1. Applications of quivers

Two rings $A$ and $B$ are Morita equivalent if the categories of $A-$ modules and $B$-modules are equivalent. A projective finitely generated $A$-module $P$ is a projective generator if any other projective finitely generated $A$-module is isomorphic to a direct summand of $P^\oplus n$ for some $n$.

**Theorem 1.1.** $A$ and $B$ are Morita equivalent iff there exists a projective generator $P$ in $A -$ mod such that $B \cong \text{End}_A (P)$. The functor $X \mapsto \text{Hom}_A (P, X)$ establishes the equivalence between $A -$ mod and $B -$ mod .

For the proof see, for example, Bass “Algebraic K-theory”.

Assume now that $C$ is a finite-dimensional algebra over algebraically closed field $k$. Let $P_1, \ldots , P_n$ be a set of representatives of isomorphism classes of indecomposable projective $C$-modules. Then $P = P_1 \oplus \cdots \oplus P_n$ is a projective generator, and $A = \text{End}_C (P)$ is Morita equivalent to $C$.

**Example 1.2.** Let $C$ be semisimple, then $C \cong \text{Mat}_{m_1} (k) \times \cdots \times \text{Mat}_{m_n} (k)$, and $A \cong k^n$. Let

$$C = \left\{ \begin{pmatrix} XY \\ 0Z \end{pmatrix} \in \text{Mat}_{p+q} (k) \mid X \in \text{Mat}_{p} (k), Y \in \text{Mat}_{p,q} (k), Z \in \text{Mat}_{q} (k) \right\} .$$

Then

$$A = \left\{ \begin{pmatrix} xy \\ 0z \end{pmatrix} \mid x, y, z \in k \right\} .$$

Let $R$ be the radical of $C$. Then each indecomposable projective $P_i$ has the filtration $P_i \supset R P_i \supset R^2 P_i \supset \cdots \supset 0$ such that $R^i P_i / R^{i+1} P_i$ is semisimple for all $j$. Recall that $P_i / R P_i$ is simple (lecture notes 9), hence $\text{Hom}_C (P_i, P_j / R P_j) = 0$ if $i \neq j$. Define the quiver $Q$ in the following way. Vertices are enumerated by indecomposable projective modules $P_1, \ldots , P_n$, the number of arrows $i \to j$ equals $\dim \text{Hom}_C (P_i, R P_j / R^2 P_j)$. We construct a surjective homomorphism $\phi: k (Q) \to A$. (This construction is not canonical). First set $\phi (e_i) = \text{Id}_{P_i}$. Let $\gamma_1, \ldots , \gamma_s$ be the set of arrows from $i$ to $j$, choose a basis $\eta_1, \ldots , \eta_s \in \text{Hom}_C (P_i, R P_j / R^2 P_j)$, each $\eta_i$ can be lifted to $\xi_i \in \text{Hom}_C (P_i, R P_j)$ as $P_i$ is projective. Define $\phi (\gamma_i) = \xi_i$. Now $\phi$ extends in the unique way to the whole $k (Q)$ since $k (Q)$ is generated by idempotents $e_i$ and arrows.

Since $\phi$ is surjective, then $A \cong k (Q) / I$ for some two-sided ideal $I \subset k (Q)$. The pair $Q$ and an ideal $I$ in $k (Q)$ is called a quiver with relations. The problem of classification of indecomposable $C$-modules is equivalent to the problem of classification.
of indecomposable representations of $Q$ satisfying relations $I$. In some cases such quiver approach is very useful.

**Example 1.3.** Let $k$ be the algebraic closure of $\mathbb{F}_3$ and $C = k[S_3]$. In lecture notes 9 we showed that $C$ has two indecomposable projectives $P_+ = \text{Ind}_{S_2}^{S_3} \text{triv}$ and $P_- = \text{Ind}_{S_2}^{S_3} \text{sgn}$. The quiver $Q$ is

$\bullet \iff \alpha \beta \bullet$

with relations $\alpha \beta \alpha = 0$, $\beta \alpha \beta = 0$. The quiver itself is $\hat{A}_2$, indecomposable representations have dimensions $(m, m)$, $(m + 1, m)$ and $(m, m + 1)$. Since we have the precise description, it is not difficult to see that only six indecomposable representations satisfy the relations. They are

$k \iff 0; 0 \iff k$; $k \iff k, \alpha = 1, \beta = 0$ or $\alpha = 0, \beta = 1$; $k \iff k^2, \alpha = (0) \beta = (10)$.

