CATEGORY OF $\mathfrak{sp}(2n)$-MODULES WITH BOUNDED WEIGHT MULTIPLICITIES

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Abstract. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra. Denote by $\mathcal{B}$ the category of all bounded weight $\mathfrak{g}$-modules, i.e. those which are direct sum of their weight spaces and have uniformly bounded weight multiplicities. A result of Fernando shows that infinite-dimensional bounded weight modules exist only for $\mathfrak{g} = \mathfrak{sl}(n)$ and $\mathfrak{g} = \mathfrak{sp}(2n)$. If $\mathfrak{g} = \mathfrak{sp}(2n)$ we show that $\mathcal{B}$ has enough projectives if and only if $n > 1$. In addition, the indecomposable projective modules can be parameterized and described explicitly. All indecomposable objects are described in terms of indecomposable representations of a certain quiver with relations. This quiver is wild for $n > 2$. For $n = 2$ we describe all indecomposables by relating the blocks of $\mathcal{B}$ to the representations of the affine quiver $A_3^{(1)}$.

1. Introduction

To classify all indecomposable objects in a category of representations is usually a challenging and difficult problem. It is often the case that there are not enough projectives or the category itself is wild. A classical example of a wild category with enough projectives is the category $\mathcal{O}$ introduced by Bernstein-Gelfand-Gelfand in 1967. The simple objects in this category are highest weight modules, and the indecomposable projectives are described by the celebrated BGG reciprocity law.

A natural generalization of the category $\mathcal{O}$ is the category of all weight (not necessarily highest weight) modules. Weight modules have attracted considerable mathematical attention in the last 20 years and appeared in works of G. Benkart, D. Britten, S. Fernando, V. Futorny, and F. Lemire, [2], [3], [4], [6], [7]. A major breakthrough was the recent classification of O. Mathieu, [9], of all simple weight modules with finite weight multiplicities over finite dimensional reductive Lie algebras. A crucial role in this classification is played by the category $\mathcal{B}$ of bounded weight modules, i.e. those for which the set of weight multiplicities is uniformly bounded. This is due to the fact that, as Fernando showed in [6], every simple weight module M with finite weight multiplicities is obtained by a parabolic induction from a simple module S in $\mathcal{B}$ (in fact S has equal weight multiplicities). An important observation of Mathieu is that the direct sum of all simple objects in a single block of $\mathcal{B}$ form a so-called coherent family which is parameterized by a highest weight module, i.e. an object in $\mathcal{O}$.

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In the present paper we initiate a study of the category of bounded modules. A result of Fernando shows that infinite dimensional bounded weight modules exist only for Lie algebras of type A and C ([6], [9]). As a first step in our project we consider the Lie algebra $\mathfrak{g} = \mathfrak{sp}(2n)$. This case is simpler in terms of the classification of Mathieu as a semisimple irreducible coherent family over $\mathfrak{sp}(2n)$ is determined uniquely by its central character. The case of $\mathfrak{sl}(n+1)$ is more delicate and one has to consider three separate cases for the central character: regular integral, singular, and nonintegral.

One of the main results in the paper is providing a complete classification of all indecomposable projective objects in $\mathcal{B}$. An interesting observation is that if $n = 1$, i.e. $\mathfrak{g} = \mathfrak{sl}(2)$, the category $\mathcal{B}$ does not contain any projective objects. The picture is totally different for the higher dimensional algebras as for $n > 1$ each simple object has a projective cover.

In order to describe the indecomposable objects of $\mathcal{B}$ we first show that this category is equivalent to the category of weight modules over the Weyl algebra $\mathcal{A}_n$ (see Lemma 3.1 and Corollary 5.3). We then conclude that each block $\mathcal{B}^\chi$ of $\mathcal{B}$ is equivalent to the category of a certain quiver with relations. This quiver is wild if and only if $n > 2$. In the case $n = 2$ indecomposable representations of the quiver can be expressed in terms of the affine quiver $A_3^{(1)}$, the theory of which is well established. In addition, in section 6 we provide an explicit description of all indecomposable bounded modules over $\mathfrak{sp}(4)$ in terms of the twisted localization correspondence.

We show also that there are not enough projectives in the category of all weight $\mathfrak{g}$-modules with finite weight multiplicities (see Example 4.10) which provides an additional motivation to focus our attention on the bounded modules only.

2. Weight modules over the Weyl algebra

The ground field is $\mathbb{C}$. By $\mathcal{A}_n$ we denote the Weyl algebra, i.e. the algebra of polynomial differential operators on $\mathbb{A}^n$. Let $t_1, \dots, t_n, \partial_1, \dots, \partial_n$ be the standard generators of $\mathcal{A}_n$. Recall that the following relations hold

$$[t_i, \partial_j] = \delta_{ij},\ [t_i, t_j] = [\partial_i, \partial_j] = 0.$$

In what follows we will consider $\mathcal{A}_n$ as a Lie algebra over $\mathbb{C}$. Let M be an $\mathcal{A}_n$-module. We say that M is a *weight module* if

$$M = \bigoplus_{\mu \in \mathbb{C}^n} M^\mu,$$

where $M^\mu := \{m \in M \mid t_i\partial_i\,(m) = \mu_i m \text{ for all } i\}$ and $\mu = (\mu_1, ..., \mu_n) \in \mathbb{C}^n$. The space M^μ is the *weight space of weight* μ and $\dim M^\mu$ is the *weight multiplicity* of M^μ. We say that M is *multiplicity free* if $\dim M^\mu \leq 1$. The *support* of M is the set $\operatorname{supp} M := \{\mu \in \mathbb{C}^n \mid M^\mu \neq 0\}$.

In this section we study weight modules over $\mathcal{A}_n$. These modules are studied by Bekkert-Benkart-Futorny in [1] in more general setting, i.e. over a generalized Weyl algebra and an arbitrary field K. Some of the results in this section are particular cases of the general statements in [1], but for the sake of simplicity we provide independent proofs.

The Lie algebra $\mathcal{A}_n$ acts on itself via the adjoint map $\mathrm{ad} : \mathcal{A}_n \to \mathrm{End}(\mathcal{A}_n), \mathrm{ad}(x)(y) := [x, y]$. The elements $t_1\partial_1, \dots, t_n\partial_n$ act diagonally on $\mathcal{A}_n$. The adjoint action induces a $\mathbb{Z}^n$-grading of $\mathcal{A}_n$ via the root decomposition:

$$\mathcal{A}_n = \bigoplus_{\alpha \in P} \mathcal{A}_n^{\alpha},$$

where $P = \mathbb{Z}^n$ is considered as a sublattice of $\mathbb{C}^n$ with the standard generators $\varepsilon_1, \dots, \varepsilon_n$. The following lemma follows by a direct verification.

Lemma 2.1. *$\mathcal{A}_n^0$ is a free commutative algebra with generators $t_1\partial_1, \dots, t_n\partial_n$ and each $\mathcal{A}_n^{\alpha}$ is a free left $\mathcal{A}_n^0$-module of rank 1.*

Example 2.2. Let $\mu \in \mathbb{C}^n$, and let t^{μ} stand for $t_1^{\mu_1} \dots t_n^{\mu_n}$. The vector space $F(\mu) = t^{\mu}\mathbb{C}\left[t_1^{\pm 1}, \dots, t_n^{\pm 1}\right]$ has a natural structure of an $\mathcal{A}_n$-module. It is an easy exercise to check that $F(\mu)$ is a multiplicity free $\mathcal{A}_n$-module with $\mathrm{supp}\, F(\mu) = \mu + P$.

