BLOCKS IN THE CATEGORY OF FINITE-DIMENSIONAL REPRESENTATIONS OF $\mathfrak{gl}(m|n)$

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Abstract. We decompose the category of finite-dimensional $\mathfrak{gl}(m|n)$-modules into the direct sum of blocks, show that two blocks are equivalent iff their degrees of atypicality are the same. Furthermore we show that a block is equivalent to the maximal atypical block in the category of $\mathfrak{gl}(k|k)$-modules.

Contents

1. Introduction

The abelian category of finite-dimensional representations of $\mathfrak{gl}(m|n)$ is not semi-simple, i.e. indecomposable objects do not coincide with irreducible. In this paper we show that it splits into the direct sum of indecomposable full subcategories called blocks. Similar situation happens in the representation theory over fields of finite characteristic. We show that each block consists of all representations of $\mathfrak{gl}(m|n)$ with fixed action of the center of a universal enveloping algebra of $\mathfrak{gl}(m|n)$. We also show that two blocks are equivalent as categories iff the degrees of atypicality of the corresponding central characters are the same. Thus the category of finite-dimensional $\mathfrak{gl}(m|n)$-modules has $\min(m,n)$ blocks up to equivalence. Moreover, a block of degree of

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atypicality k is equivalent to the block of maximal degree of atypicality in the category of finite-dimensional $\mathfrak{gl}(k|k)$-modules. This allows one to reduce many problems in representation theory (such as calculating multiplicities of irreducible modules in Kac modules and projective modules) to the case of a maximal degenerate block of $\mathfrak{gl}(k|k)$. In particular the combinatorial algorithm for the character of an irreducible representation obtained in [?] becomes clearer in this setting, as we discuss in the last chapter.

2. Category $\mathcal{F}$

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ over $\mathbb{C}$, $\mathcal{F}$ be the category of finite-dimensional $\mathfrak{g}$-modules. As it was shown in [?] all irreducible objects can be described by highest weights similarly to irreducible $\mathfrak{gl}(n)$-modules. Here we briefly recall how it works.

First we fix a Cartan subalgebra $\mathfrak{h}$ of diagonal matrices. Then $\mathfrak{g}$ has a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\oplus_{\alpha\in\Delta}\mathfrak{g}^{\alpha}\right),$$

where each root subspace $\mathfrak{g}^{\alpha}$ is 1-dimensional even or odd. According to the parity of $\mathfrak{g}^{\alpha}$ a root α is called *even* or *odd*. The set of even roots is denoted by Δ_0, the set of odd roots is denoted by Δ_1. Consider the natural basis $\{\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n\}$ in $\mathfrak{h}^*$. Then

$$\Delta_0 = \{\varepsilon_i - \varepsilon_j | i, j = 1, \dots, m\} \cup \{\delta_i - \delta_j | i, j = 1, \dots, n\},$$
$$\Delta_1 = \{\pm(\varepsilon_i - \delta_j) | i = 1, \dots, m, j = 1, \dots, n\}.$$

We fix also a $\mathfrak{g}$-invariant bilinear symmetric form $\langle , \rangle$ on $\mathfrak{g}^*$ by the condition

$$\langle\varepsilon_i, \varepsilon_j\rangle = -\langle\delta_i, \delta_j\rangle = \delta_{ij}, \ \langle\varepsilon_i, \delta_j\rangle = 0.$$

Finally we divide Δ into the set of positive and negative roots in the following way

$$\Delta_0^+ = \Delta_0 \cap \Delta^+ = \{\varepsilon_i - \varepsilon_j | 1 \le i < j \le m\} \cup \{\delta_i - \delta_j | 1 \le i < j \le n\},$$
$$\Delta_1^+ = \Delta_1 \cap \Delta^+ = \{\varepsilon_i - \delta_j | 1 \le i \le m, 1 \le j \le n\},$$
$$\Delta^- = -\Delta^+,$$

and fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \text{ where } \mathfrak{n}^{\pm} = \oplus_{\alpha\in\Delta^{\pm}}\mathfrak{g}_{\alpha}.$$

The set of weights of $\mathfrak{g}$ is a subset $P \subset \mathfrak{h}^*$ defined as

$$P = \left\{\lambda \in \mathfrak{h}^* | 2\frac{\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle} \in \mathbb{Z} \text{ for any } \alpha \in \Delta_0\right\}.$$

A weight $\lambda \in P$ is *positive* iff $2\frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle} > 0$ for any $\alpha \in \Delta_0^+$. A weight $\lambda \in P$ is *regular* iff $\langle\lambda,\alpha\rangle \neq 0$ for any $\alpha \in \Delta_0$. The set of positive weights we denote by P^+, the set of regular weights we denote by P^{reg}. In the standard coordinates

$$\lambda \in P \text{ iff } \langle\lambda,\varepsilon_i\rangle - \langle\lambda,\varepsilon_j\rangle \in \mathbb{Z},\ \langle\lambda,\delta_i\rangle - \langle\lambda,\delta_j\rangle \in \mathbb{Z};$$

$$\lambda \in P^{\text{reg}} \text{ iff } \langle\lambda,\varepsilon_i\rangle \neq \langle\lambda,\varepsilon_j\rangle, \langle\lambda,\delta_i\rangle \neq \langle\lambda,\delta_j\rangle \text{ for } i \neq j;$$

$$\lambda \in P^+ \text{ iff } \langle\lambda,\varepsilon_i\rangle > \langle\lambda,\varepsilon_j\rangle \text{ for } 1 \leq i < j \leq m \text{ and } \langle\lambda,\delta_i\rangle < \langle\lambda,\delta_j\rangle \text{ for } 1 \leq i < j \leq n.$$

Now we are able to describe irreducible finite-dimensional $\mathfrak{g}$-modules. It was shown in [?] that all irreducible modules of $\mathcal{F}$ are in one-to-one correspondence with positive weights. For each $\lambda \in P^+$ one constructs an irreducible $\mathfrak{g}$-module L_λ in the following way. Define a Verma module $M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} c_\lambda$, where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+, c_\lambda$ be the irreducible 1-dimensional $\mathfrak{b}$-module determined by the conditions $\mathfrak{n}^+ \cdot c_\lambda = 0$, $h|_{c_\lambda} = (\lambda - \rho)(h) \cdot \text{id}$ for any $h \in \mathfrak{h}, \rho = 1/2\left(\Sigma_{\alpha\in\Delta_0^+}\alpha - \Sigma_{\alpha\in\Delta_1^+}\alpha\right)$. Then M_λ has a unique irreducible quotient L_λ which is finite-dimensional. (To be honest each $\lambda \in P^+$ determines two irreducible modules since c_λ can have dimension $(1|0)$ or $(0|1)$. To avoid boring but nonessential considerations of parity we allow an isomorphism to change parity, i.e. we consider M and ΠM as isomorphic, where Π stays for the changing parity functor.)