The first two representations correspond to irreducible representations triv and sgn, the last two are projectives. Two representations of dimension (1,1) correspond to the quotients of $P_+$ and $P_-$ by the minimal submodules.

In fact one can apply the quiver approach to any category $\mathcal{C}$ which satisfies the following conditions

(1) All objects have finite length;
(2) Any object has a projective resolution;
(3) For any two objects $X, Y$, $\text{Hom}(X, Y)$ is a vector space over an algebraically closed field $k$.

We do not need the assumption that the number of simple or projective objects is finite. We illustrate this in the following example.

**Example 1.4.** Let $\Lambda$ be the Grassmann algebra with two generators, i.e. $\Lambda = k < x, y > / (x^2, y^2, xy + yx)$. Consider the $\mathbb{Z}$-grading $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$, where $\Lambda_0 = k$, $\Lambda_1$ is the span of $x$ and $y$, $\Lambda_2 = kxy$. Let $\mathcal{C}$ denote the category of graded $\Lambda$-modules. In other words, objects are $\Lambda$-modules $M = \bigoplus_{i \in \mathbb{Z}} M_i$, such that $\Lambda_i M_j \subseteq M_{i+j}$ and morphisms preserve the grading. All projective modules are free. An indecomposable projective module $P_i$ is isomorphic to $\Lambda$ with shifted grading $\text{deg} (1) = i$. Thus, the quiver $Q$ has infinitely many vertices enumerated by $\mathbb{Z}$:

$\cdots \iff \alpha \beta_i \iff \alpha \beta_{i+1} \iff \alpha \beta_{i+2} \iff \alpha \beta_{i+3} \cdots$

Here $\alpha_{i+1}, \beta_{i+1} \in \text{Hom}(P_{i+1}, P_1)$, $\alpha_{i+1} (1) = x, \beta_{i+1} (1) = y$. Relations are $\alpha_i \alpha_{i+1} = \beta_i \beta_{i+1} = 0$, $\alpha_i \beta_{i+1} + \beta_i \alpha_{i+1} = 0$.

Let us classify the indecomposable representations of above quiver. Assume first that, that there exists $v \in X_{i+1}$ such that $\alpha_i \beta_{i+1} \neq 0$. Then the subrepresentation $V$ spanned by $v$, $\alpha_{i+1} v$, $\beta_{i+1} v$, $\alpha_i \beta_{i+1} v$ splits as a direct summand in $X$. If $X$ is indecomposable, then $X = V$. The corresponding object in $\mathcal{C}$ is $P_{i+1}$. 

Now assume that $\alpha_i \beta_{i+1} X_{i+1} = 0$ for any $i \in \mathbb{Z}$. That is equivalent to putting the new relations for $Q$: every path of length 2 is zero. Consider the subspaces

$$W_i = \text{Im} \alpha_{i+1} + \text{Im} \beta_{i+1} \subset X_i, \quad Z_{i+1} = \text{Ker} \alpha_{i+1} \cap \text{Ker} \beta_{i+1} \subset X_{i+1}.$$ 

One can find $U_i \subset X_i$ and $Y_i \subset X_{i+1}$ such that $X_i = U_i \oplus W_i$, $X_{i+1} = Z_{i+1} \oplus Y_{i+1}$.

Check that $W_i \oplus Y_{i+1}$ is a subrepresentation, which splits as a direct summand in $X$. If $X$ is indecomposable and $W_i \neq 0$, then $X = W_i \oplus Y_{i+1}$. Thus, we reduced our problem to Kronecker quiver $\bullet \leftrightarrow \bullet$. There is the obvious bijection between indecomposable non-projective objects from $C$ and the pairs $(Y, i)$, where $Y$ is an indecomposable representation of Kronecker quiver, $i \in \mathbb{Z}$ (defines the grading).

**Remark 1.5.** The last example is related to the algebraic geometry as the derived category of $C$ is equivalent to the derived category of coherent sheaves on $\mathbb{P}^1$.

**Remark 1.6.** If in the last example we increase the number of generators in $\Lambda$, then the problem becomes wild (definition below).

Let $C$ be a finite-dimensional algebra. We say that $C$ is **finitely represented** if $C$ has finitely many indecomposable representations. We call $C$ **tame** if for each $d \subset \mathbb{Z}_{>0}$, there exist a finite set $M_1, \ldots, M_r$ of $C - k[x]$ bimodules (free of rank $d$ over $k[x]$) such that every indecomposable representation of $C$ of dimension $d$ is isomorphic to $M_i \otimes_{k[x]} k[x] / (x - \lambda)$ for some $i \leq r$, $\lambda \in k$. Finally, $C$ is **wild** if there exists a $C - k < x, y >$ bimodule $M$ such that the functor $X \mapsto M \otimes_{k<x,y>} X$ preserves indecomposability and is faithful. We formulate here without proof the following results.