Lemma 2.3. *The $\mathcal{A}_n$-module $F(\mu)$ is indecomposable. It is irreducible if and only if $\mu_i \notin \mathbb{Z}$ for all $i = 1, \dots, n$.*

Proof. Suppose that $F(\mu) = M_1 \oplus M_2$. Since $F(\mu)$ is multiplicity free, $\mathrm{supp}\, F(\mu)$ is a disjoint union of $\mathrm{supp}\, M_1$ and $\mathrm{supp}\, M_2$. Therefore one can find $p \in \{1, 2\}$, $\nu \in \mathrm{supp}\, M_p$, and $i \in \{1, 2, ..., n\}$ such that $\nu + \varepsilon_i \notin \mathrm{supp}\, M_p$. Then $t_i v = 0$ whenever $v \in F(\mu)^{\nu}$. This is impossible because $F(\mu)$ is free over $\mathbb{C}[t_1, \dots, t_n]$.

To prove the second statement we first assume that $\mu_i \in \mathbb{Z}$ for some i. Since $F(\mu)$ is isomorphic to $F(\mu + \gamma)$ for any $\gamma \in P$ we may assume that $\mu_i = 0$. Then one easily checks that $t^{\mu}\mathbb{C}\left[t_1^{\pm 1}, \dots, t_{i-1}^{\pm 1}, t_i, t_{i+1}^{\pm 1}, \dots, t_n^{\pm 1}\right]$ is a submodule of $F(\mu)$. Finally, if $\mu_i \notin \mathbb{Z}$ for all i, any element of $F(\mu)^{\nu}$ generates $F(\mu)$. Hence $F(\mu)$ is irreducible. $\square$

Denote by $\mathcal{F}_n$ the category of weight $\mathcal{A}_n$-modules with finite weight multiplicities.

Lemma 2.4. *The category $\mathcal{F}_n$ splits into a direct sum of blocks*

$$\bigoplus_{\bar{\nu} \in \mathbb{C}^n / P} \mathcal{F}_n^{\bar{\nu}},$$

where the sum runs over all distinct classes $\bar{\nu} := \nu + P$ in $\mathbb{C}^n/P$ and $\mathcal{F}_n^{\bar{\nu}}$ is the subcategory of all modules M such that $\mathrm{supp}\, M \subset \bar{\nu}$.

Proof. Let $M \in \mathcal{F}_n$. For any $\bar{\nu} \in \mathbb{C}^n/P$ let

$$M(\bar{\nu}) := \bigoplus_{\mu \in \bar{\nu}} M^{\mu}.$$

Obviously, $M(\bar{\nu})$ is a submodule of M and

$$M = \bigoplus_{\bar{\nu} \in \mathbb{C}^n / P} M(\bar{\nu}).$$

This proves the lemma. □

For each $\mu \in P$ put

$$P(\mu) := \mathcal{A}_n \otimes_{\mathcal{A}_n^0} C_\mu,$$

where C_μ denotes the unique 1-dimensional $\mathcal{A}_n^0$-module of weight μ.

Theorem 2.5. (1) *$P(\mu)$ is a multiplicity free module with* $\operatorname{supp} P(\mu) = \mu + P$*;*
(2) *$P(\mu)$ has a unique irreducible quotient which we denote by $L(\mu)$;*
(3) *If M is an irreducible module in $\mathcal{F}_n$ such that $\mu \in \operatorname{supp} M$, then M is isomorphic to $L(\mu)$;*
(4) *$P(\mu)$ is indecomposable;*
(5) *$P(\mu)$ is a projective module in the category $\mathcal{F}_n$;*
(6) *Every indecomposable projective module in the category $\mathcal{F}_n$ is isomorphic to $P(\mu)$ for some μ.*

Proof. The first statement follows from Lemma 2.1. To show (2) it suffices to prove that $P(\mu)$ has a unique maximal proper submodule. Indeed, N is a proper submodule of $P(\mu)$ iff $\mu \notin \operatorname{supp} N$. Since

$$\operatorname{supp}(N_1 \oplus N_2) = \operatorname{supp} N_1 \cup \operatorname{supp} N_2,$$

the sum of all proper submodules of $P(\mu)$ is proper. (3) follows from the Frobenius reciprocity theorem, and (4) follows from (2). To prove (5) consider an exact sequence

$$0 \to N \to S \xrightarrow{p} P(\mu) \to 0.$$

Since the sequence

$$0 \to N^\mu \to S^\mu \to P(\mu)^\mu \to 0$$

of $\mathcal{A}_n^0$-modules splits, there is a map $i \colon P(\mu)^\mu \cong C_\mu \to S^\mu$ such that $i \circ p = id$. By the Frobenius reciprocity theorem, the map i induces a map $j : P(\mu) \to S$ for which $j \circ p = \mathrm{Id}$. Hence $P(\mu)$ is projective. To prove (6) let S be an indecomposable projective module. Then we have a surjective map $a : S \to L(\mu)$ for some irreducible module $L(\mu)$. Let $r : P(\mu) \to L(\mu)$ be the canonical map. Then there exist $b : S \to P(\mu)$ and $c : P(\mu) \to S$ such that $a \circ c = r$ and $r \circ b = a$. Then $r \circ b \circ c = r$, and therefore $b \circ c \neq 0$. On the other hand, one can easily see that $\operatorname{End}_{\mathfrak{g}} P(\mu) = \mathbb{C}$. Hence $b \circ c$ is an automorphism. In particular, b is surjective. Then $P(\mu)$ is isomorphic to a direct summand of S. But S is indecomposable, so S is isomorphic to $P(\mu)$. □

Corollary 2.6. *Let M and N be simple modules in $\mathcal{F}_n$. Then M and N are non-isomorphic if and only if* $\operatorname{supp} M$ *and* $\operatorname{supp} N$ *are disjoint.*

Proof. If $\mu \in \operatorname{supp} M \cap \operatorname{supp} N$, then both modules are quotients of $P(\mu)$. By Theorem 2.5, (2), $P(\mu)$ has a unique simple quotient, and thus M and N are isomorphic. □

Let $M \in \mathcal{F}_n$ and $M = \oplus_{\mu \in \operatorname{supp} M} M^\mu$. Set $M^* := \oplus_{\mu \in \operatorname{supp} M} (M^\mu)^*$. Define the action of $\mathcal{A}_n$ on M^* by

$$\partial_i \cdot \tau(v) = \tau(t_i \cdot v), t_i \cdot \tau(v) = \tau(\partial_i \cdot v)$$

for any $v \in M, \tau \in M^*$. It is easy to check that $M^* \in \mathcal{F}_n$ and $\operatorname{supp} M = \operatorname{supp} M^*$. Moreover, $*$ is an exact contravariant functor on $\mathcal{F}_n$ which maps projective objects to injective ones and preserves the simple objects.

To obtain a complete description of all irreducible and indecomposable projectives in each block $\mathcal{F}_n^{\bar{\nu}}$ we observe that

$$\mathcal{A}_n \cong \mathcal{A}_1 \otimes \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_1.$$

Therefore every irreducible object in $\mathcal{F}_n$ is a tensor product of irreducibles in $\mathcal{F}_1$, and by Theorem 2.5, the same holds for the indecomposable projectives. Hence, it is enough to describe the blocks of $\mathcal{F}_1$. This description is obtained in the following lemma.

Lemma 2.7. *For any $\bar{\nu} \neq \bar{0}$, the block $\mathcal{F}_1^{\bar{\nu}}$ is semi-simple and has exactly one up to isomorphism irreducible object $F(\mu)$, $\mu \in \bar{\nu}$. The block $\mathcal{F}_1^{\bar{0}}$ has two isomorphism classes of simple objects: $L(0)$ and $L(-1)$. The structure of the indecomposable projective modules is described by the following exact sequences*

$$0 \to L(-1) \to P(0) \to L(0) \to 0,\ 0 \to L(0) \to P(-1) \to L(-1) \to 0.$$

Proof. By Lemma 2.3 $F(\mu)$ is irreducible iff $\mu = \mu_1 \notin \mathbb{Z}$. Clearly, in this case $F(\mu)$ is isomorphic to $P(\mu)$, therefore $\mathcal{F}_1^{\bar{\mu}}$ contains one up to an isomorphism indecomposable object $F(\mu)$ which is both projective and simple.

