1. **Invariant forms**

Recall that a bilinear form on a vector space \( V \) is a map
\[
B : V \times V \to k
\]
satisfying
\[
B(cv, dw) = cdB(v, w), \quad B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w), \quad B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2).
\]

One can also think about a bilinear form as a vector in \( V^* \otimes V^* \) or as a homomorphism \( B : V \to V^* \) given by the formula \( B(v)(w) = B(v, w) \). A bilinear form is symmetric if \( B(v, w) = B(w, v) \) and skew-symmetric if \( B(v, w) = -B(w, v) \). Every bilinear form is a sum \( B = B^+ + B^- \) of a symmetric and a skew-symmetric form,
\[
B^\pm(v, w) = \frac{B(v, w) \pm B(w, v)}{2}.
\]

Such decomposition corresponds to the decomposition
\[
V^* \otimes V^* = S^2V^* \oplus \Lambda^2V^*.
\]

A form is non-degenerate if \( B : V \to V^* \) is an isomorphism, in other words \( B(v, V) = 0 \) implies \( v = 0 \).

Let \( \rho : G \to GL(V) \) be a representation. We say that a bilinear form \( B \) on \( V \) is \( G \)-invariant if
\[
B(\rho_s v, \rho_s w) = B(v, w)
\]
for any \( v, w \in V, s \in G \).

The following properties of an invariant form are easy to check

1. If \( W \subset V \) is an invariant subspace, then \( W^\perp = \{ v \in V \mid B(v, W) = 0 \} \) is invariant. In particular, \( \text{Ker} B \) is invariant.
2. \( B : V \to V^* \) is invariant iff \( B \in \text{Hom}_G(V, V^*) \).
3. If \( B \) is invariant, then \( B^+ \) and \( B^- \) are invariant.

**Lemma 1.1.** Let \( \rho \) be an irreducible representation of \( G \), then any bilinear invariant form is non-degenerate. If \( k = k \), then a bilinear form is unique up to multiplication of a scalar.

**Proof.** Follows from (2) and Schur’s lemma. \( \square \)
Corollary 1.2. A representation $\rho$ of $G$ admits an invariant form iff $\chi_\rho(s) = \chi_\rho(s^{-1})$ for any $s \in G$.

Lemma 1.3. If $\bar{k} = k$, then an invariant form on an irreducible representation $\rho$ is either symmetric or skew-symmetric. Let

$$m_\rho = \frac{1}{|G|} \sum_{s \in G} \chi_\rho(s^2).$$

Then $m_\rho = 1, 0$ or $-1$. If $m_\rho = 0$, then $\rho$ does not admit an invariant form. If $m_\rho = \pm 1$, then $m_\rho$ admits a symmetric (skew-symmetric) invariant form.

Proof. Recall that $\rho \otimes \rho = \rho_{\text{alt}} \oplus \rho_{\text{sym}}$.

$$(\chi_{\text{sym}}, 1) = \frac{1}{|G|} \sum_{s \in G} \frac{\chi_\rho(s^2) + \chi_\rho(s^2)}{2},$$

$$(\chi_{\text{alt}}, 1) = \frac{1}{|G|} \sum_{s \in G} \frac{\chi_\rho(s^2) - \chi_\rho(s^2)}{2}.$$

Note that

$$\frac{1}{|G|} \sum_{s \in G} \chi_\rho(s^2) = (\chi_\rho, \chi_\rho^*).$$

Therefore

$$(\chi_{\text{sym}}, 1) = \frac{(\chi_\rho, \chi_\rho^*) + m_\rho}{2}, \quad (\chi_{\text{alt}}, 1) = \frac{(\chi_\rho, \chi_\rho^*) - m_\rho}{2}.$$

If $\rho$ does not have an invariant form, then $(\chi_{\text{sym}}, 1) = (\chi_{\text{alt}}, 1) = 0$, and $\chi_\rho^* \neq \chi_\rho$, hence $(\chi_\rho, \chi_\rho^*) = 0$. Thus, $m_\rho = 0$.