**Theorem 1.7.** Every finite-dimensional algebra over algebraically closed field $k$ is either finitely represented or tame or wild.

**Theorem 1.8.** Let $Q$ be a connected quiver without oriented cycles. Then $k(Q)$ is finitely represented iff $Q$ is Dynkin, $k(Q)$ is tame iff $Q$ is affine.

**Theorem 1.9.** Let $\text{Alg}_n$ be the algebraic variety of all $n$-dimensional algebras over $k$. Then the set of finitely represented algebras is Zariski open in $\text{Alg}_n$.

## 2. Frobenius algebras

Let $A$ be a finite-dimensional algebra over $k$. Recall that we denote by $D$ the functor $\text{mod} - A \to A - \text{mod}$, such that $D(X) = X^*$. Recall also that $D$ maps projective modules to injective and vice versa.

A finite-dimensional $A$ algebra over $k$ is called a **Frobenius algebra** if $D(A_A)$ is isomorphic to $A$, where $A_A$ is the right $A$-module over itself.

**Theorem 2.1.** The following conditions on $A$ are equivalent

1. $A$ is a Frobenius algebra;
2. There exists a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on $A$ such that $\langle ab, c \rangle = \langle a, bc \rangle$;
There exists $\lambda \in A^*$ such that $\text{Ker } \lambda$ does not contain non-trivial left or right ideals.

**Proof.** A form $\langle \cdot, \cdot \rangle$ gives an isomorphism $\mu : A \rightarrow A^*$ by the formula $x \rightarrow \langle \cdot, x \rangle$. The condition $\langle ab, c \rangle = \langle a, bc \rangle$ is equivalent to $\mu$ being a homomorphism of modules. A linear functional $\lambda$ can be constructed by $\lambda(x) = \langle 1, x \rangle$. Conversely, given $\lambda$, one can define $\langle x, y \rangle = \lambda(xy)$. The condition $\text{Ker } \lambda$ does not contain non-trivial one-sided ideals is equivalent to the condition that the left and right kernels of $\langle \cdot, \cdot \rangle$ are zero. \qed

**Lemma 2.2.** Let $A$ be a Frobenius algebra. An $A$-module $X$ is projective iff it is injective.

**Proof.** A projective module $X$ is a direct summand of a free module, but a free module is injective as $D(A_A)$ is isomorphic to $A$. Hence, $X$ is injective. By duality an injective module is projective. \qed

**Example 2.3.** A group algebra $k(G)$ is Frobenius. Take

$$\lambda \left( \sum_{g \in G} a_g g \right) = a_1.$$ 

The corresponding bilinear form is symmetric.

A Grassmann algebra $\Lambda = k < x_1, \ldots, x_n > / (x_ix_j + x_jx_i)$ is Frobenius. Put

$$\lambda \left( \sum_{i_1 < \cdots < i_k} c_{i_1 \cdots i_k} x_{i_1} \cdots x_{i_k} \right) = c_{12\ldots n}.$$ 

In a sense Frobenius algebras generalize group algebras. For example, if $T \in \text{Hom}_k(X, Y)$ for two $k(G)$-modules $X$ and $Y$ then

$$\bar{T} = \sum_{g \in G} gTg^{-1} \in \text{Hom}_G(X, Y).$$

This idea of taking average over the group is very important in representation theory. It has an analog for Frobenius algebras.

Choose a basis $e_1, \ldots, e_n$ in a Frobenius algebra $A$. Let $f_1, \ldots, f_n$ be the dual basis, i.e.

$$(2.1) \quad \langle f_i, e_j \rangle = \delta_{ij}.$$ 

Every $a \in A$ can be written

$$a = \sum \langle f_i, a \rangle e_i = \sum \langle a, e_i \rangle f_i.$$ 

and

$$\sum ae_i \otimes f_i = \sum \langle f_j, ae_i \rangle e_j \otimes f_i = \sum \langle f_ja, e_i \rangle e_j \otimes f_i = \sum e_j \otimes f_ja.$$ 

**Lemma 2.4.** Let $X$ and $Y$ be $A$-modules, $T \in \text{Hom}_k(X, Y)$. Then $\bar{T} = \sum e_i T f_i \in \text{Hom}_A(X, Y)$. 