If $\bar{\nu} = \bar{0}$, then $F(0) \cong F(n)$ for any $n \in \mathbb{Z}$ and a simple calculation leads to the exact sequence

$$0 \to L(0) \to F(0) \to L(-1) \to 0.$$

The Frobenius reciprocity implies that there is a surjective homomorphism $P(-1) \to F(0)$, which is an isomorphism because both modules are multiplicity free and have the same support. By Corollary 2.6 every simple object in $\mathcal{F}_1^{\bar{0}}$ is a subquotient of $F(0)$. Finally, by similar arguments $P(0) \cong F(0)^*$, which leads to the exact sequence for $P(0)$. □

Remark 2.8. One can use also the following geometric description. $L(0)$ is isomorphic to $\mathbb{C}[t]$, $P(-1)$ is isomorphic to $\mathbb{C}[t, t^{-1}]$, and $L(-1)$ is a module generated by the δ-function concentrated at zero on $\mathbb{C}^1$.

Corollary 2.9. *Let $\nu \in \mathbb{C}^n$ and $I(\bar{\nu}) := \{i \leq n \mid \nu_i \in \mathbb{Z}\}$. Then all indecomposable projective modules and all irreducible modules of $\mathcal{F}_n^{\bar{\nu}}$ are parameterized by the set $\mathcal{S}$ of all maps $s : I(\bar{\nu}) \to \{0, -1\}$. More precisely, $P(s)$ is the tensor product of $P(\nu_j)$ for $j \notin I(\bar{\nu})$ and $P(s(i))$ for $i \in I(\bar{\nu})$. The same description works for the irreducibles.*

Since $\mathcal{F}_n^{\bar{\nu}}$ has finitely many irreducible modules and each irreducible has a unique indecomposable projective cover, the category $\mathcal{F}_n^{\bar{\nu}}$ is equivalent to the category of finite-dimensional E^{ν}-modules, where

$$E^{\nu} := \operatorname{End}_{\mathcal{A}_n}\left(\oplus_{s\in\mathcal{S}} P\left(s\right)\right).$$

Furthermore,

$$E^{\nu} \cong E^{\nu_1} \otimes \cdots \otimes E^{\nu_n}. \tag{2.1}$$

Observe that $E^{\nu_i} \cong \mathbb{C}$ whenever $\nu_i \notin \mathbb{Z}$. Let V_1 be the quiver

$$\bullet \underset{\varphi^-}{\overset{\varphi^+}{\rightleftarrows}} \bullet$$

with relations $\varphi^+\varphi^- = \varphi^-\varphi^+ = 0$. Then one can see easily that $E^{\nu_i} \cong \mathbb{C}\left(V_1\right)$ in the case $\nu_i \in \mathbb{Z}$.

Define the quiver V_k in the following way. The vertices of V_k are the vertices of the cube in $\mathbb{R}^k$ with coordinates 1 or -1. The edges are the edges of the cube with two possible orientations. We call a path on the cube *admissible* if each coordinate function is weakly monotonic along the path. Finally, we impose the following relations: each non-admissible path is zero, every two admissible paths with the same start and the same end points are equal.

Theorem 2.10. *Let $\bar{\nu} \in \mathbb{C}^n/P$ and k be the number of all i for which $\bar{\nu}_i = \bar{0}$. Then $\mathcal{F}_n^{\bar{\nu}}$ is equivalent to the category of representations of the quiver V_k.*

Lemma 2.11. *For $k \geq 3$ the quiver V_k is wild.*

Proof. Choose a subquiver $V_3 \subset V_k$ in an arbitrary way. Then choose $W_3 \subset V_3$ to be a maximal subquiver without cycles. Every representation of W_3 can be extended to a representation of V_k trivially: every arrow of V_k which is not in W_3 is represented by the zero map. Since W_3 is wild, V_k is wild as well. □

The indecomposable representations of V_1 are easy to describe.

Lemma 2.12. *The quiver V_1 has four isomorphism classes of indecomposable representations with dimension functions* $(1,0)$, $(0,1)$, $(1,1)$, *and* $(1,1)$, *respectively.*

Proof. Consider an indecomposable representation of V_1. Let A_1 and A_2 be the spaces attached to the vertices of V_1, and let

$$\varphi^+ : A_1 \to A_2,\ \varphi^- : A_2 \to A_1$$

be the corresponding maps. We have that $\varphi^+\varphi^- = \varphi^-\varphi^+ = 0$. Choose $B_1 \subset A_1$ and $B_2 \subset A_2$, so that $A_1 = B_1 \oplus \operatorname{Ker}\varphi^+$ and $A_2 = B_2 \oplus \operatorname{Ker}\varphi^-$. Then the representation splits into the direct sum

$$(\varphi^+ : B_1 \to \operatorname{Ker}\varphi^-) \oplus (\varphi^- : B_2 \to \operatorname{Ker}\varphi^+).$$

Thus either $B_1 = 0$ or $B_2 = 0$, and the problem is reduced to the quiver

$$\bullet \longrightarrow \bullet$$

which is well-understood. $\square$

To describe the indecomposable representations of V_2 we first introduce some notation. By ρ_1 we denote the following indecomposable representation of V_2

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{C} \end{array}$$

where all the arrows are represented by the identity maps and all inverse arrows are represented by the zero maps. One obtains ρ_2, ρ_3, ρ_4 from ρ_1 by rotating the picture by 90° one, two, or three times, respectively.

We next introduce the quivers A and B:

$$\begin{array}{ccc} A_{11} & \longrightarrow & A_{12} \\ \downarrow & & \uparrow \\ A_{21} & \longleftarrow & A_{22} \end{array}$$

$$\begin{array}{ccc} B_{11} & \longleftarrow & B_{12} \\ \uparrow & & \downarrow \\ B_{21} & \longrightarrow & B_{22} \end{array}$$

Any indecomposable representation of A or B induces an indecomposable representation of V_2 if we represent all reverse arrows in V_2 by the zero maps.

Lemma 2.13. *Any indecomposable representation of V_2 is either isomorphic to one of the representations $\rho_1, \rho_2, \rho_3, \rho_4$, or induced by an indecomposable representation of A or B.*

Proof. Consider some indecomposable indecomposable representation ρ of V_2:

$$\begin{array}{ccc} C_{11} & \overset{\varphi^+}{\underset{\varphi-}{\rightleftarrows}} & C_{12} \\ \xi^+ \downarrow \uparrow \xi^- & & \eta^- \uparrow \downarrow \eta^+ \\ C_{21} & \overset{\psi^-}{\underset{\psi+}{\leftrightarrows}} & C_{22} \end{array}$$

Assume that there is $v \in C_{11}$ such that $\eta^+\varphi^+(v) \neq 0$. The relations of V_2 imply that

$$\psi^+\xi^+(v) = \eta^+\varphi^+(v).$$

One can see easily that v generates a subrepresentation ρ' of ρ isomorphic to ρ_1. Moreover, ρ' is a direct summand of ρ, since each of the vectors v, $\varphi^+(v)$, $\eta^+\varphi^+(v)$ and $\xi^+(v)$ does not belong to the sum of images of all reverse maps, i.e.

$$v \notin \operatorname{im}(\xi^-) + \operatorname{im}(\varphi^-),\ \varphi^+(v) \notin \operatorname{im}(\eta^-),\ \xi^+(v) \notin \operatorname{im}(\varphi^-).$$

Indeed, say $v = \xi^-(u) + \varphi^-(w)$. Then

$$\eta^+\varphi^+(v) = \eta^+\varphi^+\xi^-(u) \neq 0,$$

which contradicts the relations. Thus in this case $\rho \cong \rho_1$. In the same way, if we start with $v \in C_{12}$ we will conclude that $\rho \cong \rho_2$, etc.

