REPRESENTATION THEORY.
WEEK 8

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1. Representations of $\mathrm{SL}_2(\mathbb{R})$

In this section
$$G = \mathrm{SL}_2(\mathbb{R}) = \{g \in \mathrm{GL}_2(\mathbb{R}) \mid \det g = 1\}.$$
Let K be the subgroup of matrices
$$g_\theta = \begin{pmatrix} \cos\ \theta & \sin\ \theta \\ -\sin\ \theta & \cos\ \theta \end{pmatrix}.$$
The group K is a maximal compact subgroup of G, clearly K is isomorphic to S^1. If $\rho \colon G \to \mathrm{GL}(V)$ is a unitary representation of G in a Hilbert space then then $\mathrm{Res}_K\, \rho$ splits into the sum of 1-dimensional representations of V. In particular, one can find $v \in V$ such that $\rho_{g_\theta}(v) = e^{in\theta}v$. Define the matrix coefficient function $f : G \to \mathbb{C}$ given by
$$f(g) = \langle v, \rho_g v\rangle.$$
Then f satisfies the condition
$$f(gg_\theta) = e^{in\theta} f(g).$$
Thus, one can consider f as a section of a linear bundle on the space G/K (if $n = 0$, then f is a function). Thus, it is clear that the space G/K is an important geometric object, where the representations of G are "realized".

Consider the Lobachevsky plane
$$H = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$$
with metric defined by the formula $\frac{dx^2+dy^2}{y^2}$ and the volume form $\frac{dxdy}{y^2}$. Then G coincides with the group of rigid motions of H preserving orientation. The action of G on H is given by the formula
$$z \mapsto \frac{az+b}{cz+d}.$$
One can check easily that G acts transitively on H, preserves the metric and volume. Moreover, the stabilizer of $i \in H$ coincides with K. Thus, we identify H with G/K.

The first series of representations we describe is called the representations of *discrete series*. Those are the representations with matrix coefficients in $L^2(G)$. Let

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$\mathcal{H}_n^+$ be the space of holomorphic densities on H, the expressions $\varphi(z)(dz)^{n/2}$, where $\varphi(z)$ is a holomorphic function on H satisfying the condition that

$$\int |\varphi|^2 y^{n-2} dz d\bar{z}$$

is finite. Define the representation of G in $\mathcal{H}_n^+$ by

$$\rho_g\left(\varphi(z)(dz)^{n/2}\right) = \varphi\left(\frac{az+b}{cz+d}\right)\frac{1}{(cz+d)^n}(dz)^{n/2},$$

and Hermitian product on $\mathcal{H}_n$ the formula

$$\left\langle \varphi(dz)^{n/2}, \psi(dz)^{n/2}\right\rangle = \int \bar{\varphi}\psi y^{n-2} dz d\bar{z}, \tag{1.1}$$

for $n > 1$. For $n = 1$ the product is defined by

$$\left\langle \varphi(dz)^{n/2}, \psi(dz)^{n/2}\right\rangle = \int_{-\infty}^{\infty} \bar{\varphi}\psi dx, \tag{1.2}$$

in this case $\mathcal{H}_1^+$ consists of all densities which converge to L^2-functions on the boundary (real line). Check that this Hermitian product is invariant.

Let us show that $\mathcal{H}_n$ is irreducible. It is convenient to consider Poincaré model of Lobachevsky plane using the conformal map

$$w = \frac{z-i}{z+i},$$

that maps H to a unit disk $|w| < 1$. Then the group G acts on the unit disk by linear-fractional maps $w \to \frac{aw+b}{bw+\bar{a}}$ for all complex a, b

satisfying $|a|^2 - |b|^2 = 1$, and K is defined by the condition $b = 0$. If $a = e^{i\theta}$, then $\rho_{g_\theta}(w) = e^{2i\theta}w$. The invariant volume form is $\frac{dwd\bar{w}}{1-\bar{w}w}$.

It is clear that $w^k(dw)^{n/2}$ for all $k \geq 0$ form an orthogonal basis in $\mathcal{H}_n^+$, each vector $w^k(dw)^{n/2}$ is an eigen vector with respect to K, namely

$$\rho_{g_\theta}\left(w^k(dw)^{n/2}\right) = e^{(2k+n)i\theta}w^k(dw)^{n/2}.$$

It is easy to check now that $\mathcal{H}_n^+$ is irreducible. Indeed, every invariant closed subspace V has a topological basis consisting of eigenvectors of K, in other words $w^k(dw)^{n/2}$ for some positive k must form a topological basis of V. Without loss of generality assume that V contains $(dw)^{n/2}$, then by applying ρ_g one can get that $\frac{1}{(bw+a)^n}(dw)^{n/2}$, and in Taylor series for $\frac{1}{(bw+a)^n}$ all elements of the basis appear with non-zero coefficients. That implies $w^k(dw)^{n/2} \in V$ for all $k \geq 0$, hence $V = \mathcal{H}_n^+$.

