Lect 3

Cardinality of Sets

Definition: Two sets A and B have the same cardinality if there is one-to-one correspondence from A to B.

Countable set is a finite set or the set of the same cardinality as $\mathbb{Z}^+$ (the set of positive integers).

Example The set S of all odd positive integers is countable

$f: \mathbb{Z}^+ \to S$ given by $f(n) = 2n+1$.

The set of all integers is countable

$0, 1, -1, 2, -2, 3, -3, \ldots$.

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{(n-1)}{2} & \text{if } n \text{ is odd} \end{cases}$$

<u>Hilbert's Grand Hotel</u>: Countable number of rooms.

All are occupied. A new guest arrive.

Move a guest from room # n to room # $n+1$ and move the new guest into the room # 1.

<u>Def.</u> We say that cardinality of A is less than (or the same as) cardinality of B (notations $|A| \le |B|$) if there is a one-to-one function

$f: A \to B$.

The set of positive rational numbers is countable.

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} \quad \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4}
\end{array}$$

$\left\{ \frac{p}{q} \mid p+q = n \right\}$

Finitely many rational numbers

$\bigcup_{i=1}^{\infty} A_i$ if A_i are finite is countable.

A subset of countable set is countable.

<u>Further results:</u> If A and B are countable, then $A \cup B$ is countable.

$A \cup B = A \cup (B-A)$

Suppose A is finite, $(B-A)$ is finite, then $|A \cup (B-A)| = |A| + |B-A|$ is again finite.

Suppose A is infinite countable, $B-A$ is finite

$B-A = \{b_1, b_2, b_3, \ldots, b_n\}$

$A = \{a_1, a_2, \ldots, a_n, \ldots$

$$\underset{\substack{\uparrow \\ 1}}{b_1}, \ldots, \underset{\substack{\uparrow \\ n}}{b_n}, \underset{\substack{\uparrow \\ n+1}}{a_1}, \underset{n+2}{a_2}$$

$$f(k) = \begin{cases} b_k & \text{if } k \leq n \\ a_{k-1} & \text{if } k > n \end{cases}$$

Finally, let A and $B-A$ are infinite countable

$A = \{a_1, a_2, a_3, \ldots\}$

$B = \{b_1, b_2, b_3, \ldots\}$

$a_1, b_1, a_2, b_2, a_3, b_3, \ldots$

$$f(n) = \begin{cases} b_{\frac{n}{2}} & \text{if } n \text{ is even} \\ a_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

Schröder - Bernstein theorem. If A and B are sets and $|A| \leq |B|$, $|B| \leq |A|$. Then $|A| = |B|$.

Proof Too difficult. Not in this class.

Examples: Show that $|(0,1)| = |[0,1]|$

$f : (0,1) \longrightarrow [0,1]$ $\quad f(x) = x$

$g : [0,1] \longrightarrow (0,1)$ $\quad f(x) = \frac{1}{3}x + \frac{1}{3}$

both mappings are one-to-one

$|(0,1)| \leq [0,1]$

$|[0,1]| \leq (0,1)$

Hence $|(0,1)| = |[0,1]|$

An example of uncountable set: $\mathbb{R}$ is uncountable.

Sufficient to show that $(0,1)$ is uncountable. Assume the opposite

$r_1 = 0.\, d_{11}\, d_{12}\, d_{13}\, d_{14} \ldots$

$r_2 = 0.\, d_{21}\, d_{22}\, d_{23}\, d_{24} \ldots$

$r_3 = 0.\, d_{31}\, d_{32}\, d_{33}\, d_{34} \ldots$

$r = 0.\, d_1\, d_2\, d_3\, d_4 \ldots$

$$d_i = \begin{cases} 1 & \text{if } d_{ii} \neq 1 \\ 2 & \text{if } d_{ii} = 2 \end{cases}$$

Assume $r = r_i$ for some i. But then $d_i = d_{ii}$, contradiction.

Let S be a set. Then $|S| \leq |\mathcal{P}(S)|$ and $|S| \neq |\mathcal{P}(S)|$.

<u>Proof.</u> $f: S \to \mathcal{P}(S)$ defined by $f(x) = \{x\}$. f is one-to-one

Hence $|S| \leq |\mathcal{P}(S)|$

Suppose that there is a bijection $g: S \to \mathcal{P}(S)$

Let us consider the subset $A \subseteq S$ defined by

$A = \{x \in S \mid x \notin g(x)\}$, $\bar{A} = \{x \in S \mid x \in g(x)\}$

Then we claim that there is no $x \in S$ such that $g(x) = A$.