Let us assume now that ρ is not isomorphic to ρ_1, ρ_2, ρ_3 or ρ_4. Then the above argument shows that a composition of any two arrows is the zero map. Let U_{ij} be the intersection of the kernels of the two maps starting at C_{ij}. Write $C_{ij} = U_{ij} \oplus D_{ij}$ choosing D_{ij} in an arbitrary way. Then $\rho = \pi_1 \oplus \pi_2$, where π_1 is the following representation of A

$$\begin{array}{ccc} D_{11} & \longrightarrow & U_{12} \\ \downarrow & & \uparrow \\ U_{21} & \longleftarrow & D_{22} \end{array}$$

and π_2 is the following representation of B

$$\begin{array}{ccc} U_{11} & \longleftarrow & D_{12} \\ \uparrow & & \downarrow \\ D_{21} & \longrightarrow & U_{22} \end{array}$$

Hence the lemma is proved. □

Since the quivers A and B are isomorphic to affine Dynkin graph $A_3^{(1)}$, we can use the general theory of representation of tame quivers.

3. The algebra $\mathcal{A}_n^{ev}$

Let Q be a sublattice of index 2 in $P = \mathbb{Z}^n$ consisting of all $(\mu_1, \dots, \mu_n)$ such that $\mu_1 + \cdots + \mu_n \in 2\mathbb{Z}$. Define

$$\mathcal{A}_n^{ev} = \bigoplus_{\alpha \in Q} \mathcal{A}_n^{\alpha}.$$

Clearly, $\mathcal{A}_n^{ev}$ is a Lie subalgebra of $\mathcal{A}_n$.

Denote by $\mathcal{F}_n^{ev}$ the category of weight $\mathcal{A}_n^{ev}$-modules with finite weight multiplicities. As in Lemma 2.4 one has a block decomposition

$$\mathcal{F}_n^{ev} = \bigoplus_{\bar{\nu} \in \mathbb{C}^n/Q} \left(\mathcal{F}_n^{ev}\right)^{\bar{\nu}}.$$

Let $\bar{\nu} \in \mathbb{C}^n/Q$, and let $\bar{\mu} \in \mathbb{C}^n/P$ be the image of $\bar{\nu}$ under the the natural projection $\mathbb{C}^n/Q \to \mathbb{C}^n/P$. Define two functors

$$\mathrm{Ind} : (\mathcal{F}_n^{ev})^{\bar{\nu}} \to \mathcal{F}_n^{\bar{\mu}},\ \mathrm{Res}\colon \mathcal{F}_n^{\bar{\mu}} \to (\mathcal{F}_n^{ev})^{\bar{\nu}}$$

by putting

$$\mathrm{Ind}\,(M) = \mathcal{A}_n \otimes_{\mathcal{A}_n^{ev}} M,\ \mathrm{Res}\,(N) = \oplus_{\gamma\in\bar{\nu}} N^{\gamma}.$$

The following lemma is straightforward.

Lemma 3.1. *The functors* Ind *and* Res *establish an equivalence of the categories* $(\mathcal{F}_n^{ev})^{\bar{\nu}}$ *and* $\mathcal{F}_n^{\bar{\mu}}$.

4. Twisted localization of bounded modules

Let $\mathfrak{g} = \mathfrak{sp}(2n)$ or $\mathfrak{g} = \mathfrak{sl}(n+1)$, and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the root system, and Q be the root lattice of $\mathfrak{g}$. For every $\alpha \in \Delta$ fix a standard triple $\{e_\alpha, f_\alpha, h_\alpha\}$ such that $e_\alpha \in \mathfrak{g}^\alpha$, $f_\alpha \in \mathfrak{g}^{-\alpha}$ and $[e_\alpha, f_\alpha] = h_\alpha$. Let $U := U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, $Z := Z(\mathfrak{g})$ be its center, and $Z' := \mathrm{Hom}(Z, \mathbb{C})$.

By $\mathcal{B}^\chi$ we denote the category of weight $\mathfrak{g}$-modules with bounded weight multiplicities admitting generalized central character $\chi \in Z'$. In other words, $M \in \mathcal{B}^\chi$ if

$$M = \bigoplus_{\mu\in\mathfrak{h}^*} M^\mu,$$

there exists C_M such that $\dim M^\mu < C_M$ for all $\mu \in \mathfrak{h}^*$, and for each $m \in M$ and $z \in Z$ there exists N such that

$$(z - \chi(z))^N m = 0.$$

Put $\mathcal{B} := \cup_{\chi\in Z'} \mathcal{B}^\chi$. In what follows we assume that all $\mathfrak{g}$-modules are *bounded*[1], i.e. in $\mathcal{B}$. Following the approach in [9], we recall some facts about the localization of (bounded) weight modules with respect to a set of commuting roots. Let $\Gamma = \{\gamma_1, ..., \gamma_l\} \subset \Delta$ be a linearly independent subset of Q for which $\gamma_i + \gamma_j \notin \Delta$. The set $\{f_{\gamma_1}, ..., f_{\gamma_l}\}$ generates a multiplicative subset F_Γ of U which satisfies Ore's localizability conditions. Let U_{F_Γ} be the localization of U relative to F_Γ.

A $\mathfrak{g}$-module M is called Γ-*injective* (Γ-*bijective*) if f_γ acts injectively (bijectively) on M for every γ in Γ. For any $\mathfrak{g}$-module M we define the Γ-*localization* $\mathcal{D}_\Gamma M$ of M by $\mathcal{D}_\Gamma M := U_{F_\Gamma} \otimes_U M$. If M is Γ-injective, then $M \subset \mathcal{D}_\Gamma M$. Note that if $\Gamma = \Gamma_1 \cup \Gamma_2$ we have $\mathcal{D}_{\Gamma_1}\mathcal{D}_{\Gamma_2} = \mathcal{D}_{\Gamma_2}\mathcal{D}_{\Gamma_1} = \mathcal{D}_\Gamma$ over the set of all Γ-injective modules.

Example 4.1. Let $\mathfrak{g} = \mathfrak{sp}(2n)$, $\mathfrak{b}$ be the standard Borel subalgebra with basis $\{\varepsilon_1 - \varepsilon_2, ..., \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$, and $\Gamma := \{2\varepsilon_1, ..., 2\varepsilon_n\}$. Then every simple $\mathfrak{b}$-highest weight module $M = L_B(\lambda)$ is Γ-injective. Furthermore, if M is bounded, then $\mathcal{D}_\Gamma M$ has 2^n simple subquotients all of which are highest weight modules (with respect to

[1]In [9] Mathieu uses the term "admissible" weight module, but to avoid confusion with Harish-Chandra modules of finite type we prefer to use the term "bounded" weight module

different Borel subalgebras). This is proved in [5] and a detailed description of $\mathcal{D}_\Gamma M$ for $\mathfrak{g} = \mathfrak{sp}(4)$ will be provided in section 6.

The preceding example is a part of more general picture which is summarized in the following statement. The proof uses a combinations of statements (Lemma 4.5, Proposition 4.8, and Lemma 9.2) in [9] and is based on Mathieu's description of the coherent extensions of bounded $\mathfrak{sp}(2n)$-modules.

Proposition 4.2. *Let $\mathfrak{g} = \mathfrak{sp}(2n)$ and M be a simple module in $\mathcal{B}^\chi$. There is a subset Γ of Δ consisting of n long roots for which M is Γ-injective. The set of all simple subquotients of $\mathcal{D}_\Gamma M$ coincides with the set of all simple modules N in $\mathcal{B}^\chi$ for which* $\operatorname{supp} N \subset \operatorname{supp} \mathcal{D}_\Gamma M = \operatorname{supp} M + Q$.

