REPRESENTATION THEORY
WEEK 5

1. Invariant forms

Recall that a bilinear form on a vector space V is a map

$$B : V \times V \to k$$

satisfying

$$B(cv, dw) = cdB(v, w), \ B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w), \ B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2).$$

One can also think about a bilinear form as a vector in $V^* \otimes V^*$ or as a homomorphism $B : V \to V^*$ given by the formula $B_v(w) = B(v, w)$. A bilinear form is symmetric if $B(v, w) = B(w, v)$ and skew-symmetric if $B(v, w) = -B(w, v)$. Every bilinear form is a sum $B = B^+ + B^-$ of a symmetric and a skew-symmetric form,

$$B^{\pm}(v, w) = \frac{B(v, w) \pm B(w, v)}{2}.$$

Such decomposition corresponds to the decomposition

$$V^* \otimes V^* = S^2 V^* \oplus \Lambda^2 V^*. \tag{1.1}$$

A form is *non-degenerate* if $B : V \to V^*$ is an isomorphism, in other words $B(v, V) = 0$ implies $v = 0$.

Let $\rho : G \to \mathrm{GL}(V)$ be a representation. We say that a bilinear form B on V is *G-invariant* if

$$B(\rho_s v, \rho_s w) = B(v, w)$$

for any $v, w \in V$, $s \in G$.

The following properties of an invariant form are easy to check

1. If $W \subset V$ is an invariant subspace, then $W^{\perp} = \{v \in V \mid B(v, W) = 0\}$ is invariant. In particular, $\operatorname{Ker} B$ is invariant.
2. $B : V \to V^*$ is invariant iff $B \in \mathrm{Hom}_G(V, V^*)$.
3. If B is invariant, then B^+ and B^- are invariant.

Lemma 1.1. *Let ρ be an irreducible representation of G, then any bilinear invariant form is non-degenerate. If $\bar{k} = k$, then a bilinear form is unique up to multiplication on a scalar.*

Proof. Follows from (2) and Schur's lemma. □

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Corollary 1.2. *A representation ρ of G admits an invariant form iff $\chi_\rho(s) = \chi_\rho(s^{-1})$ for any $s \in G$.*

Lemma 1.3. *If $\bar{k} = k$, then an invariant form on an irreducible representation ρ is either symmetric or skew-symmetric. Let*

$$m_\rho = \frac{1}{|G|} \sum_{s \in G} \chi_\rho\left(s^2\right).$$

Then $m_\rho = 1, 0$ or -1. If $m_\rho = 0$, then ρ does not admit an invariant form. If $m_\rho = \pm 1$, then m_ρ admits a symmetric (skew-symmetric) invariant form.

Proof. Recall that $\rho \otimes \rho = \rho_{\text{alt}} \oplus \rho_{\text{sym}}$.

$$(\chi_{\text{sym}}, 1) = \frac{1}{|G|} \sum_{s \in G} \frac{\chi_\rho(s^2) + \chi_\rho(s^2)}{2},$$

$$(\chi_{\text{alt}}, 1) = \frac{1}{|G|} \sum_{s \in G} \frac{\chi_\rho(s^2) - \chi_\rho(s^2)}{2}.$$

Note that

$$\frac{1}{|G|} \sum_{s \in G} \chi_\rho\left(s^2\right) = (\chi_\rho, \chi_{\rho^*}).$$

Therefore

$$(\chi_{\text{sym}}, 1) = \frac{(\chi_\rho, \chi_{\rho^*}) + m_\rho}{2}, \ (\chi_{\text{alt}}, 1) = \frac{(\chi_\rho, \chi_{\rho^*}) - m_\rho}{2}$$

If ρ does not have an invariant form, then $(\chi_{\text{sym}}, 1) = (\chi_{\text{alt}}, 1) = 0$, and $\chi_{\rho^*} \neq \chi_\rho$, hence $(\chi_\rho, \chi_{\rho^*}) = 0$. Thus, $m_\rho = 0$.

If ρ has a symmetric invariant form, then $(\chi_\rho, \chi_{\rho^*}) = 1$ and $(\chi_{\text{sym}}, 1) = 1$. This implies $m_\rho = 1$. Similarly, if ρ admits a skew-symmetric invariant form, then $m_\rho = -1$. □

Let $k = \mathbb{C}$. An irreducible representation is called *real* if $m_\rho = 1$, *complex* if $m_\rho = 0$ and *quaternionic* if $m_\rho = -1$. Since $\chi_\rho(s^{-1}) = \bar{\chi}_\rho(s)$, then χ_ρ takes only real values for real and quaternionic representations. If ρ is complex then $\chi_\rho(s) \notin \mathbb{R}$ at least for one $s \in G$.