Here similarity between $\mathfrak{gl}(m|n)$-modules and $\mathfrak{sl}(n)$-modules ends, since the category $\mathcal{F}$ is not semi-simple and has rather complicated structure (see [?]). So the natural question arises: what are the blocks, i.e. "minimal indecomposable pieces" in the category $\mathcal{F}$. We answer this question in the next section.

3. Categories $\mathcal{F}_\chi$

Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra $U(\mathfrak{g})$. *A central character* is a homomorphism $\chi\colon Z(\mathfrak{g}) \to \mathbb{C}$. We say that a $\mathfrak{g}$-module M has a central character χ if for any $z \in Z(\mathfrak{g}), m \in M$ there is $n \in Z_{\geq 0}$ such that $(z - \chi(z)\,\text{id})^n \cdot m = 0$. Clearly any finite-dimensional indecomposable $\mathfrak{g}$-module has some central character, and any finite-dimensional $\mathfrak{g}$-module decomposes into a direct sum of submodules with central characters. Let $\mathcal{F}_\chi$ be the subcategory of $\mathcal{F}$ consisting of all modules with central character χ. Obviously, each $\mathcal{F}_\chi$ is a full subcategory of $\mathcal{F}$. Furthermore, $\mathcal{F} = \oplus\mathcal{F}_\chi$, where the summation is taken over all central characters χ for which $\mathcal{F}_\chi$ is nonempty. We are interested in a description of the structure of $\mathcal{F}_\chi$.

First, let us describe irreducible modules in $\mathcal{F}_\chi$. We use a Harish-Chandra homomorphism $HC\colon Z(\mathfrak{g}) \to \mathrm{Pol}(\mathfrak{h}^*)$. The construction of this homomorphism is the same as for semi-simple Lie algebras (see for example [?]). Thus, any $\lambda \in \mathfrak{h}^*$ defines a central character χ_λ by the rule $\chi_\lambda(z) = HC(z)(\lambda)$. Definition of HC immediately implies that an irreducible module L_λ has a central character χ_λ. Let us denote by P_χ the set of weights $\lambda \in P$ such that $\chi_\lambda = \chi$ and set $P_\chi^+ = P_\chi \cap P^+, P_\chi^{\mathrm{reg}} = P_\chi \cap P^{\mathrm{reg}}$. Then obviously irreducible objects in $\mathcal{F}_\chi$ are L_λ for all $\lambda \in P_\chi^+$. To study the category $\mathcal{F}_\chi$ we need a good description of P_χ^+. The following statement was first formulated in [?] and proved in [?].

Proposition 3.1. *Let $\lambda, \mu \in P$, W be the Weyl group of $\mathfrak{g}_0$. Then $\chi_\lambda = \chi_\mu$ iff there is a sequence of odd roots $\alpha_1, \dots, \alpha_s \in \Delta_1$ and $w \in W$ such that $\mu = w(\lambda + \alpha_1 + \dots + \alpha_s)$ and $\langle \lambda + \alpha_1 + \dots + \alpha_{i-1}, \alpha_i \rangle = 0$ for $i = 1, \dots, s$.*

We want to give more constructive condition for $\chi_\lambda = \chi_\mu$. Let $\lambda \in P$ and $A_\lambda = \{\alpha_1, \dots, \alpha_k\}$ be a maximal set of mutually orthogonal odd positive roots such that $\langle \lambda, \alpha_i \rangle = 0$ for $i = 1, \dots, k$. If $\lambda \in P^{\mathrm{reg}}$ the set A_λ is unique. Otherwise there could be several possible A_λ but they all have the same number of elements k. This number is called *degree of atypicality* of λ and is denoted by $\#\lambda$. Let $\alpha_1 = \varepsilon_{i_1} - \delta_{j_1}, \dots, \alpha_k = \varepsilon_{i_k} - \delta_{j_k}$. Since $\alpha_1, \dots, \alpha_k$ are mutually orthogonal $i_p \neq i_q$ and $j_r \neq j_s$. *The core* c_λ of λ is a pair of sets $a_\lambda = \{\langle \lambda, \varepsilon_i \rangle \,|\, i \neq i_1, \dots, i_k\}$ and $b_\lambda = \{\langle \lambda, \delta_j \rangle \,|\, j \neq j_1, \dots, j_k\}$. The core does not depend on the choice of A_λ.

Example 3.2. Let $\mathfrak{g} = \mathfrak{gl}(3|3)$, $\lambda = 5\varepsilon_1 + 3\varepsilon_2 - \varepsilon_3 + 4\delta_1 + 3\delta_2 + \delta_3$. Then $A_\lambda = \{\varepsilon_3 - \delta_3\}$, $\#\lambda = 1$, $c_\lambda = (\{5, 3\}, \{-4, -3\})$.

If λ is not regular, say $\lambda = 5\varepsilon_1 - 3\varepsilon_2 - 3\varepsilon_3 + 4\delta_1 + 3\delta_2 + \delta_3$, then A_λ is either $\{\varepsilon_2 - \delta_2\}$ or $\{\varepsilon_3 - \delta_2\}$, $\#\lambda = 1$. Meanwhile the core is uniquely defined: $c_\lambda = (\{5, -3\}, \{-4, -1\})$.

Proposition 3.1 implies the following

Corollary 3.3. *Let $\lambda, \mu \in P$. Then $\chi_\lambda = \chi_\mu$ iff $\#\lambda = \#\mu$ and $c_\lambda = c_\mu$.*

Thus for any central character χ one can define $\#\chi$ and c_χ by putting $\#\chi = \#\lambda$ and $c_\chi = c_\lambda$ for some $\lambda \in P_\chi$. Furthermore, χ is uniquely determined by $\#\chi$ and c_χ, and $P_\chi = \{\lambda \in P | \#\lambda = \#\chi, c_\lambda = c_\chi\}$.

The next question we are going to discuss is when two categories $\mathcal{F}_\chi$ and $\mathcal{F}_{\chi'}$ are equivalent. Consider the simplest example first.