Indeed, if $x \in A$, then $g(x) \neq A$ since $x \notin g(x)$.

if $x \notin A$, ~~then $g(x) \neq A$, because otherwise~~

~~$g(x) = A$~~ then $x \in \bar{A}$, then $x \in g(x) = A$. Contradiction

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<u>Uncomputable functions</u>: A function is computable if there is a program computing it.

<u>Theorem.</u> $\exists$ an uncomputable function.

<u>Proof.</u> The set of all programs is countable.
A program is a string of symbols from finite alphabet.

$|A| \leq 10^n$ $\quad f: \{a_1, \dots, a_m\} \longmapsto a_1 + 10^n a_2 + 10^{2n} a_3 + \dots + 10^{(m-1)n} a_m$

f: Strings $\longrightarrow \mathbb{Z}^+$ one to one.

The set of function $\varphi: \mathbb{Z}^+ \longrightarrow \{0,1\}$ is uncountable.
Indeed, the cardinality of this set is the same as $\mathcal{P}(\mathbb{Z}^+)$.

<u>Def.</u> <u>Continuum</u>. is the cardinality of $\mathbb{R}$ (the same as the cardinality of $\mathcal{P}(\mathbb{Z}^+)$.

<u>Continuum hypothesis</u>: There is no set with cardinality between countable and continuum, i.e. $|\mathbb{Z}^+| < |S| < |\mathbb{R}|$.

Theorem. For any set S, $|S| < |\mathcal{P}(S)|$.

Proof. 1. $|S| \leq |\mathcal{P}(S)|$. Indeed define $f : S \to \mathcal{P}(S)$ by $f(x) = \{x\}$. Then f is one-to-one.

2. $|S| \neq |\mathcal{P}(S)|$. By contradiction. Assume that there exists a bijection $f : S \to \mathcal{P}(S)$

Define $A = \{x \in S \mid x \notin f(x)\}$

If f is a bijection, then there exist $a \in S$ such that $f(a) = A$. If $a \in A$ then $A \neq f(a)$

If $a \notin A$ then $A \neq f(a)$ because $\bar{A} = \{x \in S \mid x \in f(x)\}$. Contradiction.

Continuum $|\mathcal{P}(\mathbb{Z}^+)| = |\mathbb{R}| = \mathfrak{c}$

Continuum hypothesis: There is no set S such that $|\mathbb{Z}^+| < |S| < |\mathbb{R}|$

Algorithm is a finite sequence of precise instructions for solving a problem.

- Input
- Output
- Definitness
- Correctness
- Finiteness
- Generality

comments

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Computer problem in pseudocode

procedure min ($a_1, a_2, \ldots, a_n$: integers)

```
min := a1
for i := 2 to n
        if min > ai , then min = ai

return min
```

Search (linear)

procedure linear search (x: integer, $a_1, \ldots, a_n$: dist. integers)

$i := 1$

while ($i \leq n$ and $x \neq a_i$) $i := i + 1$

if $i \leq n$ then location $:= i$

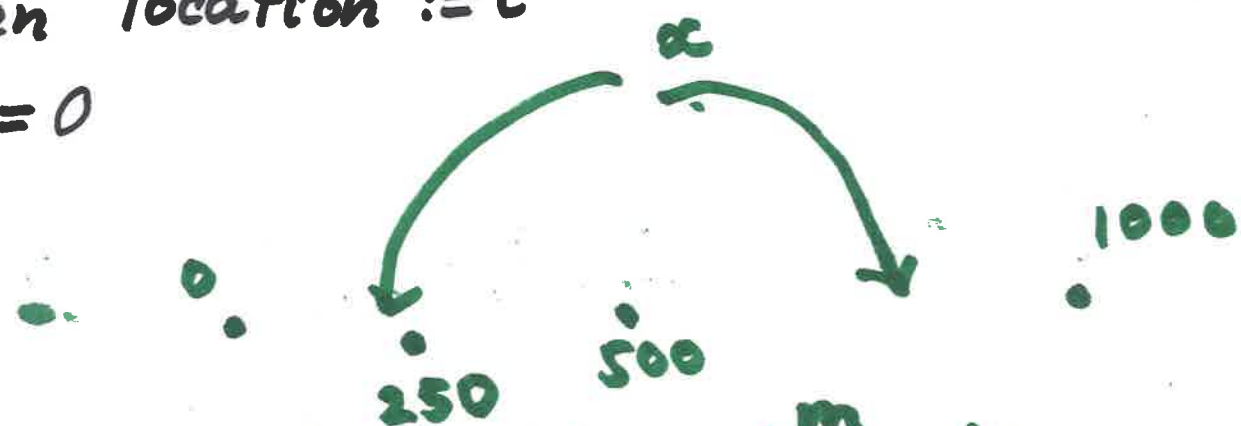
else location $:= 0$

return location

Binary search

procedure search (x: integer, $a_1, \ldots, a_n$ increasing integers)