Example. Any irreducible representation of S_4 is real. A non-trivial representation of $\mathbb{Z}_3$ is complex. The two-dimensional representation of quaternionic group is quaternionic.

Exercise. Let $|G|$ be odd. Then any non-trivial irreducible representation of G over $\mathbb{C}$ is complex.

2. Some generalities about field extension

Lemma 2.1. *If* $\operatorname{char} k = 0$ *and* G *is finite, then a representation* $\rho : G \to \mathrm{GL}(V)$ *is irreducible iff* $\operatorname{End}_G(V)$ *is a division ring.*

Proof. In one direction it is Schur's Lemma. In the opposite direction if V is not irreducible, then $V = V_1 \oplus V_2$, then the projectors p_1 and p_2 are intertwiners such that $p_1 \circ p_2 = 0$. □

For any extension F of k and a representation $\rho : G \to \mathrm{GL}(V)$ over k we define by ρ_F the representation $G \to \mathrm{GL}(F \otimes_k V)$.

For any representation $\rho : G \to \mathrm{GL}(V)$ we denote by V^G the subspace of G-invariants in V, i.e.

$$V^G = \{v \in V \mid \rho_s v = v, \forall s \in G\}.$$

Lemma 2.2. $(F \otimes_k V)^G = F \otimes_k V^G$.

Proof. The embedding $F \otimes_k V^G \subset (F \otimes_k V)^G$ is trivial. On the other hand, V^G is the image of the operator

$$p = \frac{1}{|G|} \sum_{s \in G} \tau_s,$$

in particular $\dim V^G$ equals the rank of p. Since rank p does not depend on a field, we have

$$\dim F \otimes_k V^G = \dim (F \otimes_k V)^G.$$

□

Corollary 2.3. *Let* $\rho : G \to \mathrm{GL}(V)$ *and* $\sigma : G \to \mathrm{GL}(W)$ *be two representations over* k. *Then*

$$\operatorname{Hom}_G(F \otimes_k V, F \otimes_k W) = F \otimes \operatorname{Hom}_G(V, W).$$

In particular,

$$\dim_k \operatorname{Hom}_G(V, W) = \dim_F \operatorname{Hom}_G(F \otimes_k V, F \otimes_k W).$$

Proof.

$$\operatorname{Hom}_G(V, W) = (V^* \otimes W)^G.$$

□

Corollary 2.4. *Even if a field is not algebraically closed*

$$dim \ \operatorname{Hom}_G(V, W) = (\chi_\rho, \chi_\sigma).$$

A representation $\rho : G \to \mathrm{GL}(V)$ over k is called *absolutely irreducible* if it remains irreducible after any extension of k. This is equivalent to $(\chi_\rho, \chi_\rho) = 1$. A field is *splitting* for a group G if any irreducible representation is absolutely irreducible. It is not difficult to see that some finite extension of $\mathbb{Q}$ is a splitting field for a finite group G.

3. Representations over $\mathbb{R}$

A bilinear symmetric form B is *positive definite* if $B(v,v) > 0$ for any $v \neq 0$.

Lemma 3.1. *Every representation of a finite group over $\mathbb{R}$ admits positive-definite invariant symmetric form. Two invariant symmetric forms on an irreducible representation are proportional.*

Proof. Let B' be any positive definite form. Define

$$B(v,w) = \frac{1}{|G|}\sum_{s\in G} B'(\rho_s v, \rho_s w).$$

Then B is positive definite and invariant.

Let $Q(v,w)$ be another invariant symmetric form. Then from linear algebra we know that they can be diagonalized in the same basis. Then for some $\lambda \in \mathbb{R}$, $\operatorname{Ker}(Q - \lambda B) \neq 0$. Since $\operatorname{Ker}(Q - \lambda B)$ is invariant, $Q = \lambda B$. □

Theorem 3.2. *Let $\mathbb{R} \subset K$ be a division ring, finite-dimensional over $\mathbb{R}$. Then $\mathbb{R}$ is isomorphic $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (quaternions).*