If $\rho$ has a symmetric invariant form, then $(\chi_{\text{sym}}, 1) = 1$ and $(\chi_{\text{sym}}, 1) = 1$. This implies $m_\rho = 1$. Similarly, if $\rho$ admits a skew-symmetric invariant form, then $m_\rho = -1$. \qed

Let $k = \mathbb{C}$. An irreducible representation is called real if $m_\rho = 1$, complex if $m_\rho = 0$ and quaternionic if $m_\rho = -1$. Since $\chi_\rho(s^{-1}) = \bar{\chi}_\rho(s)$, then $\chi_\rho$ takes only real values for real and quaternionic representations. If $\rho$ is complex then $\chi_\rho(s) \notin \mathbb{R}$ at least for one $s \in G$.

Example. Any irreducible representation of $S_4$ is real. A non-trivial representation of $\mathbb{Z}_3$ is complex. The two-dimensional representation of quaternionic group is quaternionic.

Exercise. Let $|G|$ be odd. Then any non-trivial irreducible representation of $G$ over $\mathbb{C}$ is complex.
2. Some generalities about field extension

Lemma 2.1. If \( \text{char } k = 0 \) and \( G \) is finite, then a representation \( \rho : G \to GL(V) \) is irreducible iff \( \text{End}_G(V) \) is a division ring.

Proof. In one direction it is Schur’s Lemma. In the opposite direction if \( V \) is not irreducible, then \( V = V_1 \oplus V_2 \), then the projectors \( p_1 \) and \( p_2 \) are intertwiners such that \( p_1 \circ p_2 = 0 \). \( \square \)

For any extension \( F \) of \( k \) and a representation \( \rho : G \to GL(V) \) over \( k \) we define by \( \rho_F \) the representation \( G \to GL(F \otimes_k V) \).

For any representation \( \rho : G \to GL(V) \) we denote by \( V^G \) the subspace of \( G \)-invariants in \( V \), i.e.
\[
V^G = \{ v \in V | \rho_s v = v, \forall s \in G \}.
\]

Lemma 2.2. \( (F \otimes_k V)^G = F \otimes_k V^G \).

Proof. The embedding \( F \otimes_k V^G \subset (F \otimes_k V)^G \) is trivial. On the other hand, \( V^G \) is the image of the operator
\[
p = \frac{1}{|G|} \sum_{s \in G} \tau_s,
\]
in particular \( \dim V^G \) equals the rank of \( p \). Since rank \( p \) does not depend on a field, we have
\[
\dim F \otimes_k V^G = \dim (F \otimes_k V)^G.
\]

Corollary 2.3. Let \( \rho : G \to GL(V) \) and \( \sigma : G \to GL(W) \) be two representations over \( k \). Then
\[
\text{Hom}_G(F \otimes_k V, F \otimes_k W) = F \otimes \text{Hom}_G(V, W).
\]

In particular,
\[
\dim_k \text{Hom}_G(V, W) = \dim_F \text{Hom}_G(F \otimes_k V, F \otimes_k W).
\]

Proof.
\[
\text{Hom}_G(V, W) = (V^* \otimes W)^G.
\]

Corollary 2.4. Even if a field is not algebraically closed
\[
\dim \text{Hom}_G(V, W) = (\chi_\rho, \chi_\sigma).
\]

A representation \( \rho : G \to GL(V) \) over \( k \) is called absolutely irreducible if it remains irreducible after any extension of \( k \). This is equivalent to \( (\chi_\rho, \chi_\rho) = 1 \). A field is splitting for a group \( G \) if any irreducible representation is absolutely irreducible. It is not difficult to see that some finite extension of \( \mathbb{Q} \) is a splitting field for a finite group \( G \).
3. Representations over \( \mathbb{R} \)

A bilinear symmetric form \( B \) is **positive definite** if \( B(v, v) > 0 \) for any \( v \neq 0 \).

