factorizations and factors of \( \det S_\nu \) are of the form \( h_2(\mu)h_1 + h_1(\mu)h_2 - h_1(\mu)h_2(\mu) - h_1(\mu) \) where \( 0 < \mu \leq \nu \). Comparing with the above expression of the leading term we conclude that

\[
\det S_\nu = \prod_{k=1}^\infty (h_2 - h_1 + k - 1)^{d_k}(h_2 + h_1 - k - 1)^{d'_k} h_1^{c_2(\nu)} h_2^{c_1(\nu)}
\]

where the multiplicities \( d_k, d'_k \) are non-negative integers which satisfy the conditions

\[
\sum_k d_k = d(\nu) = \sum_{m=1}^\infty \tau(\nu - m(\varepsilon_1 - \varepsilon_2)), \quad \sum_k d'_k = d'(\nu) = \sum_{m=1}^\infty \tau(\nu - m(\varepsilon_1 + \varepsilon_2))
\]

(in particular, only finitely many multiplicities are non-zero and thus the above product is finite). Now the standard reasoning based on a use of Jantzen filtration gives \( d_k = \tau(\nu - k(\varepsilon_1 - \varepsilon_2)) \) and \( d'_k = \tau(\nu - k(\varepsilon_1 + \varepsilon_2)) \). Finally, up to a non-zero scalar, one has

\[
\det S_\nu = \prod_{k=1}^\infty (h_2 - h_1 + k - 1)^{\tau(\nu - k(\varepsilon_1 - \varepsilon_2))}(h_2 + h_1 - k - 1)^{\tau(\nu - k(\varepsilon_1 + \varepsilon_2))}
\]

\[
\times h_1^2 \sum_{m=1}^\infty (-1)^{m+1} \tau(\nu - m \varepsilon_2)(h_2 - 1)^2 \sum_{m=1}^\infty (-1)^{m+1} \tau(\nu - m \varepsilon_1).
\]

The notion of Jantzen filtration on a Verma module was introduced in [Ja] for semisimple Lie algebras. It can be easily extended to superalgebra case. One has to take into account however that the vector $\rho$ is no longer “regular” in a sense that hypersurfaces $\det S_\nu = 0$ contain straight lines parallel to $\rho$ so that in the construction of the Jantzen filtration one should use a/any regular vector $\rho' \in \mathfrak{h}^*$ instead of $\rho$— see [G3], 7.1 for details.

6.1. Retain notation of 2.4. The Jantzen filtration on $M(\lambda)$ is a decreasing filtration with the following properties:

$$\mathcal{F}^0(M(\lambda)) = M(\lambda), \quad \mathcal{F}^1(M(\lambda)) = M(\lambda), \quad \bigcap_{r=0}^{\infty} \mathcal{F}^r(M(\lambda)) = 0,$$

(3) $$d_\nu(\lambda) = \sum_{r \geq 1} \dim \mathcal{F}^r(M(\lambda))_{\lambda - \nu},$$

where $d_\nu(\lambda)$ is the order of zero of the polynomial $\det S_\nu$ at the point $\lambda$ (if $\det S_\nu = \prod p_i^{r_i}$ where $p_i$ are irreducible then $d_\nu(\lambda) = \sum_{i : p_i(\lambda) = 0} r_i$). The formula (3) is proven in [Ja] and is called “sum formula”.

For $M(\lambda)$ being simple one has $\mathcal{F}^1(M(\lambda)) = 0$. For basic classical (except $\mathfrak{psl}(2|2)$) or $Q$-type Lie superalgebras the Jantzen filtration has length two, i.e. $\mathcal{F}^2(M(\lambda)) = 0$, if $M(\lambda)$ is a “generic” reducible Verma module. More precisely, $\mathcal{F}^2(M(\lambda)) = 0$ if $\lambda$ lies on exactly one of irreducible components of a hypersurface $\det S_\nu = 0$. Remarkably, this is far from being true in our case. We demonstrate this phenomenon on some examples below.

Set $\mathfrak{g} := \mathfrak{po}(0|2n)$. In the examples below we assume that

$$\lambda \text{ is a generic point of the hyperplane } h_\alpha = k,$$

(4) where $h_\alpha = k$ is an irreducible component of a hypersurface $\det S_\nu = 0$. For $n > 2$ one has $k = 0$ and genericity means that $h_\beta(\lambda) \neq 0$ for $\beta \in \Delta^+, \beta \neq \alpha$.

Denote by $v_\lambda$ the highest weight vector of $M(\lambda)$.

6.2. The algebra $\mathfrak{g} = \mathfrak{po}(0|4)$. If $\dim \mathfrak{g}_\alpha = 1$ (i.e., $\alpha = \varepsilon_1 \pm \varepsilon_2$) and $\lambda$ satisfies (4) one can easily deduce from the sum formula (3) that $\mathcal{F}^2(M(\lambda)) = 0$.

