The set of all even integers is countable.

\[ f(n) = \begin{cases} \frac{n-1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases} \]

Given by \( f(n) = \frac{n}{2} + 1 \).

Example: The set of all positive even integers is countable.

\[ \mathbb{Z^+} \subseteq \mathbb{R} \]

The set of positive even integers is a finite set of the set of the same cardinality.

If there is one-to-one correspondence from \( A \to B \), then \( |A| = |B| \).

Definition: Two sets \( A \) and \( B \) have the same cardinality.
A set of countable set.

\[ A \cup \bigcup_{i=1}^{\infty} A_i \]

Finite union of countable sets is countable.

For every natural number \( n \)

\[ \{ \frac{1}{n} \} \]

The set of positive rational numbers is countable.

\[ f: A \leftrightarrow B \]

\[(A \subseteq B) \lor (B \subseteq A) \]

If there is a one-to-one function of \( A \) is less than cardinality of \( B \).

We say that cardinality of \( A \) is equal to the same as \( B \).

The new guest into the room.

Move a guest from room \( n \) to room \( n+1 \) and move.

We are successful. A new guest arrive.

Heller's equation: Countable numbers of room.
Suppose $A$ is finite, is again finite.

If $A$ and $B$ are countable, then $A \cup B$ is countable.

Further results: If $A$ and $B$ are countable, then $A \cup B = A \cup (B - A)$ is countable.

$A = \{ a_1, a_2, \ldots \}$
$B = \{ b_1, b_2, b_3, \ldots \}$
$\mathbb{N} = \{ 0, 1, 2, 3, \ldots \}$
An example of uncountable set: \( \mathbb{R} \) is uncountable. Assume the opposite.

\[
\exists x, y \in \mathbb{R} \text{ such that } x < y \quad \text{and} \quad x + y = 1
\]

Since \( \mathbb{R} \) is uncountable, there must exist \( \exists z \in \mathbb{R} \) such that \( x, y, z \) are one-to-one.

We prove the following:

\[
\mathbb{R} \neq [0,1]
\]

We prove that \( \exists x \in [0,1] \) such that \( f(x) = x \).

Let \( f(x) = x^2 \) for \( x \in [0,1] \). Then \( f([0,1]) \mapsto [0,1] \).

Show that \( f([0,1]) \) is not open.

Proof: It is not difficult. Note the proof above:

\[
|A| > |B| \implies |A| > |B|
\]

Schöder-Bernstein Theorem. If \( A \) and \( B \) are sets and \( |A| > |B| \),
There is a program computing $f$. A function is computable if

\[ g(x) = \begin{cases} A \quad & \text{if } x \in A, \text{ then } g(x) = A. \\ \varnothing \quad & \text{if } x \notin A, \text{ then } g(x) = \varnothing. \end{cases} \]

Indeed, if $x \in A$, then $g(x) = A$ since $x \in g(x)$.

Then we claim that there is no $x \in S$ such that

\[ A = \{ x \in S \mid x \notin g(x) \}. \]

Let us consider the subset $A \subseteq S$ defined by

Suppose that there is a bijection $g : S \leftrightarrow \{ 0 \}$.

Hence $|S| = |\{ 0 \}|$.

\[ (S \smallsetminus \{ 0 \}) \neq \emptyset \quad \text{and} \quad |S| \geq |S| \Rightarrow |S| \geq |(S \smallsetminus \{ 0 \})| \quad \text{and} \quad |(S \smallsetminus \{ 0 \})| \neq 0. \]

Let $S$ be a set. Then

Proof: $f : S \rightarrow \{ 0 \}$. \[ \]
between countable and continuum, i.e. $1 \leq |\mathbb{Z}^+| \leq |\mathbb{R}|$.

**Continuum Hypothesis**: There is no set with cardinality

of $(\mathbb{Z}^+, \mathcal{P}(\mathbb{Z}^+))$.