**Lemma 3.1.** Every representation of a finite group over \( \mathbb{R} \) admits positive-definite invariant symmetric form. Two invariant symmetric forms on an irreducible representation are proportional.

**Proof.** Let \( B' \) be any positive definite form. Define

\[
B(v, w) = \frac{1}{|G|} \sum_{s \in G} B'(\rho_s v, \rho_s w)
\]

Then \( B \) is positive definite and invariant.

Let \( Q(v, w) \) be another invariant symmetric form. Then from linear algebra we know that they can be diagonalized in the same basis. Then for some \( \lambda \in \mathbb{R} \),

\[
\text{Ker } (Q - \lambda B) \neq 0.
\]

Since \( \text{Ker } (Q - \lambda B) \) is invariant, \( Q = \lambda B \).

**Theorem 3.2.** Let \( \mathbb{R} \subset K \) be a division ring, finite-dimensional over \( \mathbb{R} \). Then \( \mathbb{R} \) is isomorphic \( \mathbb{R} \), \( \mathbb{C} \) or \( \mathbb{H} \) (quaternions).

**Proof.** If \( K \) is a field, then \( K \cong \mathbb{R} \) or \( \mathbb{C} \), because \( \mathbb{C} = \overline{\mathbb{R}} \) and \( |\mathbb{C} : \mathbb{R}| = 2 \). Assume that \( K \) is not commutative. For any \( x \in K \setminus \mathbb{R} \), \( \mathbb{R}[x] = \mathbb{C} \). Therefore we have a chain \( \mathbb{R} \subset \mathbb{C} \subset K \). Let \( f(x) = ix^{-1} \). Obviously \( f \) is an automorphism of \( K \) and \( f^2 = id \).

Hence \( K = K^+ \oplus K^- \), where

\[
K^\pm = \{ x \in K \mid f(x) = \pm x \}.
\]

Moreover, \( K^+K^+ \subset K^+ \), \( K^-K^- \subset K^- \), \( K^+K^- \subset K^- \), \( K^-K^+ \subset K^- \). If \( x \in K^+ \), then \( \mathbb{C}[x] \) is a finite extension of \( \mathbb{C} \). Therefore \( K^+ = \mathbb{C} \). For any nonzero \( y \in K^- \) the left multiplication on \( y \) defines an isomorphism of \( K^+ \) and \( K^- \) as vector spaces over \( \mathbb{R} \). In particular \( \dim_\mathbb{R} K^- = \dim_\mathbb{R} K^+ = 2 \). For any \( y \in K^- \), \( x \in \mathbb{C} \), we have \( y\bar{x} = xy \), therefore \( y^2 \in \mathbb{R} \). Moreover, \( y^2 < 0 \). (If \( y^2 > 0 \), then \( y^2 = b^2 \) for some real \( b \) and \( (y-b)(y+b) = 0 \), which is impossible). Put \( j = \sqrt{-y^2} \). Then we have

\[
k = ij = -ji,\ k_i = (ij)i = j,\ K = \mathbb{R}[i, j] \text{ is isomorphic to } \mathbb{H}.
\]

**Lemma 3.3.** Let \( \rho : G \to \text{GL}(V) \) be an irreducible representation over \( \mathbb{R} \), then there are three possibilities:

1. \( \text{End}_G(V) = \mathbb{R} \) and \( (\chi_\rho, \chi_\rho) = 1 \);
2. \( \text{End}_G(V) \cong \mathbb{C} \) and \( (\chi_\rho, \chi_\rho) = 2 \);
3. \( \text{End}_G(V) \cong \mathbb{H} \) and \( (\chi_\rho, \chi_\rho) = 4 \).

**Proof.** Lemma 2.1 and Theorem 3.2 imply that \( \text{End}_G(V) \) is isomorphic to \( \mathbb{R}, \mathbb{C} \) or \( \mathbb{H} \), \( (\chi_\rho, \chi_\rho) = 1, 2 \) or 4 as follows from Corollary 2.4.
4. Relationship between representations over $\mathbb{R}$ and over $\mathbb{C}$

**Hermitian invariant form.** Recall that a Hermitian form satisfies the following conditions

$$H(\alpha v, \beta w) = \overline{\beta \alpha} H(v, w), \ H(w, v) = \overline{H(v, w)}.$$  

The following Lemma can be proved exactly as Lemma 3.1.