**Proof.** Direct calculation using (2.2) and (2.3).

□

**Example 2.5.** If $A = k(G)$, the dual bases can be chosen as $\{g\}_{g \in G}$ and $\{g^{-1}\}_{g \in G}$. Hence $T = \sum gTg^{-1}$.

In Frobenius algebra one can use the following criterion of projectivity.

**Theorem 2.6.** An $A$-module $X$ is injective (hence projective) if there exists $T \in \text{End}_k(X)$ such that $\bar{T} = \text{Id}$.

**Proof.** First, assume the existence of $T$. We have to show that $X$ is injective, in other words, for any embedding $\varepsilon: X \rightarrow Y$ there exists $p \in \text{Hom}_k(Y, X)$ such that $p \circ \varepsilon = \text{Id}$. Put $\pi = \sum e_i Tp f_i$. Then for any $x \in X$ we have

$$\pi(\varepsilon(x)) = \sum e_i Tp f_i (\varepsilon(x)) = \sum e_i T(p \varepsilon(f_i x)) = \sum e_i T(f_i x) = \bar{T} x = \text{Id}.$$ 

Here we use $f_i \varepsilon = \varepsilon f_i$. By Lemma 2.4 $\pi \in \text{Hom}_A(X, Y)$.

Now assume that $X$ is injective. Define the map $\delta: X \rightarrow A \otimes_k X$ by the formula

$$f(x) = \sum e_i \otimes f_i x.$$ 

Then $f \in \text{Hom}_A(X, A \otimes_k X)$ by (2.3). It is obvious that $f$ is injective. Thus, we may consider $X$ as a submodule of $X$, moreover $X$ is a direct summand because $X$ is injective. So we have a projector $\tau: A \otimes_k X \rightarrow X$. Let $S \in \text{Hom}_k(A \otimes_k X, A \otimes_k X)$ be defined by the formula

$$S(a \otimes x) = \langle 1, a \rangle 1 \otimes x.$$ 

Then

$$\bar{S}(a \otimes x) = \sum e_i S(f_i a \otimes x) = \sum \langle 1, f_i a \rangle e_i \otimes x = \sum \langle f_i, a \rangle e_i \otimes x = a \otimes x$$

due to (2.2). Put $T = \tau \circ S \circ \delta$. Then $\bar{T} = \text{Id}$. □

3. **Relative projective and injective modules in group algebra**

Let $H$ be a subgroup of a group $G$. A $k(G)$-module $X$ is $H$-injective if any exact sequence of $k(G)$-modules

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0,$$

which splits over $k(H)$, splits over $k(G)$.

In the similar way one defines $H$-projective module.

Let $\{g_1, \ldots, g_r\}$ be a set of representatives in the set of left cosets $G/H$. For any $k(G)$-modules $X, Y$, and $T \in \text{Hom}_H(X, Y)$ put

$$\bar{T} = \sum g_i Tg_i^{-1}.$$ 

Prove yourself the following
Lemma 3.1. $\bar{T}$ does not depend on a choice of representatives and $\bar{T} \in \text{Hom}_G(X,Y)$.

Theorem 3.2. The following conditions on $k(G)$-module $X$ are equivalent

1. $X$ is $H$-injective;
2. $X$ is a direct summand in $\text{Ind}_H^G X$;
3. $X$ is $H$-projective;
4. There exists $T \in \text{End}_H(X)$ such that $\bar{T} = \text{Id}$.

Proof. This theorem is very similar to Theorem 2.6. To prove $1 \Rightarrow 2$ check that $\delta: X \to \text{Ind}_H^G X$ defined by the formula
$$\delta(x) = \sum g_i \otimes g_i^{-1} x,$$
defines an embedding of $X$. By injectivity $X$ is a direct summand of $\text{Ind}_H^G X$.

To prove $3 \Rightarrow 2$ use the projection $\text{Ind}_H^G X \to X$ defined by $g \otimes x \mapsto gx$.
Now prove $2 \Rightarrow 4$. Define $S: \text{Ind}_H^G X \to \text{Ind}_H^G X$ by
$$S\left(\sum g_i \otimes x_i\right) = 1 \otimes x_1,$$
here we assume that $g_1 = 1$. Check that $S \in \text{End}_H(\text{Ind}_H^G X)$ and $\bar{S} = \text{Id}$. Then obtain $T = \tau \circ S \circ \delta$, where $\tau: \text{Ind}_H^G X \to X$ be the projection such that $\tau \circ \delta = \text{Id}$.

Prove yourself $4 \Rightarrow 1$ and $4 \Rightarrow 3$ similarly to the first part of the proof of Theorem 2.6.