Recall now the definition of a generalized conjugation in U_{F_Γ} introduced in [9]. Let $\mu = x_1\gamma_1 + ... + x_l\gamma_l \in \operatorname{Span}_{\mathbb{C}} \Gamma \subseteq \mathfrak{h}^*$. For $u \in U_{F_\Gamma}$, $v \in N$ set

$$\Theta_{(x_1,\dots,x_l)}(u) := \sum_{0 \leq i_1,\dots,i_l \leq N(u)} \tbinom{x_1}{i_1} \dots \tbinom{x_l}{i_l} \operatorname{ad}(f_{\gamma_1})^{i_1} \dots \operatorname{ad}(f_{\gamma_l})^{i_l}(u)\, f_{\gamma_1}^{-i_1} \dots f_{\gamma_l}^{-i_l},$$

where $\binom{x}{i} := x(x-1)...(x-i+1)/i!$ for $x \in \mathbb{C}$ and $i \in \mathbb{Z}_+ \cup \{0\}$. Note that for $(x_1, ..., x_l) \in \mathbb{Z}^l$ we have $\Theta_{(x_1,...,x_l)}(u) = f_{\gamma_1}^{x_1} ... f_{\gamma_l}^{x_l} u f_{\gamma_1}^{-x_1} ... f_{\gamma_l}^{-x_l}$. For a U_{F_Γ}-module N by $\Phi_\Gamma^\mu N$ we denote the U_{F_Γ}-module N twisted by the action

$$u \cdot v^\mu := (\Theta_{(x_1,\dots,x_l)}(u) \cdot v)^\mu,$$

where $u \in U_{F_\Gamma}$, $v \in N$, and v^μ stands for the element v considered as an element of $\Phi_\Gamma^\mu N$. In particular, $v^\mu \in N^{\lambda+\mu}$ whenever $v \in N^\lambda$. The following lemma is straightforward.

Lemma 4.3. (i) $\Phi_\Gamma^\mu \circ \Phi_\Gamma^\nu = \Phi_\Gamma^{\mu+\nu}$*, in particular,* $\Phi_\Gamma^\mu \circ \Phi_\Gamma^{-\mu} = \mathrm{Id}$*;*
(ii) $\Phi_\Gamma^\mu = \mathrm{Id}$ *whenever* $\mu \in Q$*;*
(iii) *M is an indecomposable U_{F_Γ}-module if and only if $\Phi_\Gamma^\mu M$ is indecomposable.*

For a Γ-injective module M and $\mu \in \mathfrak{h}^*$ we define the *twisted localization $\mathcal{D}_\Gamma^\mu M$ of M relative to* Γ *and* μ by $\mathcal{D}_\Gamma^\mu M := \Phi_\Gamma^\mu \mathcal{D}_\Gamma M$. The twisted localization plays a major role in the theory of coherent families introduced by Mathieu. An example of such family is the *coherent extension* $\mathcal{E}(M) := \oplus_{\bar{\mu} \in \mathfrak{h}^*/Q} \mathcal{D}_\Gamma^{\bar{\mu}} M$ of M. Here $\mathcal{D}_\Gamma^{\bar{\mu}} M := \mathcal{D}_\Gamma^\mu M$ for $\bar{\mu} := \mu + Q \in \mathfrak{h}^*/Q$ (see Lemma 4.3, (ii)). If M and Γ are as in Proposition 4.2 then $\mathcal{E}(M)$ contains all simple modules in $\mathcal{B}^\chi$ as subquotients.

Some of the properties of the twisted localization are described in the following proposition:

Proposition 4.4. *Let M be a* Γ*-injective* $\mathfrak{g}$*-module in* $\mathcal{B}$*.*
(i) *$\mathcal{D}_\Gamma M \simeq M$ iff M is Γ-bijective.*
(ii) $\operatorname{supp} \mathcal{D}_\Gamma^\mu M = \mu + \operatorname{supp} M + \operatorname{Span}_{\mathbb{Z}} \Gamma$. *Moreover, if* $\nu_0 \in \operatorname{supp} M$ *then* $\dim(\mathcal{D}_\Gamma^\mu M)^{\nu'} = \max\{\dim M^\nu \mid \nu \in \nu_0 + \operatorname{Span}_{\mathbb{Z}} \Gamma\}$, *whenever* $\nu' \in \mu + \nu_0 + \operatorname{Span}_{\mathbb{Z}} \Gamma$.
(iii) *Let M be a module in $\mathcal{B}$ which has a unique simple submodule. Then $\mathcal{D}_\Gamma^\mu M$ is indecomposable whenever M is indecomposable.*

Proof. Statement (i) is straightforward. (ii) follows from a generalization of Lemma 4.4 in [9]. Since $\Phi_\Gamma^\mu \circ \Phi_\Gamma^{-\mu} = \mathrm{Id}$, to prove (iii) is enough to show that $\mathcal{D}_\Gamma M$ is indecomposable. Suppose $\mathcal{D}_\Gamma M = D_1 \oplus D_2$. Then by our assumption M has trivial intersection with one of the modules D_1 or D_2, say $M \cap D_1 = 0$. We next show that $(D_1)^{\nu'} = 0$, for a fixed $\nu' \in \operatorname{supp} \mathcal{D}_\Gamma M$ (and thus $D_1 = 0$). We choose $\nu_0 \in \nu' + \operatorname{Span}_{\mathbb{Z}} \Gamma$ such that $\dim M^{\nu_0} = \max\{\dim M^{\nu} \mid \nu \in \nu' + \operatorname{Span}_{\mathbb{Z}} \Gamma\}$. Then by (ii), $M^{\nu_0} = (\mathcal{D}_\Gamma M)^{\nu_0} = (D_1)^{\nu_0} \oplus (D_2)^{\nu_0}$ and therefore $(D_1)^{\nu_0} = 0$. However, (i) implies that D_1 is Γ-bijective as a submodule of $\mathcal{D}_\Gamma M$ and thus $(D_1)^{\nu'} = 0$. □

Remark 4.5. Statement (iii) of Proposition 4.4 remains valid if we replace the condition $M \in \mathcal{B}$ by the weaker requirement that M is bounded in the Γ-directions only, i.e. that the set $\{\dim M^\lambda \mid \lambda \in \lambda_0 + \operatorname{Span}_{\mathbb{C}} \Gamma\}$ is uniformly bounded for every $\lambda_0 \in \operatorname{supp} M$.

Proposition 4.6. *Let $\mathfrak{g} = \mathfrak{sp}(2n)$ and $n > 1$. Let M be an indecomposable $\mathfrak{g}$-module with unique simple submodule. Then there is a set Γ consisted of n commuting (i.e. orthogonal) long roots such that M is Γ-injective. Moreover, any composition series of M is multiplicity free, i.e. every two distinct simple subquotients of M are nonisomorphic.*

Proof. Let us prove the first statement. Suppose that there is a long root β for which both f_β and $f_{-\beta}$ do not act injectively on M. Let

$$M_0 := \{m \in M \mid f_\alpha^N m = 0, \text{ some } N\} \oplus \{m \in M \mid f_{-\alpha}^K m = 0, \text{ some } K\}.$$

The sum is direct since for every simple $\mathfrak{g}$-module P, for every $p \in P$, and for every long root β, we have that $f_\beta^N p = f_{-\beta}^M p = 0$ implies $p = 0$. The submodule M_0 of M is a direct sum of two nonzero submodules which contradicts the initial assumption.

To prove the second statement choose $\mu \in \mathfrak{h}^*$ and $\Gamma \subset \Delta$ so that $C := \mathcal{D}_\Gamma^\mu M$ is a cuspidal module, i.e. all root vectors e_α and f_α, $\alpha \in \Delta$, act bijectively on C. We have that C is semisimple (Theorem 1 in [5]) and indecomposable (Proposition 4.4, (ii)), and hence it is simple. Let N be the simple submodule of M. Then $C \simeq \mathcal{D}_\Gamma^\mu N$, and therefore

$$\mathcal{D}_\Gamma N \simeq \Phi_\Gamma^{-\mu} C \simeq \Phi_\Gamma^{-\mu} \mathcal{D}_\Gamma^\mu M \simeq \mathcal{D}_\Gamma M.$$

By Proposition 4.2, $\mathcal{D}_\Gamma M$ has a multiplicity free compositions series, and so does its submodule M. □

Proposition 4.7. *Let $\mathfrak{g} = \mathfrak{sp}(2n)$, $n > 1$. Let M be a simple module in $\mathcal{B}$ and Γ be a set of n long roots such that M is Γ-injective. Then $\mathcal{D}_\Gamma M$ and its restricted dual $(\mathcal{D}_\Gamma M)^*$ are the injective hull and the projective cover of M in $\mathcal{B}$, respectively.*

