1. Characters

For any finite-dimensional representation $\rho : G \rightarrow \text{GL} (V)$ its character is a function $\chi_\rho : G \rightarrow k$ defined by

$$\chi_\rho (s) = \text{tr} \rho_s.$$ 

It is easy to see that characters have the following properties

1. $\chi_\rho (1) = \dim \rho$;
2. if $\rho \cong \sigma$, then $\chi_\rho = \chi_\sigma$;
3. $\chi_{\rho \oplus \sigma} = \chi_\rho + \chi_\sigma$;
4. $\chi_{\rho \otimes \sigma} = \chi_\rho \chi_\sigma$;
5. $\chi_{\rho^*} (s) = \chi_\rho (s^{-1})$;
6. $\chi_{\rho (sts^{-1})} = \chi_\rho (t)$.

**Example 1.** If $R$ is a regular representation, then $\chi_R (s) = 0$ for any $s \neq 1$ and $\chi_R (1) = |G|$.

**Example 2.** Let $\rho : G \rightarrow \text{GL} (V)$ be a representation. Recall that $\rho \otimes \rho = \text{sym} \oplus \text{alt}$, where $\text{alt} : G \rightarrow \text{GL} (\Lambda^2 V)$ and $\text{sym} : G \rightarrow \text{GL} (S^2 V)$. Let us calculate $\chi_{\text{sym}}$ and $\chi_{\text{alt}}$. For each $s \in G$ let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of $\rho_s$ taken with multiplicities. Then the eigenvalues of $\text{alt}_s$ are $\lambda_i \lambda_j$ for all $i < j$ and the eigenvalues of $\text{sym}_s$ are $\lambda_i \lambda_j$ for $i \leq j$. Hence

$$\chi_{\text{sym}} (s) = \sum_{i \leq j} \lambda_i \lambda_j, \quad \chi_{\text{alt}} (s) = \sum_{i < j} \lambda_i \lambda_j,$$

and therefore

$$\chi_{\text{sym}} (s) - \chi_{\text{alt}} (s) = \sum_i \lambda_i^2 = \text{tr} \rho s^2 = \chi_\rho (s^2).$$

On the other hand by properties (3) and (4)

$$\chi_{\text{sym}} (s) + \chi_{\text{alt}} (s) = \chi_{\rho \otimes \rho} (s) = \chi_\rho^2 (s).$$

Thus, we get

$$\chi_{\text{sym}} (s) = \frac{\chi_\rho^2 (s) + \chi_\rho (s^2)}{2}, \quad \chi_{\text{alt}} (s) = \frac{\chi_\rho^2 (s) - \chi_\rho (s^2)}{2}.$$
Starting from this point we assume that $G$ is finite and $k$ is algebraically closed of characteristic 0.

Introduce the non-degenerate symmetric bilinear form on the space of functions $\mathcal{F}(G)$ by the formula

$$(f, g) = \frac{1}{|G|} \sum_{s \in G} f(s^{-1}) g(s).$$

**Theorem 1.1.** Let $\rho, \sigma$ be irreducible. If $\rho$ and $\sigma$ are not isomorphic, then $(\chi_\rho, \chi_\sigma) = 0$. If $\rho$ and $\sigma$ are isomorphic, then $(\chi_\rho, \chi_\sigma) = 1$.

**Proof.** Let $V$ be the space of the representation $\rho$ and $W$ be the space of $\sigma$. Denote $n = \dim V$, $m = \dim W$. Choose a basis $v_1, \ldots, v_n$ in $V$, $w_1, \ldots, w_m$ in $W$. Define $P(i, j): W \to W$ by

$$P(i, j) v_k = \delta_{jk} w_i.$$ 

**Lemma 1.2.** For any $T \in \text{Hom}_k(V, W)$ let

$$\bar{T} = \frac{1}{|G|} \sum_{s \in G} \sigma_s \circ T \circ \rho_s^{-1}.$$ 

Then $\bar{T} \in \text{Hom}_G(V, W)$. If $V = W$, then $\text{tr} T = \text{tr} \bar{T}$.

**Proof.** Direct calculations. \hfill \Box

For any $T \in \text{Hom}(V, W)$ let $T_{kl}$ denote the corresponding matrix entry. For example, $P(i, j)_{kl} = \delta_{ik} \delta_{jl}$. Then

$$\bar{P}(i, j)_{kl} = \frac{1}{|G|} \sum_{s \in G} (\sigma_s)_{ki} (\rho_s^{-1})_{jl}.$$ 

If $\sigma$ and $\rho$ are not isomorphic, then by Schur’s Lemma

$$\bar{P}(i, j)_{kl} = 0$$

for all $i, j, k, l$. In particular, $\bar{P}(i, j)_{ij} = 0$ and therefore

$$\sum_{i=1}^m \sum_{j=1}^n \bar{P}(i, j)_{ij} = \frac{1}{|G|} \sum_{i=1}^m \sum_{j=1}^n \sum_{s \in G} (\sigma_s)_{ii} (\rho_s^{-1})_{jj} = 0.$$ 

But

$$\sum_{i=1}^m \sum_{j=1}^n (\sigma_s)_{ii} (\rho_s^{-1})_{jj} = \chi_\sigma(s) \chi_\rho(s^{-1}).$$

Hence

$$\frac{1}{|G|} \sum_{s \in G} \chi_\sigma(s) \chi_\rho(s^{-1}) = (\chi_\rho, \chi_\sigma) = 0.$$ 

Let now $\rho \cong \sigma$. The by Property (2), we may assume $\rho = \sigma$. Then by Schur’s Lemma

$$\bar{P}(i, j) = \lambda \text{Id}.$$
Since $\text{tr} \bar{P}(i, j) = \text{tr} P(i, j) = \delta_{ij}$, we have
\[
\bar{P}(i, j) = \frac{\delta_{ij}}{n} \text{Id}.
\]
Then
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \bar{P}(i, j)_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\delta_{ij}}{n} = 1,
\]
which implies $(\chi_\rho, \chi_\rho) = 1$. \hfill \Box

**Corollary 1.3.** Let $\rho = m_1 \rho_1 \oplus \cdots \oplus m_l \rho_l$ be decomposition into the sum of irreducibles. Then $m_i = (\chi_\rho, \chi_{\rho_i})$.