Proof. If K is a field, then $K \cong \mathbb{R}$ or $\mathbb{C}$, because $\mathbb{C} = \bar{\mathbb{R}}$ and $[\mathbb{C} : \mathbb{R}] = 2$. Assume that K is not commutative. For any $x \in K\backslash\mathbb{R}$, $\mathbb{R}[x] = \mathbb{C}$. Therefore we have a chain $\mathbb{R} \subset \mathbb{C} \subset K$. Let $f(x) = ixi^{-1}$. Obviously f is an automorphism of K and $f^2 = id$. Hence $K = K^+ \oplus K^-$, where

$$K^{\pm} = \{x \in K \mid f(x) = \pm x\}.$$

Moreover, $K^+K^+ \subset K^+$, $K^-K^- \subset K^+$, $K^+K^- \subset K^-$, $K^-K^+ \subset K^-$. If $x \in K^+$, then $\mathbb{C}[x]$ is a finite extension of $\mathbb{C}$. Therefore $K^+ = \mathbb{C}$. For any nonzero $y \in K^-$ the left multiplication on y defines an isomorphism of K^+ and K^- as vector spaces over $\mathbb{R}$. In particular $\dim_{\mathbb{R}} K^- = \dim_{\mathbb{R}} K^+ = 2$. For any $y \in K^-$, $x \in \mathbb{C}$, we have $y\bar{x} = xy$, therefore $y^2 \in \mathbb{R}$. Moreover, $y^2 < 0$. (If $y^2 > 0$, then $y^2 = b^2$ for some real b and $(y-b)(y+b) = 0$, which is impossible). Put $j = \frac{y}{\sqrt{-y^2}}$. Then we have $k = ij = -ji$, $ki = (ij)i = j$, $K = \mathbb{R}[i,j]$ is isomorphic to $\mathbb{H}$. □

Lemma 3.3. *Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation over $\mathbb{R}$, then there are three possibilities:*

1. *$\mathrm{End}_G(V) = \mathbb{R}$ and $(\chi_\rho, \chi_\rho) = 1$;*
2. *$\mathrm{End}_G(V) \cong \mathbb{C}$ and $(\chi_\rho, \chi_\rho) = 2$;*
3. *$\mathrm{End}_G(V) \cong \mathbb{H}$ and $(\chi_\rho, \chi_\rho) = 4$.*

Proof. Lemma 2.1 and Theorem 3.2 imply that $\mathrm{End}_G(V)$ is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $(\chi_\rho, \chi_\rho) = 1, 2$ or 4 as follows from Corollary 2.4.

□

4. Relationship between representations over $\mathbb{R}$ and over $\mathbb{C}$

Hermitian invariant form. Recall that a Hermitian form satisfies the following conditions

$$H(av, bw) = \bar{a}bH(v, w), \; H(w, v) = \bar{H}(v, w).$$

The following Lemma can be proved exactly as Lemma 3.1.

Lemma 4.1. *Every representation of a finite group over $\mathbb{C}$ admits positive-definite invariant Hermitian form. Two invariant Hermitian forms on an irreducible representation are proportional.*

Let $\rho : G \to \mathrm{GL}(V)$ be a representation over $\mathbb{C}$. Denote by $V^{\mathbb{R}}$ a vector space V as a vector space over $\mathbb{R}$ of double dimension. Denote by $\rho^{\mathbb{R}}$ the representation of G in $V^{\mathbb{R}}$. Check that

$$\chi_{\rho^{\mathbb{R}}} = \chi_\rho + \bar{\chi}_\rho. \tag{4.1}$$

Theorem 4.2. *Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation over $\mathbb{C}$.*

1. *If ρ can be realized by matrices with real entries, then ρ admits an invariant symmetric form.*
2. *If $\mathrm{End}_G\left(V^{\mathbb{R}}\right) = \mathbb{C}$, then ρ is complex, i.e. ρ does not admit a bilinear invariant symmetric form.*
3. *If $\mathrm{End}_G\left(V^{\mathbb{R}}\right) = \mathbb{H}$, then ρ admits an invariant skew-symmetric form.*

Proof. (1) follows from Lemma 3.1. For (2) use (4.1). Since $(\chi_\rho, \chi_\rho) = 2$ by Lemma 3.3, then $\chi_\rho \neq \bar{\chi}_\rho$, and therefore ρ is complex.

Finally let us prove (3). Let $j \in \mathrm{End}_G\left(V^{\mathbb{R}}\right) = \mathbb{H}$, then $j(bv) = \bar{b}v$ for any $b \in \mathbb{C}$. Let H be a positive-definite Hermitian form on V. Then

$$Q(v, w) = H(jw, jv)$$

is another invariant positive-definite Hermitian form. By Lemma 4.1 $Q = \lambda H$ for some $\lambda > 0$. Since $j^2 = -1$, $\lambda^2 = 1$ and therefore $\lambda = 1$. Thus,

$$H(v, w) = H(jw, jv).$$

Set

$$B(v, w) = H(jv, w).$$

Then B is a bilinear invariant form, and

$$B(w, v) = H(jw, v) = H\left(jv, j^2 w\right) = -H(jv, w) = -B(v, w),$$

hence B is skew-symmetric. □

Corollary 4.3. *Let σ be an irreducible representation of G over $\mathbb{R}$. There are three possibilities for σ*

σ is absolutely irreducible and $\chi_\sigma = \chi_\rho$ for some real representation ρ of G over $\mathbb{C}$;
$\chi_\sigma = \chi_\rho + \bar{\chi}_\rho$ for some complex representation ρ of G over $\mathbb{C}$;
$\chi_\sigma = 2\chi_\rho$ for some quaternionic representation ρ of G over $\mathbb{C}$.