Proof. We first show that $\mathcal{D}_\Gamma M$ is injective, i.e. any exact sequence

$$0 \to \mathcal{D}_\Gamma M \to M' \to N \to 0$$

splits in $\mathcal{B}$. It suffices to prove this in the case when N is simple. Assume that a sequence does not split. Then M' satisfies Proposition 4.6. Since $\operatorname{supp} N \subset \operatorname{supp} \mathcal{D}_\Gamma M$

and and N has the same central character as M, by Proposition 4.2, N is isomorphic to some simple subquotient of $\mathcal{D}_\Gamma M$. Therefore N is a subquotient of M' with multiplicity higher than one, which contradicts to Proposition 4.6. The second statement follows by duality. □

Corollary 4.8. *Let $\mathfrak{g} = \mathfrak{sp}(2n)$, $n > 1$. Then every simple object in $\mathcal{B}$ has a unique projective indecomposable cover and a unique injective hull.*

Remark 4.9. Propositions 4.6 and 4.7 are false for $n = 1$. In fact, in this case the category $\mathcal{B}$ does not have injective and projective modules. To see this, let Ω denote the Casimir operator of $\mathfrak{sl}(2)$ and H be the standard element in the Cartan subalgebra. Let P be an indecomposable projective module in $\mathcal{B}$, M be some simple quotient and $\mu \in \operatorname{supp} M$. There exists an integer p and $\nu \in \mathbb{C}$ such that $(\Omega - \nu)^p$ acts by zero on P.

Let for $s \in \mathbb{Z}$, I_s be the left ideal in $U(\mathfrak{g})$ generated by $H - \mu$ and $(\Omega - \nu)^s$, and let $F(s, \mu, \nu) := U(\mathfrak{g})/I_s$. Then $\operatorname{supp} F(s, \mu, \nu) = \mu + Q$ and every weight has multiplicity s. Moreover, $F(s, \mu, \nu)$ is indecomposable with unique simple quotient isomorphic to M. Hence there exists a surjective homomorphism $P \to F(s, \mu, \nu)$. However, if $s > p$ such homomorphism can not be surjective which leads to a contradiction.

Example 4.10. Let $\mathcal{FIN}$ be the category of all weight $\mathfrak{sp}(2n)$-modules with finite weight multiplicities and locally finite action of the center of $U(\mathfrak{g})$. It is not difficult to show that every indecomposable module in $\mathcal{FIN}$ has finite length. However, Corollary 4.8 does not hold if we replace the category $\mathcal{B}$ by $\mathcal{FIN}$. Here is a counterexample. Choose a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that a Levi subalgebra $\mathfrak{s}$ of $\mathfrak{p}$ is isomorphic to $\mathfrak{sl}(2)$. Choose $H \in \mathfrak{s}$, $\Omega \in U(\mathfrak{s})$ and $\mu, \nu \in \mathbb{C}$ as in the previous remark, so that $F(s, \mu, \nu)$ is a simple $\mathfrak{s}$-module. Endow $F(s, \mu, \nu)$ with a structure of a $\mathfrak{p}$-module by letting the radical to act by zero. Let

$$M^s := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F(s, \mu, \nu).$$

Then M^s is indecomposable and belongs to $\mathcal{FIN}$. It is not difficult to see that M^s has a unique simple quotient which we denote by L. We claim that L does not have a projective cover in $\mathcal{FIN}$. This follows by reasoning similar to the one in the previous remark. Indeed, if P is a projective cover of L, then there is a surjective map $P \to M^s$ for any s. Since P has finite length, this is impossible.

5. From bounded weight $\mathfrak{sp}\,(2n)$-modules to weight $\mathcal{A}_n$-modules

Let $\mathfrak{g} = \mathfrak{sp}\,(2n)$ with $n \geq 2$. Every element $X \in \mathfrak{g}$ can be written in a block matrix form

$$\begin{bmatrix} A & B \\ C & -A^t \end{bmatrix}$$

where A is an arbitrary $n \times n$-matrix, and B and C are symmetric $n \times n$-matrices. The maps

$$B \mapsto \sum_{i \leq j} b_{ij} t_i t_j,\ C \mapsto \sum_{i \leq j} c_{ij} \partial_i \partial_j$$

can be extended to a homomorphism of Lie algebras

$$\mathfrak{g} \to \mathcal{A}_n$$

which induces a homomorphism

$$\omega\colon U(\mathfrak{g}) \to \mathcal{A}_n.$$

It is easy to see that the image of ω coincides with $\mathcal{A}_n^{ev}$. If we fix the standard basis of $\mathcal{A}_n$ we verify that $\omega\colon U(\mathfrak{h}) \to \mathcal{A}_n^0$ is an isomorphism. The representation of $\mathcal{A}_n^{ev}$ in the subspace W of even functions in $\mathbb{C}[t_1, \dots, t_n]$ is called the *Weil representation*. One can check that W is irreducible. If $I := \operatorname{Ker} \omega$, then clearly $I = \operatorname{Ann} W$ is a primitive ideal in $U(\mathfrak{g})$. The center Z of $U(\mathfrak{g})$ acts on W via the central character σ of W.

The next theorem follows from Proposition 12.1 in [9].

Theorem 5.1. *Let χ be a central character such that $\mathcal{B}^\chi$ is non-empty. Then $\mathcal{B}^\chi$ is equivalent to $\mathcal{B}^\sigma$, with equivalence given by a translation functor.*

(For the definition and properties of the translation functor see [8].)

Theorem 5.2. *Let M be any module from the category $\mathcal{B}^\sigma$. Then* $\operatorname{Ann} M = I$.

Proof. It is sufficient to check that the statement holds for injective modules. The latter follows form the fact that all injectives are obtained via a localization as shown in Proposition 4.7. ☐

Corollary 5.3. *The categories $\mathcal{F}_n^{ev}$ and $\mathcal{B}^\chi$ are equivalent.*

6. Explicit description of all bounded $\mathfrak{sp}(4)$-modules

In this section we explicitly describe all indecomposable objects in $\mathcal{B}$ for $\mathfrak{g} := \mathfrak{sp}(4)$. We use the same notations as in Section 4.

Let $\Delta = \{\pm\alpha_i, \pm\beta_i \mid i = 1, 2\}$ be the root system of $\mathfrak{g}$ where α_1, α_2, and β_1, β_2 are the positive short and long roots, respectively. Denote by $B := \{\alpha_1, \beta_2\}$ the standard basis of Δ and let $\Gamma := \{\beta_1, \beta_2\}$. There is an orthonormal basis $\{\varepsilon_1, \varepsilon_2\}$ of $\mathfrak{h}^*$ for which $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\beta_2 = 2\varepsilon_2$. Let W be the Weyl group of $\mathfrak{g}$, and let $s_\alpha \in W$ denote the reflection corresponding to the root α.

For a $\mathfrak{g}$-module M we denote by M^* the restricted dual of M. Note that M^* is isomorphic to the twist $M^{s_{\beta_1} s_{\beta_2}}$ of M by $s_{\beta_1} s_{\beta_2} \in W$. For a basis B' of Δ and a weight $\lambda \in \mathfrak{h}^*$, by $L_{B'}(\lambda)$ we denote the simple highest weight module with highest weight λ relative to the Borel subalgerbra corresponding to B'. Put $\rho_{B'}$ for the half sum of the B'-positive roots in Δ.

For $\mathfrak{g}$-submodules A_1 and A_2 of a $\mathfrak{g}$-module A, as usual, the A-*diagonal* in $A_1 \oplus A_2$ is:

$$D(A) := \{(a,a) \in A_1 \oplus A_2 \mid a \in A\}.$$

For the purpose of our construction we need a more general notion. If L is an endomorphism of $\mathbb{C}^k$, we define the (A,L)-*diagonal* in $A_1^{\oplus k} \oplus A_2^{\oplus k}$ by

$$D_L(A) := \{(a, L(a)) \mid a \in A^{\oplus k}\}.$$

In particular, for $k=1$ and $L = \mathrm{Id}$ we have $D_L(A) = D(A)$.

