PROBLEM SET # 6
MATH 252

Due October 19.

In this assignment G is a compact group and all representations are over $\mathbb{C}$.

1. Let H be a closed subgroup of G and $\rho : H \to U(V)$ is a unitary representation of H. We define the induced representation $\tilde{\rho} = \mathrm{Ind}_H^G \rho$ as follows. Let $\tilde{V}$ be the space of functions $f : G \to V$ satisfying

- $f(hg) = \rho_h f(g)$ for all $g \in G$ and $h \in H$;
- $\int_G (f(g)|f(g))dg$ is finite.

Furthemore, for all $f \in \tilde{V}$, $g, x \in G$ set

$$\tilde{\rho}_g f(x) = f(xg).$$

(a) Define the hermitian product on $\tilde{V}$ such that $\tilde{\rho}$ is a unitary representation of G.

(b) Prove Frobenius reciprocity. If $\sigma : G \to U(W)$ is a unitary representation of G. Then

$$\mathrm{Hom}_G(\tilde{V}, W) \simeq \mathrm{Hom}_H(V, W).$$

2. Consider the representation of $G \times G$ in $L^2(G)$ given by

$$T_{(g_1,g_2)} f(x) = f(g_1^{-1} x g_2).$$

(a) Check that T is a unitary representation.

(b) Show that $L^2(G)$ contains a dense $G \times G$-invariant subspace isomorphic to

$$\bigoplus_{\rho \in \hat{G}} \rho^* \boxtimes \rho.$$

3. Let $G = SU_n$ and T be the subgroup of diagonal matrices in G.

(a) Check that T is an $n-1$ dimensional torus and identify T with the set

$$\{(z_1, \dots, z_n) \in \mathbb{C}^n \,|\, |z_1| = \dots = |z_n| = 1,\ z_1 \dots z_n = 1\}.$$

(b) Let $\mathcal{C}_c(G)$ denote the space of continuous class function on G, $\mathcal{F}_c(T)$ be the space of continuous functions on T and $r : \mathcal{C}_c(G) \to \mathcal{F}_c(T)$ be the restriction map. Check that r is injective and the image coincides with the subspace of symmetric functions on T, i.e. functions f satisfying

$$f(z_1, \dots, z_n) = f(z_{s(1)}, \dots, z_{s(n)})$$

for any permutation of s.

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(c) Let $\mathcal{R}(G)$ be the set of all linear combinations of irreducible characters of G. Check that $\mathcal{R}(G)$ is a subring of $\mathcal{C}_c(G)$.

(d) Let $\mathbb{C}^{S_n}[z_1, \ldots, z_n]$ denote the ring of symmetric polynomials. Prove that the restriction of r to $\mathcal{R}(G)$ establishes an isomorphism

$$\mathcal{R}(G) \simeq \mathbb{C}^{S_n}[z_1, \ldots, z_n]/(z_1 \ldots z_n - 1).$$

Hint: check that elementary symmetric polynomials are characters of some representations.