**Lemma 4.1.** Every representation of a finite group over $\mathbb{C}$ admits positive-definite invariant Hermitian form. Two invariant Hermitian forms on an irreducible representation are proportional.

Let $\rho : G \to \text{GL}(V)$ be a representation over $\mathbb{C}$. Denote by $V^\mathbb{R}$ a vector space $V$ as a vector space over $\mathbb{R}$ of double dimension. Denote by $\rho^\mathbb{R}$ the representation of $G$ in $V^\mathbb{R}$. Check that

$$(4.1) \quad \chi_{\rho^\mathbb{R}} = \chi_\rho + \overline{\chi_\rho}.$$  

**Theorem 4.2.** Let $\rho : G \to \text{GL}(V)$ be an irreducible representation over $\mathbb{C}$.

1. If $\rho$ can be realized by matrices with real entries, then $\rho$ admits an invariant symmetric form.

2. If $\text{End}_G(V^\mathbb{R}) = \mathbb{C}$, then $\rho$ is complex, i.e. $\rho$ does not admit a bilinear invariant symmetric form.

3. If $\text{End}_G(V^\mathbb{R}) = \mathbb{H}$, then $\rho$ admits an invariant skew-symmetric form.

**Proof.** (1) follows from Lemma 3.1. For (2) use (4.1). Since $(\chi_\rho, \chi_\rho) = 2$ by Lemma 3.3, then $\chi_\rho \neq \overline{\chi_\rho}$, and therefore $\rho$ is complex.

Finally let us prove (3). Let $j \in \text{End}_G(V^\mathbb{R}) = \mathbb{H}$, then $j(bv) = \overline{bv}$ for any $b \in \mathbb{C}$. Let $H$ be a positive-definite Hermitian form on $V$. Then

$$Q(v, w) = H(jw, jv)$$

is another invariant positive-definite Hermitian form. By Lemma 4.1 $Q = \lambda H$ for some $\lambda > 0$. Since $j^2 = -1$, $\chi^2 = 1$ and therefore $\lambda = 1$. Thus,

$$H(v, w) = H(jw, jv).$$

Set

$$B(v, w) = H(jv, w).$$

Then $B$ is a bilinear invariant form, and

$$B(v, w) = H(jw, v) = H(jv, j^2 w) = -H(jv, w) = -B(v, w),$$

hence $B$ is skew-symmetric. \qed

**Corollary 4.3.** Let $\sigma$ be an irreducible representation of $G$ over $\mathbb{R}$. There are three possibilities for $\sigma$

- $\sigma$ is absolutely irreducible and $\chi_\sigma = \chi_\rho$ for some real representation $\rho$ of $G$ over $\mathbb{C}$;
- $\chi_\sigma = \chi_\rho + \overline{\chi_\rho}$ for some complex representation $\rho$ of $G$ over $\mathbb{C}$;
- $\chi_\sigma = 2\chi_\rho$ for some quaternionic representation $\rho$ of $G$ over $\mathbb{C}$. 
5. REPRESENTATIONS OF SYMMETRIC GROUP

Let $A$ denote the group algebra $Q(S_n)$. We will see that $Q$ is a splitting field for $S_n$. We realize irreducible representation of $S_n$ as minimal left ideals in $A$.