For $\eta \in \mathfrak{h}^*$ we set

$$\mathcal{B}^\chi[\eta] := \{M \in \mathcal{B}^\chi \mid \operatorname{supp} M \subset \eta + Q\}.$$

We next describe the simple objects of the subcategory $\mathcal{B}^\chi[\eta]$ of $\mathcal{B}^\chi$. There are three types of categories $\mathcal{B}^\chi[\eta]$ depending on the image $\eta + Q$ of η in the torus $\mathfrak{h}^*/Q$.

• *Highest weight type:* $\eta + Q \in \mathcal{HW}(\chi)$. The simple objects of $\mathcal{B}^\chi[\eta]$ are highest weight modules. There are two elements $\eta + Q$ in $\mathfrak{h}^*/Q$ with this property. If $L_B(\lambda^+)$ and $L_B(\lambda^-)$ are the two B-highest weight modules in $\mathcal{B}^\chi$ then $\lambda^\pm + \rho_B = m_1\varepsilon_1 \pm m_2\varepsilon_2$ for $m_i \in \frac{1}{2} + \mathbb{Z}$ (note that $\lambda^- = s_{\beta_2}\lambda^+$). We fix $\lambda^\pm$ so that $m_2 \geq -1/2$. Then the four highest weight modules in $\mathcal{B}^\chi[\lambda^\pm]$ are:

$$N^\pm := L_{s_{\beta_1}s_{\beta_2}(B)}(s_{\beta_1}s_{\beta_2}(\lambda^\pm))$$
$$W^\pm := L_{s_{\beta_2}(B)}(\lambda^\pm + \beta_2), \quad E^\pm := L_{s_{\beta_1}(B)}(s_{\beta_1}s_{\beta_2}(\lambda^\pm) - \beta_2)$$
$$S^\pm := L_B(\lambda^\pm)$$

(standing for north, west, east, and south, respectively). Let $\mathcal{A}^\pm := \{N^\pm, E^\pm, S^\pm, W^\pm\}$. In future we will consider modules either in $\mathcal{A}^+$ or in $\mathcal{A}^-$. For simplicity we will omit the superscripts and will write $\mathcal{A}, N, W, E, S$.

• *Cuspidal type:* $\eta + Q \in \mathcal{CUSP}(\chi)$. In this case there is only one simple object in $\mathcal{B}^\chi[\eta]$ isomorphic to $\mathcal{D}^{\eta-\lambda^+}_{-\beta_1,-\beta_2} L_B(\lambda^+)$, where $\eta - \lambda^+ = x_1\alpha_1 + x_2\alpha_2$ with $x_i \notin \mathbb{Z}$.

• *Semi-plane type:* $\eta + Q \in \mathcal{SEMI}(\chi)$. There are two simple objects in $\mathcal{B}^\chi[\eta]$ whose supports are semi-planes. In this case $\eta + Q$ equals $\lambda^+ + x\varepsilon_1 + Q$ or $\lambda^+ + x\varepsilon_2 + Q$ for $x \notin \mathbb{Z}$ which we will call *NW-ES type* and *NE-SW type*, respectively. The two simple objects are isomorphic to $\mathcal{D}^{\eta-\lambda^+}_{-\beta_1} L_B(\lambda^+)$ and its dual for the NW-ES type and $\mathcal{D}^{\eta-\lambda^+}_{-\beta_2} L_B(\lambda^+)$ and its dual for the NE-SW type. Here $\eta - \lambda \in x\varepsilon_i + Q$ for $x \notin \mathbb{Z}$ and $i = 1$ (respectively, $i = 2$) for the NW-ES (resp., NE-SW) type.

Example 6.1. In the special case when χ equals the central character χ_0 of the Weyl modules $L_B(\omega^+)$ or $L_B(\omega^-)$, where $\omega^+ = -\frac{1}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2$ and $\omega^- = -\frac{1}{2}\varepsilon_1 - \frac{3}{2}\varepsilon_2$, we have that all simple objects in $\mathcal{B}^{\chi_0}$ have one-dimensional weight spaces. We may simplify our considerations if we first restrict our attention to the category $\mathcal{B}^{\chi_0}$ and then apply the translation functor $\theta^\chi_{\chi_0} : \mathcal{B}^{\chi_0} \to \mathcal{B}^\chi$, $\theta^\chi_{\chi_0}(M) := \mathrm{pr}_\chi(M \otimes L_B(\lambda^+ - \omega^+))$, where pr_χ is the projection onto $\mathcal{B}^\chi$ (note that $L_B(\lambda^+ - \omega^+)$ is a finite dimensional module).

The highest weight part $\mathcal{B}^{\chi_0}[\omega^-]$ of $\mathcal{B}^{\chi_0}$ is described on Figure 1. The other highest weight part, $\mathcal{B}^{\chi_0}[\omega^+]$, can be pictured by rotating Figure 1 by 90°.

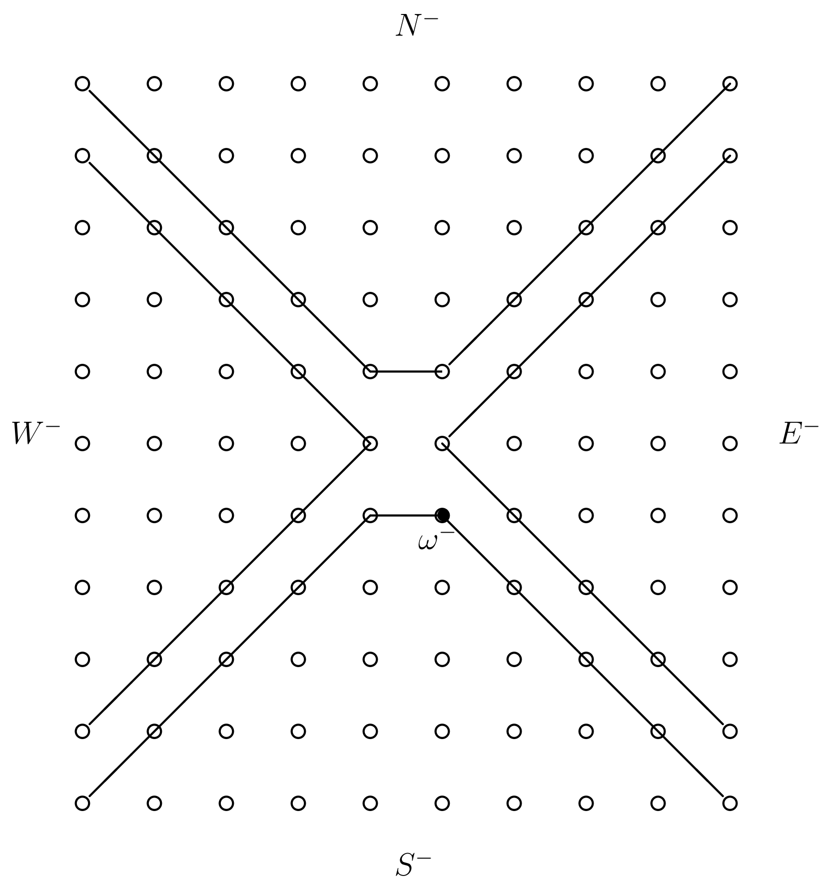

FIGURE 1

Remark 6.2. Let $\mathcal{M}_\chi$ be the unique semisimple coherent family with central character χ which is irreducible, i.e. for which $\mathcal{M}_\chi[\lambda] := \oplus_{\mu\in\lambda+Q}(\mathcal{M}_\chi)^\mu$ is irreducible for some λ. Another way to describe the three types of cosets $\eta + Q$ is via the generalized Shapovalov map $S_\chi : \mathfrak{h}^* \to \mathbb{C}$ defined by $\lambda \mapsto \det(f_{\beta_1}f_{\beta_2}e_{\beta_1}e_{\beta_2})_{|(\mathcal{M}_\chi)^\lambda}$. We have that for $\eta \in \mathfrak{h}^*$ the zero set of the restriction $S_{\chi|\eta+Q}$ of S_χ is either empty, a line, or a union of two lines. These three cases for $\eta + Q$ correspond to cuspidal, semi-plane, and highest weight type, respectively.