One can construct another series of representations $\mathcal{H}_n^-$ by considering holomorphic densities in the lower half-plane $\operatorname{Im} z < 0$.

Principal series. These representations are parameterized by a continuous parameter $s \in \mathbb{R}i\,(s \neq 0)$. Consider now the action of G on a real line by linear fractional transformations $x \mapsto \frac{ax+b}{cx+d}$. Let $\mathcal{P}_s^+$ denotes the space of densities $\varphi(x)(dx)^{\frac{1+s}{2}}$ with G-action given by

$$\rho_g\left(\varphi(x)(dx)^{\frac{1+s}{2}}\right) = \varphi\left(\frac{ax+b}{cx+d}\right)|cx+d|^{-s-1}(dx)^{\frac{1+s}{2}}.$$

The Hermitian product given by

$$\langle\varphi,\psi\rangle = \int_{-\infty}^{\infty}\bar{\varphi}\psi dx \tag{1.3}$$

is invariant. The property of invariance justify the choice of weight for the density as $(dx)^{\frac{1+s}{2}}(dx)^{\frac{1+\bar{s}}{2}} = dx$, thus the integration is invariant. To check that the representation is irreducible one can move the real line to the unit circle as in the example of discrete series and then use $e^{ik\theta}(d\theta)^{\frac{1+s}{2}}$ as an orthonormal basis in $\mathcal{P}_s^+$. Note that the eigen values of ρ_{g_θ} in this case are $e^{2ki\theta}$ for all integer k.

The second principal series $\mathcal{P}_s^-$ can be obtained if instead of densities we consider the pseudo densities which are transformed by the law

$$\rho_g\left(\varphi(x)(dx)^{\frac{1+s}{2}}\right) = \varphi\left(\frac{ax+b}{cx+d}\right)|cx+d|^{-s-1}\operatorname{sgn}(cx+d)\,dx^{\frac{1+s}{2}}.$$

Complementary series. Those are representations which do not appear in the regular representation $L^2(G)$. They can be realized as the representations in $\mathcal{C}_s$ of all densities $\varphi(x)(dx)^{\frac{1+s}{2}}$ for real $0 < s < 1$. An invariant Hermitian product is

$$\langle\varphi,\psi\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{\varphi}(x)\psi(y)|x-y|^{s-1}dxdy. \tag{1.4}$$

2. Semisimple modules and density theorem

We assume that R is a unital ring. Recall that an R-module is *semi-simple* if for any submodule $N \subset M$ there exists a submodule N' such that $M = N \oplus N'$ and R-module M is *simple* if any submodule of M is either M or 0.

Lemma 2.1. *Every submodule and every quotient of a semisimple module is semisimple.*

Proof. If Let N be a submodule of a semisimple module M, and let P be a submodule of N. Since $M = P \oplus P'$, then there exists an R-invariant projector $p : M \to P$. The restriction of p to N defines the projector $N \to P$. □

Lemma 2.2. *Any semisimple R-module contains a simple submodule.*

Proof. Let M be semisimple, $m \in M$. Let N be a maximal submodule in Rm which does not contain m (exists by Zorn's lemma, take all submodules which do not contain m). Then Rm is semisimple and $Rm = N \oplus N'$. We claim that N' is simple. Indeed, if N' is not simple, then it contains a proper submodule P. But $m \notin P \oplus N$, since $P \oplus N \neq Rm$. That contradicts maximality of N. □

Lemma 2.3. *The following conditions on a module M are equivalent*

1. *M is semisimple;*
2. *$M=\sum_{i\in I}M_i$ for some simple submodules M_i;*
3. *$M=\oplus_{j\in J}M_j$ for some simple submodules M_j.*

Proof. (1) $\Rightarrow$ (2) Let $\{M_i\}_{i\in I}$ be the collection of all simple submodules. Let $N=\sum_{i\in I}M_i$, assume that $N\neq M$, then $M=N\oplus N'$ and N' contains a simple submodule. Contradiction.

To prove (2) $\Rightarrow$ (3) let $J\subset I$ be minimal such that $M=\sum_{j\in J}M_j$ (check that it exists by Zorn's lemma). By minimality of J for any $k\in J$, M_k does not belong to $\sum_{j\in J-k}M_j$. Therefore $M=\oplus_{j\in J}M_j$.