Conjugacy classes are enumerated by partitions $m_1 \geq \cdots \geq m_k > 0$, $m_1 + \cdots + m_k = n$. To each partition we associate the table of $n$ boxes with rows of length $m_1, \ldots, m_k$, it is called a Young diagram. Young tableau is a Young diagram with entries $1, \ldots, n$ in boxes. Given a Young tableau $\lambda$, we denote by $P_\lambda$ the subgroup of permutations preserving rows and by $Q_\lambda$ the subgroup of permutations preserving columns. Introduce the following elements in $A$

$$a_\lambda = \sum_{p \in P_\lambda} p, \quad b_\lambda = \sum_{q \in Q_\lambda} (-1)^q q, \quad c_\lambda = a_\lambda b_\lambda.$$ 

The element $c_\lambda$ is called Young symmetrizer.

**Theorem 5.1.** $V_\lambda = Ac_\lambda$ is a minimal left ideal in $A$, therefore $V_\lambda$ is irreducible. $V_\lambda$ is isomorphic to $V_\mu$ iff the Young tableaux $\mu$ and $\lambda$ have the same Young diagram. Any irreducible representation of $S_n$ is isomorphic to $V_\lambda$ for some Young tableau $\lambda$.

Note that the last assertion of Theorem follows from the first two, since the number of Young diagrams equals the number of conjugacy classes.

**Examples.** For partition $(n)$, $c_\lambda = a_\lambda = \sum_{s \in S_n} s$, and the representation is trivial.

For $(1, \ldots, 1)$, $c_\lambda = b_\lambda = \sum_{s \in S_n} (-1)^s s$.

Let us consider partition $(n - 1, 1)$. Then

$$c_\lambda = \left( \sum_{s \in S_{n-1}} s \right) (1 - (1n)).$$

Clearly, $a_\lambda c_\lambda = c_\lambda$, therefore $\text{Res}_{S_{n-1}} V_\lambda$ contains the trivial representation. Let

$$V = \text{Ind}_{S_{n-1}}^{S_n} (\text{triv}).$$

Note that $V$ is the permutation representation of $S_n$. By Frobenius reciprocity we have a homomorphism $V \to V_\lambda$. Therefore $V = V_\lambda \oplus \text{triv}$.

Now we will prove Theorem 5.1. First, note that $S_n$ acts on the Young tableaux of the same shape, and

$$a_{s(\lambda)} = sa_\lambda s^{-1}, \quad b_{s(\lambda)} = sb_\lambda s^{-1}, \quad c_{s(\lambda)} = sc_\lambda s^{-1}.$$ 

Check yourself the following

**Lemma 5.2.** If $s \in S_n$, but $s \notin P_\lambda Q_\lambda$, then there exists two numbers $i$ and $j$ in the same row of $\lambda$ and in the same column of $s(\lambda)$.

It is clear also that for any $p \in P_\lambda, q \in Q_\lambda$

$$pa_\lambda = a_\lambda p = a_\lambda, \quad qb_\lambda = b_\lambda q = (-1)^q b_\lambda, \quad pc_\lambda q = (-1)^q c_\lambda.$$
Lemma 5.3. Let $y \in A$ such that for any $p \in P_\lambda, q \in Q_\lambda$

$$pyq = (-1)^q y.$$ 

Then $y \in Qc_\lambda$.

Proof. It is clear that $y$ has a form

$$\sum_{s \in P_\lambda \setminus S_n/Q_\lambda} d_s \sum_{p \in P_\lambda, q \in Q_\lambda} (-1)^q psq = \sum_{s \in P_\lambda \setminus S_n/Q_\lambda} d_s a_\lambda sb_\lambda,$$

for some $d_s \in \mathbb{Q}$. We have to show that if $s \notin P_\lambda Q_\lambda$ then $a_\lambda sb_\lambda = 0$. That follows from Lemma 5.2. There exists $(ij) \in P_\lambda \cap Q_{s(\lambda)}$. Then

$$a_\lambda sb_\lambda s^{-1} = a_\lambda b_{s(\lambda)} = a_\lambda (ij) b_{s(\lambda)} = a_\lambda b_{s(\lambda)} = -a_\lambda b_{s(\lambda)} = 0.$$

□

Corollary 5.4. $c_\lambda A c_\lambda \subset Qc_\lambda$.