Example 3.4. Consider the case when the degree of atypicality of a central character χ is zero. Assume that $\mathcal{F}_\chi$ is non-empty. By corollary 3.3 P^+_χ consists of one element λ, and therefore $\mathcal{F}_\chi$ has only one irreducible module L_λ. Moreover, L_λ is isomorphic to so called Kac module

$$V_\lambda = U(\mathfrak{g}) \otimes_{U\left(\mathfrak{g}_0 \oplus \mathfrak{g}_1^+\right)} L^0_\lambda,$$

where $\mathfrak{g}_1^+ = \oplus_{\alpha \in \Delta_1^+} \mathfrak{g}_\alpha$ and L^0_λ is the irreducible $\mathfrak{g}_0$-module with the highest weight λ and with trivial action of $\mathfrak{g}_1^+$. Now note that for any $\mathfrak{g}_0$-module M^0 one can consider the induced $\mathfrak{g}$-module

Ind $M^0 = U(\mathfrak{g}) \otimes_{U\left(\mathfrak{g}_0 \oplus \mathfrak{g}_1^+\right)} M^0$.

Obviously Ind is a functor from the category of finite-dimensional $\mathfrak{g}_0$-modules to the category $\mathcal{F}$. Let $\mathcal{F}^0_\lambda$ be the category consisting of finite-dimensional $\mathfrak{g}_0$-modules with all irreducible subquotients isomorphic to L^0_λ. Then it is not hard to show that Ind: $\mathcal{F}^0_\lambda \to \mathcal{F}_\chi$ is an equivalence of categories. The inverse functor is Inv, which maps M to $M^{\mathfrak{g}_1^+} = \left\{m \in M | \mathfrak{g}_1^+ \cdot m = 0\right\}$.

To complete the picture note that $\mathfrak{g}_0 \cong \mathfrak{sl}(m) \oplus \mathfrak{sl}(n) \oplus \mathfrak{z}$, where $\mathfrak{z}$ is two dimensional center of $\mathfrak{g}_0$. Then two functors $M \mapsto \mathrm{Hom}_{\mathfrak{g}_0}(M, L^0_\lambda)$ and $N \mapsto N \otimes_{\mathbb{C}} L^0_\lambda$ establish the equivalence between the category $\mathcal{F}^0_\lambda$ and the category of finite-dimensional nilpotent $\mathfrak{z}$-modules. Thus, all $\mathcal{F}_\chi$ with $\#\chi = 0$ are equivalent.

If we proceed to the case $\#\chi > 0$, the problem becomes more interesting. Our main results can be formulated in the following three theorems.

Theorem 3.5. *Categories $\mathcal{F}_\chi$ and $\mathcal{F}_{\chi'}$ are equivalent iff $\#\chi = \#\chi'$.*

Theorem 3.6. *Let $\#\chi = k > 0$. Then the category $\mathcal{F}_\chi$ is equivalent to the category $\mathcal{F}^k$ of finite-dimensional $\mathfrak{gl}(k|k)$-modules with the (most atypical) central character χ_ρ.*

Theorem 3.7. *For any central character χ the category $\mathcal{F}_\chi$ is indecomposable.*

The rest of the paper consist of proofs of these theorems.

4. Combinatorial preparations

The case $\#\chi = 0$ is already done in example 3.4. Thus we can assume that $\#\chi = k > 0$. Denote by $M^{p,q}$ the set of all pairs (a, b), where a is a p-element subset in $\mathbb{Z}$, b is a q-element subset in $\mathbb{Z}$, and $a \cap b = \varnothing$. Then $c_\chi \in M^{p,q}$ where $p = m - k, q = n - k$.

We say that $c' = (a', b') \in M^{p,q}$ is *a shift* of $c = (a, b) \in M^{p,q}$ iff one of the following conditions holds:

(1) there is $t \in b$ such that $b' = (b \setminus \{t\}) \cup \{t \pm 1\}$ and $a' = a$;

(2) there is $t \in a$ such that $a' = (a \setminus \{t\}) \cup \{t \pm 1\}$ and $b' = b$.

We say that $c' = (a', b') \in M^{p,q}$ is *a reflection* of $c = (a, b) \in M^{p,q}$ iff there is $t \in a$ such that $t \pm 1 \in b$ and $a' = (a \setminus \{t\}) \cup \{t \pm 1\}, b' = (b \setminus \{t \pm 1\}) \cup \{t\}$.

Example 4.1. Let $c = (\{3, 5\}, \{2, 4\})$. By shifting one can obtain from c either $(\{3, 6\}, \{2, 4\})$ or $(\{3, 5\}, \{1, 4\})$. By reflecting one can obtain from c three elements : $(\{2, 5\}, \{3, 4\}), (\{3, 4\}, \{2, 5\})$ and $(\{4, 5\}, \{2, 3\})$.

We will need the following combinatorial

Lemma 4.2. *For any two* c,$c' \in M^{p,q}$ *there exist* $c^0, \dots, c^s \in M^{p,q}$ *such that* c^{i+1} *is either a shift or a reflection of* c^i *and* $c^0 = c, c^s = c'$.

Proof. For an arbitrary $c = (a, b) \in M^{p,q}$ let $m(c) = a \cup b$. Then clearly a reflections does not change $m(c)$ and a shift increases or decreases one element in $m(c)$ by 1. Thus, using only shifts one can convert $m(c)$ into $\{1, 2, \dots, p + q\}$. Now each reflection increases or decreases one element in a by 1, thus one can convert a into $\{1, \dots, p\}$. ☐

5. Translation functors

In this section we prove Theorem 3.5 by an explicit construction of functors giving equivalence between $\mathcal{F}_\chi$ and $\mathcal{F}_{\chi'}$. These functors are similar to translation functors for category $\mathcal{O}$ (see [**?**]).

Let E be a finite-dimensional $\mathfrak{g}$-module. Let $T_E : \mathcal{F} \to \mathcal{F}$ be a functor defined by $T_E M = M \otimes_{\mathbb{C}} E$. If χ and χ' are two central characters we can define a functor $T_E^{\chi,\chi'} : \mathcal{F}_\chi \to \mathcal{F}_{\chi'}$ by putting $T_E^{\chi,\chi'} M = p_{\chi'}(T_E M)$, where $p_{\chi'} : \mathcal{F} \to \mathcal{F}_{\chi'}$ is the natural projection.