Finally, let us prove (3) $\Rightarrow$ (1). Let $N\subset M$ be a submodule and $S\subset J$ be a maximal subset such that $N\cap\oplus_{j\in S}M_j=0$. Let $M'=N\oplus(\oplus_{j\in S}M_j)$. We claim that $M'=M$. Indeed, assume that the statement is false. Then there exists k such that $M_k\cap M'=0$. But then $N\cap\oplus_{j\in S+k}M_j=0$. Contradiction. □

Lemma 2.4. *Let M be a semisimple module. Then M is simple iff $\mathrm{End}_R(M)$ is a division ring.*

Proof. In one direction this is Shur's lemma. In the opposite direction if $M=M_1\oplus M_2$, then the projectors p_1, p_2 satisfy $p_1p_2=0$ and therefore p_1,p_2 are not invertible. □

Lemma 2.5. *Let $\mathrm{End}_R(M)=K$, $\mathrm{End}_K(M)=S$. Then $\widehat{K}=\mathrm{End}_R(M^{\oplus n})\cong\mathrm{Mat}_n(K)$ and $\mathrm{End}_{\widehat{K}}(M^{\oplus n})\cong S$, the last isomorphism is given by the diagonal action*

$$s(v_1,\dots,v_n)=(sv_1,\dots,sv_n).$$

Proof. See similar statement in lecture notes 3. □

Theorem 2.6. *(Jacobson-Chevalley density theorem). Let M be a semisimple R-module, $K=\mathrm{End}_R(M)$, $S=\mathrm{End}_K(M)$. Then for any $v_1,\dots,v_n\in M$ and $X\in S$ there exists $r\in R$ such that $rv_i=Xv_i$ for all $i=1,\dots,n$.*

Proof. First, let us prove it for $n=1$. It suffices to show that Rv is S-invariant. Indeed, $M=Rv\oplus N$, and p be the projector on N with kernel Rv. Then $p\in K$, hence $\mathrm{Ker}\,p$ is S-invariant.

For arbitrary n, note that $M^{\oplus n}$ is semisimple and use Lemma 2.5. Then for any $X\in S$, $v=(v_1,\dots,v_n)$ there exists $r\in R$ such that

$$r(v_1,\dots,v_n)=X(v_1,\dots,v_n).$$

□

Corollary 2.7. *Let M be a semisimple R-module, $K=\mathrm{End}_R(M)$, and M is finitely generated over K. Then the natural map $R\to\mathrm{End}_K(M)$ is surjective.*

Corollary 2.8. *Let R be an algebra over an algebraically closed field k, and $\rho\colon R\to\mathrm{End}_k(V)$ be an irreducible finite-dimensional representation. Then ρ is surjective.*

Corollary 2.9. *Let R, ρ and V be as in previous statement but k is not algebraically closed. Then $\rho(R) \cong \operatorname{End}_D(V)$ for some division ring D containing k.*

3. Semisimple rings

A ring R is *semi-simple* if every R-module is semisimple. For example, a group algebra $k(G)$ for a finite group G such that $\operatorname{char} k$ does not divide $|G|$ is semisimple.

Lemma 3.1. *Let R be semisimple, then R is a finite direct sum of its minimal left ideals.*

Proof. R is semisimple over itself. Hence $R = \oplus_{i\in I} L_i$, where L_i are minimal left ideals. But $1 = l_1 + \cdots + l_n$, hence I is finite. □

Theorem 3.2. *(Wedderburn-Artin) Every semisimple ring is isomorphic to $\operatorname{Mat}_{n_1}(D_1) \times \cdots \times \operatorname{Mat}_{n_k}(D_k)$ for some division rings $D_1, \dots, D_k$.*

Proof. Consider the submodules $J_1, \dots, J_k$, each J_i is the sum of all isomorphic minimal left ideals. Then J_i is a two-sided ideal and

$$R = J_1 \oplus \cdots \oplus J_k,$$

where $J_i = l_i^{\oplus n_i}$. Let $D_i^{\mathrm{op}} = \operatorname{End}_R(l_i)$, then

$$R^{\mathrm{op}} \cong \operatorname{End}_R(R) = \operatorname{End}_R(J_1) \times \cdots \times \operatorname{End}_R(J_k) = \operatorname{Mat}_{n_1}(D_1^{\mathrm{op}}) \times \cdots \times \operatorname{Mat}_{n_k}(D_k^{\mathrm{op}}).$$

□