6.2.1. The case $\alpha := \varepsilon_2$. Since $\varepsilon_2$ is simple, one has $S_{\varepsilon_2} = D_{\varepsilon_2}$ (see 5.2 for the explicit formula). One has $\dim \mathcal{U}(\mathfrak{n}^-)_{-2\varepsilon_2} = 1$ and the Shapovalov matrix $S_{2\varepsilon_2}$ is equal to $h_1^2$.

Let $\lambda$ satisfy (4); since $h_\alpha = h_1$ one has $h_1(\lambda) = 0$. Set $f_{\varepsilon_2} := \xi_1 \eta_1 \eta_2$. If $h_0(\lambda) \neq 0$ the vector $f_{\varepsilon_2} v_\lambda$ lies in $\mathcal{F}^2(M(\lambda))$. Now using the “genericity” of $\lambda$ one can deduce from the
sum formula (3) that \( f_{\varepsilon_2}v_\lambda \) generates \( \mathcal{F}^1(M(\lambda)) = \mathcal{F}^2(M(\lambda)) \) and that \( \mathcal{F}^3(M(\lambda)) = 0 \). Note that a Jordan-Hölder series of \( M(\lambda) \) has length two.

If \( h_0(\lambda) = 0 \) the term \( \mathcal{F}^1(M(\lambda)) \) is generated by \( M(\lambda)_{\lambda - \alpha} \) (\( \mathcal{F}^1(M(\lambda)) \) is isomorphic to the sum of two quotients of \( M(\lambda - \alpha) \) and \( \mathcal{F}^2(M(\lambda)) \cong V(\lambda - 2\alpha) \); one has \( \mathcal{F}^3(M(\lambda)) = 0 \) as before.

Hence in a generic point of the hyperplane \( h_1 = 0 \) the Jantzen filtration has length three and \( \mathcal{F}^1(M(\lambda)) = \mathcal{F}^2(M(\lambda)) \) iff \( h_0(\lambda) = 0 \).

6.3. The algebra \( \mathfrak{g} = \mathfrak{p}(0|2n), n > 2 \).

6.3.1. Claim. Let \( \alpha \) be a simple even root and \( \lambda \) be such that \( h_\alpha(\lambda) = 0 \). Then the Jantzen filtration of \( M(\lambda) \) is infinite.

Proof. Fix any homogeneous (with respect to the \( \mathbb{Z} \)-grading) bases in \( \mathfrak{g}_\alpha \) and \( \mathfrak{g}_{-\alpha} \). The matrix \( S_\alpha = D_\alpha \) written in these bases has a column with only non-zero entry: this column corresponds to \( f_\alpha \in \mathfrak{g}_{-\alpha} \) having the maximal degree and the non-zero entry corresponds to \( e_\alpha \in \mathfrak{g}_\alpha \) having the minimal degree; the non-zero entry is equal to \( h_\alpha \). As a consequence, the matrix \( S_k\alpha \) also has a column with only non-zero entry: this column corresponds to \( f_\alpha^k \) and the entry is \( h_\alpha^k \). This gives \( f_\alpha^kv_\lambda \in \mathcal{F}^k(M(\lambda)) \). Hence the Jantzen filtration is infinite. \( \square \)

6.3.2. Notice that a submodule generated by \( f_\alpha^k v_\lambda \) is isomorphic to \( M(\lambda - k\alpha) \); denote this submodule by \( M_k \). Clearly, \( M_k \subset \mathcal{F}^k(M(\lambda)) \).

If \( \dim \mathfrak{g}_\alpha = 1 \), the sum formula (3) implies that \( \mathcal{F}^k(M(\lambda)) = M_k \) for \( \lambda \) satisfying (4).

If \( \dim \mathfrak{g}_\alpha > 1 \) one has \( \mathcal{F}^k(M(\lambda)) \neq M_k \) for \( k = 1 \) or for \( k = 2 \), since the sum formula gives \( \sum_{r \geq 1} \dim \mathcal{F}^r(M(\lambda))_{\lambda - \alpha} = \dim \mathfrak{g}_\alpha \).

For example, let \( \alpha \) be a simple even root and \( \dim \mathfrak{g}_\alpha = 2 \). Then

\[
S_\alpha = \begin{pmatrix}
h' & h_\alpha \\
- & - \\
h_\alpha & 0
\end{pmatrix}
\]

for some \( h' \in \mathfrak{b}_{2n-6}^* \). If \( h_\alpha(\lambda) = 0 \) and \( h' \neq 0 \) one has \( f_\alpha v_\lambda \in \mathcal{F}^2(M(\lambda)) \) that is \( M_1 \subset \mathcal{F}^2(M(\lambda)) \). However, a natural guess that \( M_1 \subset \mathcal{F}^{\dim \mathfrak{g}_\alpha}(M(\lambda)) \) is wrong. The example \( \dim \mathfrak{g}_\alpha = 4 \) shows that in this case \( \mathcal{F}^1(M(\lambda))_{\lambda - \alpha} = \mathcal{F}^2(M(\lambda))_{\lambda - \alpha} \) is a two dimensional subspace and so \( \mathcal{F}^3(M(\lambda))_{\lambda - \alpha} = 0 \); in particular, \( M_1 \) lies in \( \mathcal{F}^2(M(\lambda)) \) and does not lie in \( \mathcal{F}^3(M(\lambda)) \).