We need the following facts, proved in [**?**].

Lemma 5.1. *The dimension of* $\mathfrak{b}$*-invariant subspace of the weight* μ *in* $T_E(L_\lambda)$ *is not bigger than the multiplicity of the weight* $\mu - \lambda$ *in* E.

Lemma 5.2. *(a) A functor* $T_E^{\chi,\chi'}$ *is exact.*

(b) Functors $T_E^{\chi,\chi'}$ *and* $T_{E^*}^{\chi',\chi}$ *are adjoint, i.e. there are canonical isomorphisms between* $\operatorname{Hom}_{\mathfrak{g}}\left(T_E^{\chi,\chi'} M, N\right)$ *and* $\operatorname{Hom}_{\mathfrak{g}}\left(M, T_{E^*}^{\chi',\chi} N\right)$ *and between* $\operatorname{Hom}_{\mathfrak{g}}\left(M, T_E^{\chi,\chi'} N\right)$ *and* $\operatorname{Hom}_{\mathfrak{g}}\left(T_{E^*}^{\chi',\chi} M, N\right)$.

Now we are going to show that for some special χ, χ' and E the functor $T_E^{\chi,\chi'}$ is an equivalence of categories. Denote by $P(E)$ the multiset of weights of E, i.e. the multiplicity of μ in $P(E)$ is equal to the multiplicity of the weight μ in E.

Lemma 5.3. *Let χ, χ' be central characters and $\lambda \in P^+_\chi$ be such that $|(\lambda + P(E)) \cap P^+_{\chi'}| = 1$. Then $T_E^{\chi,\chi'} L_\lambda = L_{\lambda'}$ is irreducible.*

Proof. It follows from lemma 5.1 that the dimension of a maximal $\mathfrak{b}$-invariant subspace in $T_E^{\chi,\chi'} L_\lambda$ is at most 1. This implies irreducibility of $T_E^{\chi,\chi'} L_\lambda$. □

Proposition 5.4. *Let central characters χ and χ' and a finite-dimensional $\mathfrak{g}$-module E satisfy the following conditions:*

(1) *For any $\lambda \in P^+_\chi, (\lambda + P(E)) \cap P^{reg}_{\chi'} = (\lambda + P(E)) \cap P^+_{\chi'}$ is a one-element multiset $\{\lambda'\}$;*
(2) *For any $\mu \in P^+_{\chi'}, (\mu - P(E)) \cap P^{reg}_{\chi'} = (\mu - P(E)) \cap P^+_{\chi'}$ is a one-element multiset $\{\mu'\}$.*

Then the functors $T_E^{\chi,\chi'}$ and $T_{E^*}^{\chi',\chi}$ establish an equivalence of categories $\mathcal{F}_\chi$ and $\mathcal{F}_{\chi'}$.

Proof. Note that by lemma 5.2 (b) it suffices to check that the functors $T_E^{\chi,\chi'}$ and $T_{E^*}^{\chi',\chi}$ are faithful. By symmetry of conditions (1) and (2) it is sufficient to check that $T_E^{\chi,\chi'}$is faithful. By lemma 5.2 (a) it is sufficient to show that $T_E^{\chi,\chi'} L_\lambda \neq 0$.

Here we recall the superanalogue of Borel-Weil-Bott theorem (for details see [**?**]). Consider the flag supermanifold G/B. We will work in the category of G-sheaves on G/B, i.e. the sheaves obtained by induction from B-modules. Let $\mathcal{O}_\lambda$ be the invertible sheaf on G/B induced by the infinitesimal character $\lambda - \rho$. It was shown in [**?**] that if $\lambda \in P^+$ then the space of global sections $\Gamma(\mathcal{O}_\lambda)$ is a $\mathfrak{g}$-module, which contains an irreducible submodule isomorphic to L_λ. It was also shown there that if $\lambda \notin P^{\text{reg}}$ then all cohomologies of $\mathcal{O}_\lambda$ vanish. Let $\mathcal{L}_E$ be the sheaf on G/B induced by G-module E considered as B-module. Then $\Gamma(\mathcal{L}_E) = E$ and $\Gamma(\mathcal{L} \otimes_\mathcal{O} \mathcal{L}_E) = \Gamma(\mathcal{L}) \otimes_\mathbb{C} E$ for any G-sheaf $\mathcal{L}$. Finally since the action of $U(\mathfrak{g})$ is defined on any G-sheaf $\mathcal{L}$ on G/B one can define a canonical projection $p_\chi(\mathcal{L})$ on the component with central character χ. By definition $\Gamma(p_\chi(\mathcal{L})) = p_\chi(\Gamma(\mathcal{L}))$.

Consider the sheaf $p_{\chi'}(\mathcal{O}_\lambda \otimes_\mathcal{O} \mathcal{L}_E)$. It has a G-filtration with invertible quotients $\mathcal{O}_\nu$ where $\nu \in (\lambda + P(E)) \cap P_{\chi'}$. By condition (1) any $\nu \in (\lambda + P(E)) \cap P_{\chi'}, \nu \neq \lambda'$ is not regular. Therefore all cohomologies of $\mathcal{O}_\nu$ vanish for $\nu \neq \lambda'$. Hence

$$\Gamma(p_{\chi'}(\mathcal{O}_\lambda \otimes_\mathcal{O} \mathcal{L}_E)) = \Gamma(\mathcal{O}_{\lambda'}).$$

On the other hand

$$\Gamma\left(p_{\chi'}\left(\mathcal{O}_\lambda \otimes_{\mathcal{O}} \mathcal{L}_E\right)\right) = p_{\chi'}\left(\Gamma\left(\mathcal{O}_\lambda \otimes_{\mathcal{O}} \mathcal{L}_E\right)\right) = p_{\chi'}\left(\Gamma\left(\mathcal{O}_\lambda\right) \otimes_{\mathbb{C}} E\right).$$

Thus,

$$T_E^{\chi,\chi'}\Gamma\left(\mathcal{O}_\lambda\right) = \Gamma\left(\mathcal{O}_{\lambda'}\right).$$