5. Representations of symmetric group

Let $\mathcal{A}$ denote the group algebra $\mathbb{Q}(S_n)$. We will see that $\mathbb{Q}$ is a splitting field for S_n. We realize irreducible representation of S_n as minimal left ideals in $\mathcal{A}$.

Conjugacy classes are enumerated by partitions $m_1 \geq \cdots \geq m_k > 0$, $m_1 + \cdots + m_k = n$. To each partition we associate the table of n boxes with rows of length $m_1, \ldots, m_k$, it is called a *Young diagram. Young tableau* is a Young diagram with entries $1, \ldots, n$ in boxes. Given a Young tableau λ, we denote by P_λ the subgroup of permutations preserving rows and by Q_λ the subgroup of permutations preserving columns. Introduce the following elements in $\mathcal{A}$

$$a_\lambda = \sum_{p \in P_\lambda} p,\ b_\lambda = \sum_{q \in Q_\lambda} (-1)^q\, q,\ c_\lambda = a_\lambda b_\lambda.$$

The element c_λ is called *Young symmetrizer.*

Theorem 5.1. *$V_\lambda = \mathcal{A}c_\lambda$ is a minimal left ideal in $\mathcal{A}$, therefore V_λ is irreducible. V_λ is isomorphic to V_μ iff the Young tableaux μ and λ have the same Young diagram. Any irreducible representation of S_n is isomorphic to V_λ for some Young tableau λ.*

Note that the last assertion of Theorem follows from the first two, since the number of Young diagrams equals the number of conjugacy classes.

Examples. For partition (n), $c_\lambda = a_\lambda = \sum_{s \in S_n} s$, and the representation is trivial. For (1,...,1), $c_\lambda = b_\lambda = \sum_{s \in S_n} (-1)^s\, s$.

Let us consider partition $(n-1, 1)$. Then

$$c_\lambda = \left(\sum_{s \in S_{n-1}} s \right) (1 - (1n)).$$

Clearly, $a_\lambda c_\lambda = c_\lambda$, therefore $\mathrm{Res}_{S_{n-1}} V_\lambda$ contains the trivial representation. Let

$$V = \mathrm{Ind}_{S_{n-1}}^{S_n} (\mathrm{triv}).$$

Note that V is the permutation representation of S_n. By Frobenius reciprocity we have a homomorphism $V \to V_\lambda$. Therefore $V = V_\lambda \oplus \mathrm{triv}$.

Now we will prove Theorem 5.1. First, note that S_n acts on the Young tableaux of the same shape, and

$$a_{s(\lambda)} = s a_\lambda s^{-1},\ b_{s(\lambda)} = s b_\lambda s^{-1},\ c_{s(\lambda)} = s c_\lambda s^{-1}.$$

Check yourself the following

Lemma 5.2. *If $s \in S_n$, but $s \notin P_\lambda Q_\lambda$, then there exists two numbers i and j in the same row of λ and in the same column of $s(\lambda)$.*

It is clear also that for any $p \in P_\lambda$, $q \in Q_\lambda$

$$p a_\lambda = a_\lambda p = a_\lambda,\ q b_\lambda = b_\lambda q = (-1)^q\, b_\lambda,\ p c_\lambda q = (-1)^q\, c_\lambda.$$

Lemma 5.3. *Let $y \in \mathcal{A}$ such that for any $p \in P_\lambda$, $q \in Q_\lambda$*

$$pyq = (-1)^q y.$$

Then $y \in \mathbb{Q}c_\lambda$.

Proof. It is clear that y has a form

$$\sum_{s \in P_\lambda \backslash S_n / Q_\lambda} d_s \sum_{p \in P_\lambda, q \in Q_\lambda} (-1)^q psq = \sum_{s \in P_\lambda \backslash S_n / Q_\lambda} d_s a_\lambda s b_\lambda,$$

for some $d_s \in \mathbb{Q}$. We have to show that if $s \notin P_\lambda Q_\lambda$ then $a_\lambda s b_\lambda = 0$. That follows from Lemma 5.2. There exists $(ij) \in P_\lambda \cap Q_{s(\lambda)}$. Then

$$a_\lambda s b_\lambda s^{-1} = a_\lambda b_{s(\lambda)} = a_\lambda (ij)(ij) b_{s(\lambda)} = a_\lambda b_{s(\lambda)} = -a_\lambda b_{s(\lambda)} = 0.$$

□

Corollary 5.4. $c_\lambda \mathcal{A} c_\lambda \subset \mathbb{Q}c_\lambda$.