The following corollary is important for us. Let $p$ be prime. Recall that if $|G| = p^s r$ with $(p, r) = 1$, then there exists a subgroup $P$ of order $p^s$. It is called a Sylow subgroup. Two Sylow $p$-subgroups are conjugate in $G$.

Corollary 3.3. Let $\text{char } k = p$ and $P$ be a Sylow $p$-subgroup. Then every $k(G)$-module $X$ is $P$-injective.

Proof. We have to check condition (4) from Theorem 3.2. But $r = [G : P]$ is invertible in $k$. So we can put $T = \frac{1}{r} \text{Id}$.

4. Finitely represented group algebras

Let $\text{char } k = p$, $|G| = p^s r$ with $(p, r) = 1$.

Lemma 4.1. Let $H$ be a cyclic $p$-group, i.e. $|H| = p^s$. Then there are exactly $p^s$ isomorphism classes of indecomposable representations of $H$ over $k$, exactly one for each dimension. More precisely each indecomposable $L_m$ of dimension $m \leq p^s$ is isomorphic to $k(H)/(g - 1)^m$, where $g$ is a generator of $H$.

Proof. Since $k(H) \cong k[\alpha]/\alpha^{p^s}$, where $\alpha = g - 1$, the corresponding quiver is the loop quiver with one relation $\alpha^{p^s} = 0$. Hence $\alpha$ is a nilpotent Jordan block of order $\leq p^s$. □
Theorem 4.2. If a Sylow $p$-subgroup of $G$ is cyclic, then $k(G)$ is finitely represented. Moreover, the number of indecomposable $k(G)$-modules is not greater than $|G|$.

Proof. By Corollary 3.3 every indecomposable $k(G)$-module is $P$-injective. Therefore, any indecomposable $X$ is a direct summand in $\text{Ind}^G_P L_i$ for some $i$. Clearly, the number of such direct summands is finite. Now we will obtain the upper bound on the number of indecomposable representations. Let $X$ be an indecomposable $k(G)$-module, then by injectivity of $X$, $X$ is a direct summand in $\text{Ind}^G_P X$. Decompose $X$ into a direct sum of indecomposable $k(P)$-modules, then $X$ must be a direct summand in $\text{Ind}^G_P L_i$ for some $P$-indecomposable summand $L_i$ of $X$. Hence $\dim X \geq \dim L_i = i$. So if $\dim X = i$, then $X$ can be realized as a summand in $\text{Ind}^G_P (L_j)$ for some $j \leq i$. To calculate the total number of non-isomorphic indecomposable $k(G)$-modules, we can count in each $\text{Ind}^G_P L_i$ only indecomposable $k(G)$-components of dimension $\geq i$ since others are realized in $\text{Ind}^G_P L_j$ for $j < i$. Since there is no more than $r$ such components for each $i$, the total number of non-isomorphic indecomposable $k(G)$-modules is not greater than $p^r = |G|$. □

Lemma 4.3. If $P$ is a non-cyclic $p$-group, then $P$ contains a normal subgroup $N$ such that $P/N \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Proof. If $P$ is abelian, the statement follows from the classification of finite abelian groups. If $P$ is not abelian, then $P$ has a non-trivial center $Z$, and $P/Z$ is not cyclic. The statement follows by induction on $|P|$. □

Lemma 4.4. The group $S = \mathbb{Z}_p \times \mathbb{Z}_p$ has an indecomposable representation of dimension $n$ for each $n \in \mathbb{Z}_{\geq 0}$.

Proof. Let $g$ and $h$ be two generators of $S$, $\alpha = g - 1$, $\beta = h - 1$. Then $A = k(S)/(\alpha^2, \beta^2, \alpha \beta, \beta \alpha)$ is the subalgebra of $k(Q)$ for Kronecker quiver $Q$. In particular, one can see easily that every indecomposable representation of $Q$ remains indecomposable after restriction to $A$. This implies the Lemma. □

Theorem 4.5. If a $p$-Sylow subgroup of $G$ is not cyclic, then $G$ has an indecomposable representation of arbitrary high dimension.

Proof. By Lemma 4.3 and Lemma 4.4, $P$ has an indecomposable representation $Y$ of dimension $n$ for any positive integer $n$. Decompose $\text{Ind}^G_P Y$ into direct sum of indecomposable $k(G)$-modules. At least one component $X$ contains $Y$ as an indecomposable $k(P)$ component. Hence $\dim X \geq n$. □

Corollary 4.6. The group algebra $k(G)$ is finitely represented over a field of characteristic $p$ iff a Sylow $p$-subgroup of $G$ is cyclic.