Now note that conditions (1) and (2) imply that the mapping $\varepsilon\colon P_\chi^+ \to P_{\chi'}^+$, which sends λ to λ' is a bijection. For any irreducible subquotient L_κ of $\Gamma\left(\mathcal{O}_\lambda\right), \kappa \neq \lambda, T_E^{\chi,\chi'} L_\kappa$ is either $L_{\kappa'}$ $(\kappa' \neq \lambda')$ or zero by lemma 5.3. Since by lemma 5.2 (a) $T_E^{\chi,\chi'}$ is an exact functor and $\Gamma\left(\mathcal{O}_{\lambda'}\right)$ has a submodule isomorphic to $L_{\lambda'}$, there should be an irreducible subquotient L_κ in $\Gamma\left(\mathcal{O}_\lambda\right)$ such that $T_E^{\chi,\chi'} L_\kappa = L_{\lambda'}$. But then $\kappa = \varepsilon^{-1}\left(\lambda'\right) = \lambda$. □

Lemma 5.5. *Let χ and χ' be central characters with $\#\chi = \#\chi'$.*

(a) If $c_{\chi'}$ is a shift of c_χ, then for one of $E = L_{\varepsilon_1}$ or $E = L_{-\delta_n}$ the functor $T_E^{\chi,\chi'}$ is an equivalence of categories ;

(b) If $c_{\chi'}$ is a reflection of c_χ, then for one of $E = L_{2\varepsilon_1}$ or $E = L_{-\delta_{n-1}-\delta_n} \cong L_{2\varepsilon_1}^*$ the functor $T_E^{\chi,\chi'}$ is an equivalence of categories.

Proof. (a) Let $c_\chi = c = (a,b), c_{\chi'} = c' = (a',b')$. We consider the case $a' = (a \setminus \{t\}) \cup \{t+1\}$, $b' = b$. (All other variants of a shift can be done in the similar way.) In our case $E = L_{\varepsilon_1}$, and $P(E) = \{\varepsilon_1, \dots, \varepsilon_m, \delta_1, \dots, \delta_n\}$. It is sufficient to prove that χ, χ' and E satisfy the conditions (1) and (2) of proposition 5.4. Let $\lambda \in P_\chi^+$. Since $t \in a$ there is i such that $\langle\lambda, \varepsilon_i\rangle = t$. Then $\langle\lambda, \varepsilon_{i-1}\rangle \geq t+1$. First, consider the case when $\langle\lambda, \varepsilon_{i-1}\rangle > t+1$. Then $t+1 \notin a_\mu$ if $\mu = \lambda + \varepsilon_k\,(k \neq i)$ or $\mu = \lambda + \delta_j$, therefore $(\lambda + P(E)) \cap P_{\chi'} = \{\lambda + \varepsilon_i\}$. Thus, the condition (1) for λ in this case is true. Next, assume that $\langle\lambda, \varepsilon_{i-1}\rangle = t+1$. Since $t+1 \notin a$, there is j such that $\langle\lambda, \delta_j\rangle = t+1$. Then $(\lambda + P(E)) \cap P_{\chi'} = \{\lambda + \varepsilon_i, \lambda + \delta_j\}$. Obviously, $\lambda + \varepsilon_i \notin P^{\text{reg}}$. On the other hand, $\lambda + \delta_j \in P^+$, since $t \in a$ and therefore $\langle\lambda, \delta_l\rangle \neq t$ for any $l = 1, \dots, n$. Therefore the condition (1) holds. The condition (2) can be done in the same manner: just change in above argument t to $t+1, t+1$ to t and $i-1$ to $i+1$.

(b) We consider the case when $a' = (a \setminus \{t\}) \cup \{t+1\}$, $b' = (b \setminus \{t+1\}) \cup \{t\}$. (The other cases are completely similar). Take $E = L_{2\varepsilon_1}$. Then $P(E) = \{\varepsilon_i + \varepsilon_j | 1 \leq i \leq j \leq m\} \cup \{\varepsilon_i + \delta_j | 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{\delta_i + \delta_j | 1 \leq i < j \leq n\}$. Since $t \in a$, $t+1 \in b$, for any $\lambda \in P_\chi^+$ there are i and j such that $\langle\lambda, \varepsilon_i\rangle = t, \langle\lambda, \delta_j\rangle = t+1$ and $\langle\lambda, \varepsilon_l\rangle \neq t+1$ for $l = 1, \dots, m, \langle\lambda, \delta_l\rangle \neq t$ for $l = 1, \dots, n$. Then one can easily see that $(\lambda + P(E)) \cap P_{\chi'} = \{\lambda + \varepsilon_i + \delta_j\}$ and $\lambda + \varepsilon_i + \delta_j \in P^+$. Thus the condition (1) holds. Condition (2) can be checked in the same manner. □

Now lemmas 4.2 and 5.5 imply theorem 3.5.

6. Induction functor

Here we prove Theorem 3.7. Intuitively it is clear that one should use an induction from $\mathfrak{gl}(k|k) \subset \mathfrak{gl}(m|n)$ to establish an equivalence of categories $\mathcal{F}^k$ and $\mathcal{F}_\chi$. But a straightforward construction does not work. We have to cut both categories, prove an equivalence between cut pieces and then go to a limit.

Fix k, denote $\mathfrak{gl}(k|k)$ by $\mathfrak{g}'$. Put $p = m - k, q = n - k$. We consider $\mathfrak{g}'$ as a subalgebra in $\mathfrak{g}$ with the root system $\Delta' \subset \Delta$, where

$$\Delta' = \{\varepsilon_i - \varepsilon_j | i, j = p+1, \dots, m\} \cup \{\varepsilon_i - \delta_j | i = p+1, \dots, m, j = 1, \dots, k\} \cup \{\delta_i - \delta_j | i, j = 1, \dots k$$

We use notations $\mathcal{F}'$, $(\Delta^+)'$, $(P^+)'$, ρ', L'_λ e.t.c. for objects related to the Lie superalgebra $\mathfrak{g}'$.