$i := 1$
$j := n$
while $i < j$
$m := \lfloor (i+j)/2 \rfloor$
if $x > a_m$ then $i := m+1$
else $j := m$
if $x = a_i$ then location $:= i$
else location $:= 0$
return location



You find m such that $2^m = n$

$m = \lceil \log_2 n \rceil$.

The bubble sorting

procedure sort ($a_1, \ldots, a_n$: real numbers , $n \geq 2$)

for $i := 1$ to $n-1$

for $j := 1$ to $n-i$

if $a_j > a_{j+1}$ then interchange a_j and a_{j+1}

return $\{a_1, \ldots, a_n\}$

Given a finite sequence of real numbers, write them in increasing order

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3, 2, 5, 1 2, 3, 5, 1 2, 3, 1, 5

2, 3, 1, 5 2, 1, 3, 5 1, 2, 3, 5

Insertion sorting

procedure isort ($a_1, \ldots, a_n$: real number, $n \geq 2$)

```
for j := 2 to n
    i := 1 to j-1
  while a_j > a_i
        i := i+1

    m := a_j
      for k := 0 to j-i-1
      a_{j-k} := a_{j-k-1}
      a_i = m
  return {a_1, ..., a_n}
```

3, 2, 5, 1

2, 3, 5, 1

1, 2, 3, 5

Greedy algorithm

change $(c_1 > c_2 > \dots > c_r$: values of coins,
n positive integers

```
for i := 1 to r
    d_i := 0
    while n ≥ c_i
        d_i := d_i + 1
        n := n - c_i
return {d_1, ..., d_r}
```

$r = 4$
$c_1 = 25$
$c_2 = 10$
$c_3 = 5$
$c_4 = 1$

Example $25 > 10 > 5 > 1$ works

Minimal number of coins
- at most 2 dimes
- at most one nickel
- at most four pennies

- can not have two dimes and nickel
- dimes, nickels, pennies ≤ 24

Let n has as few coins as possible

$n-25$ as well

$n \leq 24$

Does not work $25 > 10 > 1$

For 30 cents greedy algorithm gives

$30 = 25 + 1 + 1 + 1 + 1 + 1$ (6 coins)

$30 = 10 + 10 + 10$ (better)

Computable functions is a function whose values can be computed by computer programs.

- There exist uncomputable functions

There are uncountaby many functions $f: \mathbb{Z}^+ \to \{0,1\}$

\# of computer programs is infinite countable

The halting problem (Turing)

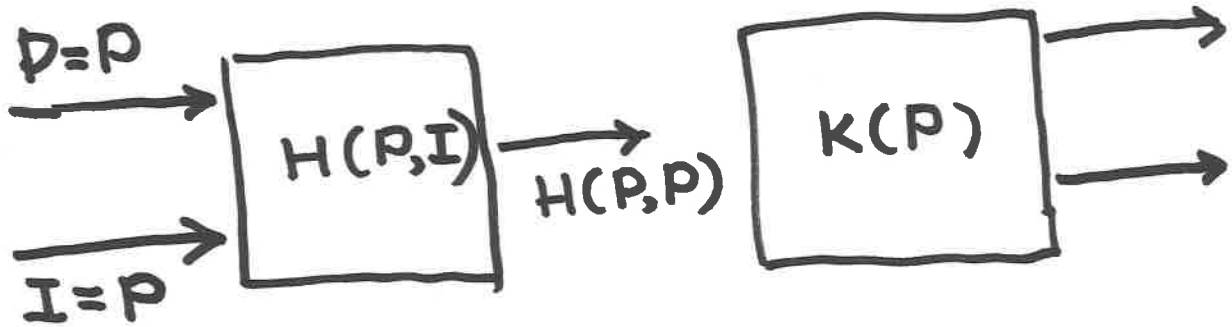
$H(P, I)$

P - program
I - input

halt

loops forever

H does not exist



H(P,P) halts
K(P) loops forever

H(P,P) loops forever
K(P) halts

Contradiction: K(K) halts
loops forever