The number $m_i$ is called the **multiplicity** of an irreducible representation $\rho_i$ in $\rho$.

**Corollary 1.4.** A representation $\rho$ is irreducible iff $(\chi_\rho, \chi_\rho) = 1$.

**Corollary 1.5.** Every irreducible representation $\rho$ appears in the regular representation with multiplicity $\dim \rho$.

**Proof.**
\[
(\chi_\rho, \chi_R) = \frac{1}{|G|} \chi_\rho(1) \chi_R(1) = \dim \rho.
\]
\hfill \Box

**Corollary 1.6.** Let $\rho_1, \ldots, \rho_l$ be all (up to isomorphism) irreducible representations of $G$ and $n_i = \dim \rho_i$. Then
\[
n_1^2 + \cdots + n_l^2 = |G|.
\]

**Example 3.** Let $G$ act on a finite set $X$ and
\[
k(X) = \left\{ \sum_{x \in X} b_x x \mid b_x \in k \right\}.
\]
Define $\rho : G \to \text{GL}(k(X))$ by
\[
\rho_s \sum_{x \in X} b_x x = \sum_{x \in X} b_x s(x).
\]
It is easy to check that $\rho$ is a representation and
\[
\chi_\rho(s) = |\{ x \in X \mid s(x) = x \}|.
\]
Clearly, $\rho$ contains a trivial subrepresentation. To find the multiplicity of a trivial representation in $\rho$ we have to calculate $(1, \chi_\rho)$.
\[
(1, \chi_\rho) = \frac{1}{|G|} \sum_{s \in G} \chi_\rho(s) = \frac{1}{|G|} \sum_{s \in G} \sum_{s(x) = x} 1 = \frac{1}{|G|} \sum_{x \in X} \sum_{s \in G_x} 1 = \frac{1}{|G|} \sum_{x \in X} |G_x|,
\]
where
\[ G_x = \{ s \in G \mid s(x) = x \}. \]

Let \( X = X_1 \cup \cdots \cup X_m \) be the disjoint union of orbits. Then \(|G_x| = \frac{|G|}{|X_i|}\) for each \( x \in X_i \) and therefore
\[
(1, \chi) = \frac{1}{|G|} \sum_{i=1}^{m} \sum_{x \in X_i} \frac{|G|}{|X_i|} = m.
\]

Now let us evaluate \((\chi_\rho, \chi_\rho)\).
\[
(\chi_\rho, \chi_\rho) = \frac{1}{|G|} \sum_{s \in G} \left( \sum_{s(x)=x} 1 \right)^2 = \frac{1}{|G|} \sum_{s \in G} \sum_{s(x)=x, s(y)=y} 1 = \frac{1}{|G|} \sum_{(x,y) \in X \times X} |G_x \cap G_y|.
\]

Let \( \sigma \) be the representation associated with the action of \( G \) on \( X \times X \). Then the last formula implies
\[
(\chi_\rho, \chi_\rho) = (1, \chi_\sigma).
\]

Thus, \( \rho \) is irreducible iff \(|X| = 1\), and \( \rho \) has two irreducible components iff the action of \( G \) on \( X \times X \) with removed diagonal is transitive or \(|X| = 2\).

Let
\[
C(G) = \{ f \in \mathcal{F}(G) \mid f(sts^{-1}) = f(t) \}.
\]

It is easy to check that the restriction of \( (,) \) on \( C(G) \) is non-degenerate.

**Theorem 1.7.** The characters of irreducible representations of \( G \) form an orthonormal basis in \( C(G) \).

**Proof.** We have to show that if \( f \in C(G) \) and \((f, \chi_\rho) = 0\) for any irreducible \( \rho \), then \( f = 0\). The following lemma is straightforward.

**Lemma 1.8.** Let \( \rho : G \to \text{GL}(V) \) be a representation, \( f \in C(G) \) and
\[
T = \frac{1}{|G|} \sum_{s \in G} f(s^{-1}) \rho_s.
\]

Then \( T \in \text{End}_G V \) and \( \text{tr} T = (f, \chi_\rho) \).

Thus, for any irreducible \( \rho \) we have
\[
(1.1) \quad \frac{1}{|G|} \sum_{s \in G} f(s^{-1}) \rho_s = 0.
\]

But then the same is true for any representation \( \rho \), since any representation is a direct sum of irreducibles. Apply \((?\text{equ}?)\) for the case when \( \rho = R \) is the regular representation. Then
\[
\frac{1}{|G|} \sum_{s \in G} f(s^{-1}) R_s 1 = \frac{1}{|G|} \sum_{s \in G} f(s^{-1}) s = 0.
\]

Hence \( f(s^{-1}) = 0 \) for all \( s \in G \), i.e. \( f = 0\) \( \square \).
Corollary 1.9. The number of isomorphism classes of irreducible representations equal the number of the conjugacy classes.