Clearly, $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}'$ is a Cartan subalgebra of $\mathfrak{g}'$, and $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$, where $\mathfrak{h}''$ is the orthogonal complement to $\mathfrak{h}$. Note that the natural basis of $\mathfrak{h}'$ is $\{\varepsilon_{p+1}, \dots, \varepsilon_m, \delta_1, \dots, \delta_k\}$, and the natural basis of $\mathfrak{h}''$ is $\{\varepsilon_1, \dots, \varepsilon_p, \delta_{k+1}, \dots, \delta_n\}$, and $[\mathfrak{h}'', \mathfrak{g}'] = 0$. Let $\mathfrak{p} = \mathfrak{g}' + \mathfrak{b}$. Then $\mathfrak{p}$ is a parabolic subalgebra in $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{g}' \oplus \mathfrak{h}'' \oplus \mathfrak{r}$, where $\mathfrak{r} = \oplus_{\alpha \in \Delta^+ \setminus \Delta'} \mathfrak{g}_\alpha$. For any $M \in \operatorname{Ob} \mathcal{F}, \mu \in (\mathfrak{h}'')^*$ set $M^\mu = M \otimes_{\mathbb{C}} c''_\mu$, where c''_μ is the 1-dimensional $\mathfrak{h}''$-module defined by the character $\mu - \rho$. We define a $\mathfrak{p}$-module structure on M^μ by putting $\mathfrak{r} \cdot M^\mu = 0$. Then a generalized Verma module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^\mu$ has a unique maximal finite-dimensional quotient which we denote by $\operatorname{Ind}^\mu M$. Clearly one can consider Ind^μ as a functor from the category $\mathcal{F}'$ to the category $\mathcal{F}$. Geometrically $\operatorname{Ind}^\mu M$ can be defined as $\Gamma\left(\mathcal{L}_{(M^\mu)^*}\right)^*$ where $\mathcal{L}_N$ is a G-sheaf on G/P induced by P-module N.

On the other hand one can construct a functor $\operatorname{Inv}^\mu \colon \mathcal{F} \to \mathcal{F}'$ by putting $\operatorname{Inv}^\mu M = \{m \in M | \mathfrak{r} m = 0, hm = (\mu - \rho)(h) m \forall h \in \mathfrak{h}''\}$.

Lemma 6.1. *(a) For any $M \in \operatorname{Ob} \mathcal{F}', N \in \operatorname{Ob} \mathcal{F}$ there is a canonical isomorphism*

$$\operatorname{Hom}_{\mathfrak{g}}(Ind^\mu M, N) \simeq \operatorname{Hom}_{\mathfrak{g}'}(M, \operatorname{Inv}^\mu N);$$

(b) Inv^μ is exact on the left;

(c) Ind^μ is exact on the right;

(d) If M is indecomposable $\mathfrak{g}'$-module then $Ind^\mu M$ is indecomposable $\mathfrak{g}$-module.

Proof. To show (a) note that since N is finite dimensional

$$\operatorname{Hom}_{\mathfrak{g}}(\operatorname{Ind}^\mu M, N) \cong \operatorname{Hom}_{\mathfrak{g}}\left(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^\mu, N\right).$$

On the other hand by Frobenius reciprocity

$$\operatorname{Hom}_{\mathfrak{g}}\left(U(\mathfrak{g})\otimes_{U(\mathfrak{p})} M^{\mu}, N\right) \cong \operatorname{Hom}_{\mathfrak{p}}\left(M^{\mu}, N\right).$$

Finally by definition of M^{μ}

$$\operatorname{Hom}_{\mathfrak{p}}\left(M^{\mu}, N\right) \cong \operatorname{Hom}_{\mathfrak{g}'}\left(M, \operatorname{Inv}^{\mu} N\right).$$

Statements (b) and (c) are trivial.

Let us prove (d). Assume that M is an indecomposable $\mathfrak{g}'$-module and $\operatorname{Ind}^{\mu} M = N' \oplus N''$ with $N', N'' \neq 0$. As easily follows from (a)

$$\operatorname{Inv}^{\mu}\operatorname{Ind}^{\mu} M \simeq M \simeq \operatorname{Inv}^{\mu} N' \oplus \operatorname{Inv}^{\mu} N''.$$

This implies that $\operatorname{Inv}^{\mu} N' \simeq M, \operatorname{Inv}^{\mu} N'' = 0$. On the other hand M generates $\operatorname{Ind}^{\mu} M$, and therefore N' generates $\operatorname{Ind}^{\mu} M$, which implies $N'' = 0$. Contradiction. ☐

Next step is to construct subcategories in $\mathcal{F}$ and $\mathcal{F}^k$ such that the functors Ind and Inv give an equivalence of these subcategories.

Let us fix some central character χ with $\#\chi = k$, and

$$c_{\chi} = \left(\{a_1 > \cdots > a_p\}, \{b_1 < b_2 < \cdots < b_q\}\right).$$

Let

$$\mu = a_1\varepsilon_1 + \cdots + a_p\varepsilon_p - b_1\delta_{k+1} - \cdots - b_q\delta_n \in (\mathfrak{h}'')^*.$$

Let $\mathcal{F}^k(\mu)$ be the full subcategory of $\mathcal{F}^k$ consisting of $\mathfrak{g}'$-modules, irreducible subquotients L'_{λ} of which satisfy the condition $(\mu, \lambda) \in P^+$. In the same way let $\mathcal{F}_{\chi}(\mu)$ be the full subcategory of $\mathcal{F}_{\chi}$ consisting of modules whose irreducible subquotients have the form $L_{(\mu,\lambda)}$.

The next lemma is trivial and we leave the proof of it to the reader.

Lemma 6.2. *(a) Let* $L'_{\lambda} \in \operatorname{Ob}\mathcal{F}^k(\mu)$*, then there is a surjection* $\operatorname{Ind}^{\mu} L'_{\lambda} \to L_{(\mu,\lambda)}$*;*

(b) Let $L_{(\mu,\lambda)} \in \operatorname{Ob}\mathcal{F}_{\chi}(\mu)$*, then* $\operatorname{Inv}^{\mu} L_{(\mu,\lambda)} \simeq L'_{\lambda}$*;*

(c) If $M \in \operatorname{Ob}\mathcal{F}_{\chi}(\mu)$*, then* $\operatorname{Inv}^{\mu} M = M(\mu) = \{m \in M \mid hm = (\mu - \rho)(h)\, m \forall h \in \mathfrak{h}''\}$.

Now we are able to prove the following

Lemma 6.3. *The functors* Ind^{μ} *and* Inv^{μ} *establish an equivalence of the categories* $\mathcal{F}^k(\mu)$ *and* $\mathcal{F}_{\chi}(\mu)$*.*

Proof. Note that by Lemma 6.2 (c) Inv^{μ} is an exact functor on $\mathcal{F}_{\chi}(\mu)$. Therefore by Lemma 6.2 (b) it maps a module from $\mathcal{F}_{\chi}(\mu)$ to a module from $\mathcal{F}^k(\mu)$ and it is faithful on $\mathcal{F}_{\chi}(\mu)$. Furthermore by Lemma 6.1 (a) $N \simeq \operatorname{Ind}^{\mu}\operatorname{Inv}^{\mu} N$ for any $N \in \operatorname{Ob}\mathcal{F}_{\chi}(\mu)$.