Lemma 6.3. *(i) The module $\mathcal{D}_{\beta_1}N$ (respectively, $\mathcal{D}_{\beta_2}N$) has length two and $(\mathcal{D}_{\beta_1}N)/N \simeq W$, (resp., $(\mathcal{D}_{\beta_2}N)/N \simeq E$).*

(ii) The module $\mathcal{D}_{\beta_1,\beta_2}N$ has length 3 and:

$$0 \subset L_1 \oplus L_2 \subset L_3 = (\mathcal{D}_{\beta_1,\beta_2}N)/N$$

where $L_1 \simeq E$, $L_2 \simeq W$, and $L_3/(L_1 \oplus L_2) \simeq S$.

Let $T = (T_1, ..., T_k)$ be an ordered k-tuple of elements in $\mathcal{A}$. We call T *admissible* if T_i and T_{i+1} are *successive* in $\mathcal{A}$, i.e. for $T_i = N$, we have either $T_{i+1} = E$ or $T_{i+1} = W$, etc. For $X \in \mathcal{A}$ and $T = (T_1, ..., T_k)$ for which (X, T) is admissible we construct an indecomposable extension X_T of X for which $(X_T/X)^* \simeq (T_1)_{T_2,...,T_k}$. A convenient way to represent X_T is by a graph with a set of vertices $T \cup \{X\}$ and oriented edges $T_{2i+1} \to T_{2i}$ and $T_{2i+1} \to T_{2i+2}$, $i \geq 0$, where $T_0 := X$. As an immediate application of Lemma 6.3 we define $W_{S,E} = E_{S,W} := \mathcal{D}_{\beta_1,\beta_2}N$. In a similar way we set $N_{W,S} := (\mathcal{D}_{\beta_1,-\beta_2}E)/E, N_{E,S} := (\mathcal{D}_{-\beta_1,\beta_2}W)/W$.

Since W is a submodule of both W_N and $E_{S,W}$ and E is a submodule of both $E_{S,W}$ and E_N we may define:

$$N_{W,S,E} := ((W_{S,E} \oplus W_N)/D_{i_W,j_W}(W))^*, N_{E,S,W} := ((E_{S,W} \oplus E_N)/D_{i_E,j_E}(E))^*.$$

We might think of $N_{W,S,E}$ as the β_1-localization of the "W-part" of $W_{S,E}$. With similar reasoning we set:

$$N^1 = N_{W,S,E,N} := ((N^*_{W,S,E} \oplus E_N)/D(E))^* \simeq ((N^*_{E,S,W} \oplus W_N)/D(W))^*.$$

We easily generalize the above constructions and for X and Y in $\mathcal{A}$ define $X_{(Y,T)}$ using a "partial localization" of Y_T. Also, if $T = (T_0, T_1)$ where T_0 has l copies of each element of $\mathcal{A}$ we set for simplicity $X^l_{T_1} := X_T$ (we allow $T_1 = \varnothing$ as a 0-tuple writing simply X^l in this case). We put also $X^0_0 := X$, $X^0_T := X_T$ for a k-tuple T, $0 \leq k \leq 3$,

We next notice that E and W are submodules of $W_{N,E}$ and $W_{S,E}$, so for every $c \in \mathbb{C}$ and a positive integer k we define

$$N^k_\lambda := (W^{\oplus k}_{N,E} \oplus W^{\oplus k}_{S,E})/(D_{\mathrm{Id}}(W^{\oplus k}) \oplus D_{J^k_c}(W^{\oplus k})),$$

where $J^k_c \in End(\mathbb{C}^k)$ is represented by a single Jordan block with c on the diagonal. Note that $N^k_0 \simeq N^{k-1}_{E,S,W}$ for $k \geq 1$.

In similar fashion we construct X^k_c for every X in $\mathcal{A}$. We set $A^l_c := N^l_c \simeq S^l_c$ and $B^l_c := E^l_c \simeq W^l_c$, $l \geq 1$. Finally, denote by P_X the projective cover of X. Note that P_X is the Γ_X-localization of X where Γ_X is the set of those two long roots for which X is Γ_X-localizable.

Proposition 6.4. *Up to an isomorphism, the complete list of the indecomposable objects in $\mathcal{B}^\chi$ includes:*

(i) Highest weight type: P_X, X^k_T, $(X^k_T)^*$, A^l_c, $B^l_c \simeq (A^l_c)^*$, where $X \in \mathcal{A}$, T is an n-tuple, $0 \leq n \leq 3$, $k \geq 0$, $l \geq 1$ $c \in \mathbb{C}$. Up to a twist of an element of the Weyl group

we have five types (the projective and four series) of modules: P_N, N^k, N^k_E, $N^k_{E,S}$, *and* N^k_c.

(ii) Cuspidal type: $\mathcal{D}^{\mu}_{\beta_1,\beta_2}N$ *with* $\mu = x_1\alpha_1 + x_2\alpha_2$, $x_i \notin \mathbb{Z}$.

(iii) Semi-plane type: $\mathcal{D}^{\mu}_{\beta_1}N$, $(\mathcal{D}^{\mu}_{\beta_1}N)^*$, $\mathcal{D}^{\nu}_{\beta_2}N$, $(\mathcal{D}^{\nu}_{\beta_2}N)^*$, $\mathcal{D}^{\mu}_{\beta_1,\beta_2}N$, $(\mathcal{D}^{\mu}_{\beta_1,\beta_2}N)^*$, $\mathcal{D}^{\nu}_{\beta_1,\beta_2}N$, $(\mathcal{D}^{\nu}_{\beta_1,\beta_2}N)^*$, *where* $\mu = x_1\varepsilon_1 + x_2\varepsilon_2$ *and* $\nu = y_1\varepsilon_1 + y_2\varepsilon_2$ *are such that* $x_1 \notin \mathbb{Z}$, $x_2 \in \mathbb{Z}$, $y_1 \in \mathbb{Z}$, $y_2 \notin \mathbb{Z}$. *Up to a twist of the Weyl group there are two types:* $\mathcal{D}^{\mu}_{\beta_1}N$ *and* $\mathcal{D}^{\mu}_{\beta_1,\beta_2}N$.

References

[1] V. Bekkert, G. Benkart, V. Futorny, Weight modules for Weyl algebras, in "Kac-Moody Lie algebras and related topics", *Contemp. Math.* **343** Amer. Math. Soc., Providence, RI, 2004, 17–42.

[2] G. Benkart, D. Britten, F. Lemire, Modules with bounded weight multiplicities for simple Lie algebras. *Math. Z.* **225** (1997), no. 2, 333–353.

[3] D. Britten, F. Lemire, A classification of simple Lie modules having a 1-dimensional weight space. *Trans. Amer. Math. Soc.* **299** (1987), no. 2, 683–697.

[4] D. Britten, F. Lemire, Tensor product realizations of simple torsion free modules. *Canad. J. Math.* **53** (2001), no. 2, 225–243.

[5] D. Britten, O. Khomenko, F. Lemire, V. Mazorchuk, Complete reducibility of torsion free C_n-modules of finite degree, *J. Algebra* **276** (2004), 129–142.

[6] S. Fernando, Lie algebra modules with finite dimensional weight spaces I, *Trans. Amer. Math. Soc.* **322** (1990), 757-781.

[7] V. Futorny, The weight representations of semisimple finite dimensional Lie algebras, Ph. D. Thesis, Kiev University, 1987.

[8] J. C. Jantzen, Moduln mit einen hochsten Gewitch, *Lect. Notes Math.* **750** 1979.

[9] O. Mathieu, Classification of irreducible weight modules, *Ann. Inst. Fourier* **50** (2000), 537–592.

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