Lemma 5.5. *Let W be a left ideal in a group algebra $k(G)$ ($\operatorname{char} k = 0$). Then $W^2 = 0$ implies $W = 0$.*

Proof. Since $k(G)$ is completely reducible $k(G) = W \oplus W'$, where W' is another left ideal. Let $y \in \operatorname{End}_G(k(G))$ such that $y_{|W} = \mathrm{Id}$, $y(W') = 0$. But we proved that any $y \in \operatorname{End}_G(k(G))$ is a right multiplication on some $u \in k(G)$ (see lecture notes 3). Then we have $u^2 = u$, $W = \mathcal{A}u$, in particular $u \in W$. If $W \neq 0$, then $u \neq 0$ and $u^2 = u \neq 0$. Hence $W^2 \neq 0$. □

Corollary 5.6. *$\mathcal{A}c_\lambda$ is a minimal left ideal.*

Proof. Let $W \subset \mathcal{A}c_\lambda$ be a left ideal. Then either $c_\lambda W = \mathbb{Q}c_\lambda$ or $c_\lambda W = 0$ by Corollary 5.4. In the former case $W = \mathcal{A}c_\lambda W = \mathcal{A}c_\lambda$. In the latter case $W^2 \subset \mathcal{A}c_\lambda W = 0$, and $W = 0$ by Lemma 5.5.

□

Corollary 5.7. *$c_\lambda^2 = n_\lambda c_\lambda$, where $n_\lambda = \frac{n!}{\dim V_\lambda}$.*

Proof. From the proof of Lemma 5.5, $c_\lambda = n_\lambda u$ for some idempotent $u \in \mathbb{Q}(S_n)$. Therefore $c_\lambda = n_\lambda u$. To find n_λ note that $\operatorname{tr}_{k(G)} u = \dim V_\lambda$, $\operatorname{tr}_{k(G)} c_\lambda = |S_n| = n!$. □

Lemma 5.8. *Order partitions lexicographically. If $\lambda > \mu$, then there exists i, j in the same row of λ and in the same column of μ.*

Proof. Check yourself. □

Corollary 5.9. *If $\lambda < \mu$, then $c_\lambda \mathcal{A} c_\mu = 0$.*

Proof. Sufficient to check that $c_\lambda s c_\mu = 0$ for any $s \in S_n$, which is equivalent to

$$c_\lambda s c_\mu s^{-1} = c_\lambda c_{s(\mu)} = 0.$$

Let $(ij) \in Q_\lambda \cap P_{s(\mu)}$. Then

$$c_\lambda (ij)(ij) c_{s(\mu)} = c_\lambda c_{s(\mu)} = -c_\lambda c_{s(\mu)} = 0.$$

□

Lemma 5.10. *V_λ and V_μ are isomorphic iff λ and μ have the same Young diagram.*

Proof. If λ and μ have the same diagram, then $\lambda = s(\mu)$ for some $s \in S_n$ and $\mathcal{A}c_\lambda = \mathcal{A}sc_\mu s^{-1} = \mathcal{A}c_\mu s^{-1}$. Assume $\lambda > \mu$, then $c_\lambda \mathcal{A} c_\mu = 0$ and $c_\lambda \mathcal{A} c_\lambda \neq 0$. Therefore $\mathcal{A}c_\lambda$ and $\mathcal{A}c_\mu$ are not isomorphic. □

Corollary 5.11. *If λ and μ have different diagrams, then $c_\lambda \mathcal{A} c_\mu = 0$.*

Proof. If $c_\lambda \mathcal{A} c_\mu \neq 0$, then $\mathcal{A}c_\lambda \mathcal{A} c_\mu = \mathcal{A}c_\mu$. On the other hand $\mathcal{A}c_\lambda\mathcal{A}$ has only components isomorphic to V_λ. Contradiction. □

Lemma 5.12. *Let $\rho : S_n \to \mathrm{GL}(V)$ be an arbitrary representation. Then the multiplicity of V_λ in V equals the rank of $\rho(c_\lambda)$.*

Proof. The rank of c_λ is 1 in V_λ and 0 in any V_μ with another Young diagram. □