On the other hand Lemma 6.1 (c) and Lemma 6.2 (a) imply that Ind^{μ} is faithful on $\mathcal{F}^k(\mu)$, and therefore by lemma 6.1 (a) $\operatorname{Inv}^{\mu}\operatorname{Ind}^{\mu} M \simeq M$ for any $M \in \operatorname{Ob}\mathcal{F}^k(\mu)$. Now to finish the proof we only have to show that $\mathrm{Ind}^{\mu} M \in \mathrm{Ob}\,\mathcal{F}_{\chi}(\mu)$ for any $M \in \mathrm{Ob}\,\mathcal{F}^{k}(\mu)$. Let M have irreducible subquotients $L'_{\lambda_1}, \dots, L'_{\lambda_r}$. Then $P(\mathrm{Ind}^{\mu} M \subseteq \cup_{i=1}^{r}(\lambda_i + P(U(\mathfrak{n}^-)))$. In particular, if L_{ν} is an irreducible subquotient of $\mathrm{Ind}^{\mu} M$, then $\nu = \lambda_s - \Sigma_{\alpha\in\Delta^+} m_{\alpha}\alpha$ for some $s \leq r$, $m_{\alpha} \in \mathbb{Z}_{\geq 0}$. Furthermore, Lemma 6.1 (d) and Lemma 6.2 (a) imply that $\chi_{\nu} = \chi$. One can easily check that the last two conditions together with $\nu \in P^+$ imply that $\nu = (\mu, \kappa)$ for some $\kappa \in (P')^+$. Thus we have proved the lemma. □

Now we can proceed to the proof of theorem 3.6. Note that $\mathcal{F}^k = \cup_{\mu\to\infty}\mathcal{F}^k(\mu)$. Therefore we can try to get an equivalence functor as a limit of Inv^{μ} as $\mu \to \infty$. Let us do it formally.

Let us fix a central character χ with $\#\chi = k$ and $c_{\chi} = (\{1, \dots, p\}, \{p+1, \dots, q\})$. Let us fix a sequence $\mu^{(n)}$ such that:

(1) $\mu^{(0)} = p\varepsilon_1 + \dots + \varepsilon_p - (p+1)\delta_{k+1} - \dots - (p+q)\delta_n$;
(2) $\mu^{(i)}$ is either $\mu^{(i-1)} + \varepsilon_s$ for some $s \in \{1, \dots, p\}$ or $\mu^{(i-1)} - \delta_s$ for some $s \in \{k+1, \dots, n\}$;
(3) $\mathrm{c}^{(i)}\left(a^{(i)}, b^{(i)}\right) = \left(\left\{\left\langle \mu^{(i)}, \varepsilon_s\right\rangle | 1 \leq s \leq p\right\}, \left\{\langle \mu, \delta_s\rangle | k+1 \leq s \leq n\right\}\right) \in M^{p,q}$;
(4) $\mathrm{a}^{(i)} \cup b^{(i)} = \{a, \dots, a+p+q+1\} \setminus \{b\}$ for some a and $b \in \mathbb{Z}$.

Note that each $c^{(i)}$ is obtained from $c^{(i-1)}$ by shifting. Let $\chi^{(i)}$ be the central character with $\#\chi^{(i)} = k, c_{\chi^{(i)}} = c^{(i)}$. Let $\mathcal{F}^{(i)} = \mathcal{F}_{\chi^{(i)}}$ and let $\Phi^{(i)} : \mathcal{F}^{(i)} \to \mathcal{F}^{(i+1)}$ be the functor $T_E^{\chi^{(i)},\chi^{(i+1)}}$ constructed in lemma 5.5 (a).

Lemma 6.4. *Let* $M \in \mathrm{Ob}\,\mathcal{F}^{(i)}\left(\mu^{(i)}\right)$. *Then* $\Phi^{(i)}(M) \in \mathrm{Ob}\,\mathcal{F}^{(i+1)}\left(\mu^{(i+1)}\right)$ *and* $\mathrm{Inv}^{\mu^{(i)}}(M) \simeq \mathrm{Inv}^{\mu^{(i+1)}}\Phi^{(i)}M$.

Proof. Since all functors in question are exact it suffices to check the first statement of lemma for an irreducible $M = L_{\left(\mu^{(i)},\lambda\right)}$. Indeed,

$$\Phi^{(i)}L_{\left(\mu^{(i)},\lambda\right)} = L_{\left(\mu^{(i+1)},\lambda\right)} \in \mathrm{Ob}\,\mathcal{F}^{(i+1)}(\mu),$$

and

$$\mathrm{Inv}^{\mu^{(i)}} L_{\left(\mu^{(i)},\lambda\right)} = \mathrm{Inv}^{\mu^{(i+1)}} L_{\left(\mu^{(i+1)},\lambda\right)} = L'_{\lambda}.$$

To show that the second statement of lemma is true consider $M \in \mathrm{Ob}\,\mathcal{F}^{(i)}\left(\mu^{(i)}\right)$. Recall that $\Phi^{(i)}M = T_E^{\chi^{(i)},\chi^{(i+1)}}M \subseteq (M \otimes E)$, where E is either the standard $\mathfrak{g}$-module or its dual. Therefore

$$\mathrm{Inv}^{\mu^{(i+1)}}\Phi^{(i)}M \subseteq (M \otimes E)\left(\mu^{(i+1)}\right).$$

We claim that $(M \otimes E)\left(\mu^{(i+1)}\right) \simeq \dim M\left(\mu^{(i)}\right)$ as $\mathfrak{g}'$-modules. Indeed, it is sufficient to check this claim for $M = L_{\left(\mu^{(i)},\lambda\right)}$. Assume that $\mu^{(i+1)} = \mu^{(i)} + \varepsilon_r$, then E is the standard module. (The case $\mu^{(i+1)} =$