**Continuum is the cardinality of $\mathbb{R}$** (the same as the cardinality of $\mathbb{Q}$) - the continuum of $\mathbb{R}$ is the same as the size of $\mathbb{R}$.

Indeed, the cardinality of this set is the same as $(\mathbb{Z}^+, \mathcal{P}(\mathbb{Z}^+))$ as uncountable.

The set of function $f: \mathbb{Z} \to \mathbb{Z}$ is one to one.

A program is a string of symbols from finite alphabet.

**Proof**: The set of all programs is countable.

In an uncomputable function.

**Theorem**
Continuum Hypothesis: There is no set $S$ such that $1 < |S| < |\mathbb{R}|$.

Define $A = \{ x \in S | f(x) \}$.

If $A \neq A$, then $A \neq f(a)$ because $A = A$.

If $f$ is a bijection, then there exists an $a$ such that $f(a) = A$.

Define $A = \{ x \in S | x \neq f(x) \}$.

If there exists a bijection $f : S \rightarrow \mathbb{R}$, then $f$ is one-to-one.

Let $f(x) = f(x)$. Then $f$ is one-to-one.

Indeed define $f : S \rightarrow \mathbb{R}$.

For any set $S$, $|S| > |\mathbb{R}|$. Proof: $f(\mathbb{R}) = \mathbb{R}$, then $f$ is one-to-one.
Algorithm is a finite sequence of instructions for solving a problem.

procedure min (a1, a2, ..., an : integers) {
    min := a1
    for i := 2 to n {
        if a[i] < min then min := a[i]
    }
    return min
}

Comments

General: Generality
Finiteness
Coherency
Definition
Output
Input

Computer problem in pseudocode
procedure Linear search (x:integer, a[1..n] : array)

q := 0
if x in a[1..n] then
  q := location
else
  q := 0

where (\exists i \in \mathbb{N} \text{ and } x \neq a[i]) \ni i = q + 1

q

else

"dist. integer"

3
You find in such that

\[ m = \lceil \log_2 n \rceil \]

2

m = \lceil \log_2 n \rceil

[2, 50, 100]

\[ \text{procedure search} (x; \text{integers}, a_1, \ldots, \text{an in sequence}) \]

\[ \text{if } x = a_i \text{ then location := } i \]

\[ \text{else location := 0} \]

\[ \text{while } i < j \]

\[ m := \frac{i + j}{2} \]

\[ \text{if } x \geq a_m \text{ then } j := m \]

\[ \text{else } i := m + 1 \]

\[ \text{return location} \]
Given a finite sequence of real numbers, write them in increasing order.

```
for i := 1 to n-
  for j := i+1 to n
    if a[i] > a[j] then interc_echange a[i] and a[j]
```

The bubble sorting procedure sort (a_1, ... , a_n : real numbers, n > 2)
```
procedure insert_sort (a : array (1..n) of real, n : real number, n > 2);

q := 1;
while q > n do
    i := i + 1
    for k := q to n - i do
        if a[k] < a[k - 1] then
            t := a[k]
            a[k] := a[k - 1]
            a[k - 1] := t
        end
    end
end
```
Example

Minimal number of coins at most 2 dimes

\[ 25 > 10 > 5 > 1 \]

works

```
return false, ..., d = 3
```

\[ r = 2 \]

```
for i = 1 to 2
    d[i] = 0
    n[i] = n - c[i]

n[2] = 25
n[1] = 10
n[0] = 1
```

\[ h = 2 \]

Greedy algorithm

change (c_1 < c_2 < ... < c_r : values in positive integers to coins)
So = 10 + 10 + 10 (better)
30 = 25 + 1 + 1 + 1 + 1 + 1 (6 coins)

For 30 cans greedy algorithm gives

Does not work

n = 25

n > 25 as well.

Let n have as few coins as possible
There are uncomputably many functions. There exist uncomputable functions. Computer programs whose values can be computed by a function are called computable functions.
The halting problem (Turing)

Input: p - program

H(p, I)