6.4. Element \( T \). The enveloping algebra of \( \mathfrak{g} := \mathfrak{p}(0|2n) \) contains a special even element \( T \) which commutes with the even elements of \( \mathfrak{g} \) and anticommutes with the odd one, see [G1]. Recall that \( U(\mathfrak{g}) \) admits the canonical filtration and that the associated graded algebra is \( S(\mathfrak{g}) \). The algebra \( S(\mathfrak{g}) \) contains \( \Lambda \mathfrak{g}_T \). It turns out that the image of \( T \) in
The element $T$ acts on a Verma module in the following way: it acts by $HC(T)(\lambda)$ id on the $\mathbb{Z}_2$-homogeneous component containing a highest weight vector and by $-HC(T)(\lambda)$ id on another $\mathbb{Z}_2$-homogeneous component.

6.4.1. Take $n > 2$. By Corollary 3.2 (ii), $M(\lambda)$ contains a primitive vector of weight $\lambda - \alpha$ if $h_\alpha(\lambda) = 0$. One can deduce from this statement that the polynomial $HC(T)$ is divisible by $h_\alpha$ for $\alpha \in \Delta^+_1$.

Conjecture: $HC(T) = \prod_{\alpha \in \Delta^+_1} h_\alpha^{\dim g_\alpha}$ up to a non-zero scalar for $n > 2$.

6.4.2. Claim. For $g := \mathfrak{po}(0|4)$ one has $HC(T) = h_1^2(h_2 - 1)^2$ up to a non-zero scalar.

Proof. First, let us show that $t := HC(T)$ is divisible by $h_1^2$. Set $\alpha := \varepsilon_2$ and let $f_1, f_2$ (resp., $e_1, e_2$) be a basis of $g_{-\alpha}$ (resp., $g_\alpha$). Write $T = t + \sum_{i,j=1,2} f_i \phi_{ij} e_j + \sum y_r x_r$, where $y_r \in \mathcal{U}(g)$, $x_r \in \mathfrak{n}_{\mu(r)}^+$ for some $\mu(r) \neq -\alpha_2$. Let $v$ be a primitive vector. Then $Tv = tv$ and $Tfv = -fTv$ and

$$Tfv = tfv + \sum_{i,j=1,2} f_i \phi_{ij} e_j f_r v = tfv + \sum_{i=1,2} f_i (\Phi S)_{ir} v$$

where $\Phi = (\phi_{ij})$ and $S := S_\alpha$ is the Shapovalov matrix written with respect to the above base. Putting $f_1 := \eta_2, f_2 := \xi_1 \eta_1 \eta_2$ we get

$$tf_1 = f_1(t - \frac{\partial t}{\partial h_2}) - f_2 \frac{\partial t}{\partial h_1 \partial h_2}, \quad tf_2 = f_2(t - \frac{\partial t}{\partial h_2}).$$

Hence

$$\Phi S = \begin{pmatrix} -2t + \frac{\partial t}{\partial h_2} & 0 \\ \frac{\partial t}{\partial h_1} & -2t + \frac{\partial t}{\partial h_2} \end{pmatrix}$$

Now substituting $S = S_\alpha$ (see 5.2) we conclude that $t$ is divisible by $h_1^2$ (this reflects the fact that for $\lambda$ being a generic point of the hyperplane $h_1 = 0$ one has $\mathcal{F}^2(M(\lambda))_{\lambda - \alpha} \neq 0$).

It remains to show that $t$ is divisible by $(h_2 - 1)^2$. Take $\lambda$ such that $\lambda(h_1 - h_2) = k \in \mathbb{Z}_{\geq 0}$. Then $M(\lambda)$ has a primitive vector of the weight $\lambda - (k + 1)(\varepsilon_1 - \varepsilon_2)$ and so $t(\lambda) = t(\lambda - (k + 1)(\varepsilon_1 - \varepsilon_2))$. As a consequence, $t$ is stable under the involution of the algebra $S(\mathfrak{h})$ which acts by id on $\mathfrak{h}_{-2} + \mathfrak{h}_2$ and acts on $\mathfrak{h}_0$ by mapping $h_1$ to $h_2 - 1$. Since $t$ is divisible by $h_1^2$, $t$ is divisible by $(h_2 - 1)^2$ as well. The claim follows.

References


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