$\mu^{(i)}-\delta_r$ and E is the dual to the standard module is completely similar). All weights of $M\otimes E$ belong to $P(M)+\{\varepsilon_1,\dots,\varepsilon_m,\delta_1,\dots,\delta_n\}$. A simple direct checking shows that for any $\nu\in P(M)$ one has $\mu+\varepsilon_s=\left(\mu^{(i+1)},\lambda\right)$ implies $s=r$ and $\nu=\left(\mu^{(i)},\lambda\right)$, and $\nu+\delta_s\neq\left(\mu^{(i+1)},\lambda\right)$. Thus $(M\otimes E)\left(\mu^{(i+1)}\right)=p_{\chi'}(M\otimes E)\left(\mu^{(i+1)}\right)\simeq M\left(\mu^{(i)}\right)$. □

Now we construct a functor $\Phi:\mathcal{F}_\chi\to\mathcal{F}'_0$ by putting

$$\Phi M=\mathrm{Inv}^{\mu^{(i)}}\circ\Phi^{(i-1)}\circ\dots\circ\Phi^{(0)}M$$

for sufficiently large i. More precisely one can choose i such that $\Phi^{(i-1)}\circ\dots\circ\Phi^{(0)}M\in\mathrm{Ob}\,\mathcal{F}^{(i)}\left(\mu^{(i)}\right)$. Lemma 6.4 implies that Φ is well defined. Lemma 6.3 together with $\cup_i\mathcal{F}^k\left(\mu^{(i)}\right)=\mathcal{F}^k$ imply that Φ gives an equivalence of categories.

7. Indecomposability of $\mathcal{F}_\chi$

What remains is to prove theorem 3.7. Theorem 3.6 implies that it is sufficient to prove indecomposability for $\mathcal{F}_{\chi_\rho}$ and $\mathfrak{g}=\mathfrak{g}l(n|n)$.

Lemma 7.1. *Let $\lambda\in P^+,\alpha\in\Delta_1$ such that $\langle\lambda,\alpha\rangle=0$ and $\lambda+\alpha\in P^+$. Then there is an indecomposable module which has subquotients isomorphic to L_λ and $L_{\lambda+\alpha}$.*

Proof. One can assume without lost of generality that $\alpha\in\Delta_1^-$. It was shown in [**?**] that the Kac module V_λ has subquotients L_λ and $L_{\lambda+\alpha}$. Since V_λ is indecomposable the statement is proved. □

Now theorem 3.7 will follow immediately from lemma 7.1 and the following simple combinatorial statement.

Lemma 7.2. *Let $\lambda,\mu\in P^+$ and $\chi_\lambda=\chi_\mu=\chi_{\rho'}$. Then one can find a sequence $\alpha_1,\dots,\alpha_s\in\Delta'_1$ such that $\langle\lambda+\alpha_1+\dots+\alpha_i,\alpha_{i+1}\rangle=0,\lambda+\alpha_1+\dots+\alpha_i\in(P^+)'$ for any $i<s$ and $\lambda+\alpha_1+\dots+\alpha_s=\mu$.*

Proof. Let $\|\lambda-\mu\|=\Sigma_{i=1}^k|\langle\lambda-\mu,\varepsilon_i\rangle|$. We prove the statement by induction on $\|\lambda-\mu\|$. Let us choose the minimal i such that $\langle\lambda,\varepsilon_i\rangle\neq\langle\mu,\varepsilon_i\rangle$. We have two cases

(1) If $\langle\lambda,\varepsilon_i\rangle<\langle\mu,\varepsilon_i\rangle$, then $\lambda'=\lambda+\varepsilon_i-\delta_i\in(P^+)'$ and $\langle\lambda,\varepsilon_i-\delta_i\rangle=0$. Since $\|\lambda'-\mu\|=\|\lambda-\mu\|-1$, the statement is true for λ' and μ. But then it is clearly true for λ and μ;
(2) If $\langle\lambda,\varepsilon_i\rangle>\langle\mu,\varepsilon_i\rangle$, then $\mu'=\mu+\varepsilon_i-\delta_i\in(P^+)'$ and $\langle\mu,\varepsilon_i-\delta_i\rangle=0$. Since $\|\lambda-\mu'\|=\|\lambda-\mu\|-1$, the statement is true for λ and μ'. But then it is clearly true for λ and μ.

□

8. Some applications

Let us apply theorem 3.6 to the problem of calculating of the character of L_λ. It was shown in [?] that the character can be written as the infinite linear combination:

$$\operatorname{ch} L_\lambda = \Sigma_{\mu\leq\lambda} a_{\lambda\mu} \operatorname{ch} V_\mu.$$

Here the coefficient $a_{\lambda\mu}$ is defined as the multiplicity of V_μ in a resolution of L_λ by Kac modules. Therefore if $a_{\lambda\mu} \neq 0$ then L_λ and L_μ should belong to the same block. Moreover, the coefficients are invariant with respect to an equivalence of blocks. Thus, by theorem 3.6 it is sufficient find the coefficients $a_{\lambda\mu}$ only for the most atypical block $\mathcal{F}_{\chi_\rho}$ for the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(n|n)$. This simplifies the problem combinatorially because the most atypical weight λ is defined only by n parameters. Indeed, $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n - \lambda_n\delta_1 - \cdots - \lambda_1\delta_n$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$ and $\lambda_1 > \cdots > \lambda_n$. Denote by Q the set of all such $(\lambda_1, \ldots, \lambda_n)$. The algorithm for calculating $a_{\lambda\mu}$ defined in [?] can be reformulated in the following way.

We identify λ with $(\lambda_1, \ldots, \lambda_n)$ and consider a free $\mathbb{Z}[q]$-module $\mathcal{H}$ with basis $\lambda \in Q$. Let $d_i(\lambda_1, \ldots, \lambda_n) = (\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_n)$ and $[f(q)]_+ = f(q) - f(0)$ for any $f(q) \in \mathbb{Z}[q]$. Define $\mathbb{Z}[q]$-linear operators $S_1, \ldots, S_n$ on $\mathcal{H}$ by the following recurrent relation:

$$S_n(\lambda) = qd_n(\lambda);$$
$$S_i(\lambda) = qd_i(\lambda) + \left[q^{-1}S_i(d_i(\lambda))\right]_+ \text{ if } d_i(\lambda) \in M;$$
$$S_i(\lambda) = qd_i(S_{i-1}(\lambda)) \text{ if } d_i(\lambda) \notin M.$$

Let $S = \Pi_{i=1}^n (1 + S_i)^{-1}$ and $s_{\lambda,\mu}(q) \in \mathbb{Z}[q]$ be the matrix coefficients of the operator S in the basis $\{\lambda\}$. Then $a_{\lambda,\mu} = s_{\lambda,\mu}(-1)$.

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