Lemma 5.5. Let $W$ be a left ideal in a group algebra $k(G)$ (char $k = 0$). Then $W^2 = 0$ implies $W = 0$.

Proof. Since $k(G)$ is completely reducible $k(G) = W \oplus W'$, where $W'$ is another left ideal. Let $y \in \text{End}_G(k(G))$ such that $y|_W = \text{Id}$, $y(W') = 0$. But we proved that any $y \in \text{End}_G(k(G))$ is a right multiplication on some $u \in k(G)$ (see lecture notes 3). Then we have $u^2 = u$, $W = Au$, in particular $u \in W$. If $W \neq 0$, then $u \neq 0$ and $u^2 = u \neq 0$. Hence $W^2 \neq 0$.

□

Corollary 5.6. $Ac_\lambda$ is a minimal left ideal.

Proof. Let $W \subset Ac_\lambda$ be a left ideal. Then either $c_\lambda W = Qc_\lambda$ or $c_\lambda W = 0$ by Corollary 5.4. In the former case $W = Ac_\lambda W = Ac_\lambda$. In the latter case $W^2 \subset Ac_\lambda W = 0$, and $W = 0$ by Lemma 5.5.

□

Corollary 5.7. $c_\lambda^2 = n_\lambda c_\lambda$, where $n_\lambda = \frac{n!}{\dim V_\lambda}$.

Proof. From the proof of Lemma 5.5, $c_\lambda = n_\lambda u$ for some idempotent $u \in Q(S_n)$. Therefore $c_\lambda = n_\lambda u$. To find $n_\lambda$ note that $\text{tr}_{k(G)} u = \dim V_\lambda$, $\text{tr}_{k(G)} c_\lambda = |S_n| = n!$.

□

Lemma 5.8. Order partitions lexicographically. If $\lambda > \mu$, then there exists $i, j$ in the same row of $\lambda$ and in the same column of $\mu$.

Proof. Check yourself.

□

Corollary 5.9. If $\lambda < \mu$, then $c_\lambda Ac_\mu = 0$.

Proof. Sufficient to check that $c_\lambda sc_\mu = 0$ for any $s \in S_n$, which is equivalent to

$$c_\lambda sc_\mu s^{-1} = c_\lambda c_{s(\mu)} = 0.$$
Let \((ij) \in Q_\lambda \cap P_{s(\mu)}\). Then
\[
c_{\lambda} (ij) c_{s(\mu)} = c_{\lambda} c_{s(\mu)} = -c_{\lambda} c_{s(\mu)} = 0.
\]

Lemma 5.10. \(V_\lambda\) and \(V_\mu\) are isomorphic iff \(\lambda\) and \(\mu\) have the same Young diagram.

Proof. If \(\lambda\) and \(\mu\) have the same diagram, then \(\lambda = s(\mu)\) for some \(s \in S_n\) and \(A_{c_\lambda} = A_{sc_\mu} s^{-1} = A_{c_\mu} s^{-1}\). Assume \(\lambda > \mu\), then \(c_\lambda A_{c_\mu} = 0\) and \(c_\lambda A_{c_\lambda} \neq 0\). Therefore \(A_{c_\lambda}\) and \(A_{c_\mu}\) are not isomorphic.

Corollary 5.11. If \(\lambda\) and \(\mu\) have different diagrams, then \(c_\lambda A_{c_\mu} = 0\).

Proof. If \(c_\lambda A_{c_\mu} \neq 0\), then \(A_{c_\lambda} A_{c_\mu} = A_{c_\mu}\). On the other hand \(A_{c_\lambda} A\) has only components isomorphic to \(V_\lambda\). Contradiction.

Lemma 5.12. Let \(\rho : S_n \to GL(V)\) be an arbitrary representation. Then the multiplicity of \(V_\lambda\) in \(V\) equals the rank of \(\rho (c_\lambda)\).

Proof. The rank of \(c_\lambda\) is 1 in \(V_\lambda\) and 0 in any \(V_\mu\) with another Young diagram.