REPRESENTATION THEORY.
WEEK 13.

1. Preprojective and preinjective representations.

The goal of these notes is to obtain a classification of indecomposable representations of affine quivers. It is rather technical and long.

A representation X is called *preprojective* if $\left(\Phi^{+}\right)^{s} X=0$ for some s, *preinjective* if $\left(\Phi^{-}\right)^{s}(X)=0$ for some s and *regular* if it is not preprojective or preinjective.

Example 1.1. For Kronecker quiver all preinjective indecomposable representations have dimension $\alpha_1+m\delta$, all preprojective representations have dimension $-\alpha_1+m\delta$, and regular indecomposable representations have dimension $m\delta$.

Lemma 1.2. *If X is preprojective, then $X=\left(\Phi^{-}\right)^{s} P$ for some projective P. If X is preinjective, then $X=\left(\Phi^{+}\right)^{s} I$ for some injective I.*

Proof. Suppose $\left(\Phi^{+}\right)^{s} X \neq 0$, and $\left(\Phi^{+}\right)^{s+1} X=0$. Then

$$F_{i-1}^{+} \dots F_{1}^{+}\left(\Phi^{+}\right)^{s} X=L_i,\ X \cong\left(\Phi^{-}\right)^{s} F_{1}^{-} \dots F_{i-1}^{-}\left(L_i\right)$$

as in Corollary 4.3 (week 12). Therefore by Corollary 5.4 (week12)

$$X \cong\left(\Phi^{-}\right)^{s}\left(A e_i\right).$$

For preinjective similarly. □

Corollary 1.3. *If X is an indecomposable preprojective or preinjective, then* $\dim X$ *is a real root. If X and Y are preprojective indecomposable representations of the same dimension, then $X \cong Y$. An indecomposable preprojective representation is a brick with trivial self-extensions.*

We see from above corollary that preprojective and preinjective indecomposables can be described precisely in terms of reflection functors in the same way as it was done for Dynkin quivers. The next lemma allows "to separate" preinjective, preinjective and regular indecomposable representations.

Lemma 1.4. *If X, Y are indecomposable and X is preprojective, Y is not, then* $\operatorname{Hom}_Q(Y,X)=\operatorname{Ext}^1(X,Y)=0$. *If X is preinjective, Y is not, then* $\operatorname{Hom}_Q(X,Y)=\operatorname{Ext}^1(Y,X)=0$.

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Proof. Let X be preprojective. Then $X = (\Phi^-)^s P$ for some projective P. Then

$$\operatorname{Ext}^1\left(\left(\Phi^-\right)^s P, Y\right) = \operatorname{Ext}^1\left(P, \left(\Phi^+\right)^s Y\right) = 0$$

by Corollary 5.2 (week12). On the other hand,

$$\left(\Phi^+\right)^{s+1} X = 0,\ Y \cong \left(\Phi^-\right)^{s+1}\left(\Phi^+\right)^{s+1} Y$$

and

$$\operatorname{Hom}_Q\left(X, \left(\Phi^-\right)^{s+1}\left(\Phi^+\right)^{s+1} Y\right) = \operatorname{Hom}_Q\left(\left(\Phi^+\right)^{s+1} X, \left(\Phi^+\right)^{s+1} Y\right) = 0.$$

For preinjective use duality. □

Let now Q be affine and define *defect* X by

$$\operatorname{def}(X) = \langle \delta, x\rangle = -\langle x, \delta\rangle.$$

We write $x \leq y$ if $y - x \in \mathbb{Z}^n_{\geq 0}$

Lemma 1.5. *If $x < \delta$ and X is indecomposable of dimension x, then X is a brick, x is a root and* $\operatorname{Ext}^1(X, X) = 0$.

Proof. If X is not a brick, then there is a brick $Y \subset X$ such that $\operatorname{Ext}^1(Y, Y) \neq 0$ (proven week 11). But then $q(y) \leq 0$, which is impossible as $y < \delta$. Hence X is a brick. Since $q(x) > 0$, we have $\operatorname{Ext}^1(X, X) = 0$ and $q(x) = 1$. □

Lemma 1.6. *There is an indecomposable representation of dimension δ.*

Proof. Pick an orbit O_Z in $\operatorname{Rep}(\delta)$ of maximal dimension. Then Z is indecomposable, because otherwise $Z = X_1 \oplus \cdots \oplus X_p$, where X_i are as in previous lemma, $\operatorname{Ext}^1(X_i, X_j) = 0$ and then $q(z) = p$. □

Lemma 1.7. *If X is regular, then there is s such that $c^s(x) = x$.*

Proof. One can find s such that $c^s(x) = x + l\delta$. But if $l \neq 0$, then $c^{ds}(x) < 0$ for some $d \in \mathbb{Z}$. This contradicts regularity of X. □

Theorem 1.8. *Let X be indecomposable.*

1. *If X is preprojective, then* $\operatorname{def}(X) < 0$*;*
2. *If X is regular, then* $\operatorname{def}(X) = 0$*;*
3. *If X is preinjective, then* $\operatorname{def}(X) > 0$.

Proof. Let X be preprojective, Z as in Lemma 1.6. Then $\operatorname{Ext}^1(X, Z) = 0$ by Lemma 1.4. On the other hand, $X = (\Phi^-)^s Ae_i$. Hence

$$\operatorname{Hom}_Q(X, Z) = \operatorname{Hom}_Q\left(\left(\Phi^-\right)^s Ae_i, Z\right) = \operatorname{Hom}_Q\left(Ae_i, \left(\Phi^+\right)^s Z\right) \neq 0,$$

and $\langle x, \delta\rangle > 0$. For preinjective X use duality.

Finally, let X be regular. Assume $\operatorname{def}(X) \neq 0$, say $\operatorname{def}(X) > 0$. Since x is regular $c^s(x) = x$ for some s. Then $y = x + c(x) + \cdots + c^{s-1}(x)$ is c-invariant, therefore $x + c(x) + \cdots + c^{s-1}(x) = m\delta$ by Lemma 3.2 (week 12). But $\langle \delta, c^i(x)\rangle = \langle \delta, x\rangle > 0$ for all $i < s$, hence $\langle \delta, m\delta\rangle > 0$. But $\langle \delta, \delta\rangle = q(\delta) = 0$. Contradiction. □

2. Regular representations

In this section we describe indecomposable regular representations.

We say that a representation is regular if it is a direct sum of indecomposable regular representations.

Theorem 2.1. *If X, Y are regular and $\varphi \in \mathrm{Hom}_Q(X, Y)$ then $\mathrm{Im}\,\varphi$, $\mathrm{Ker}\,\varphi$, $\mathrm{Coker}\,\varphi$ are regular. If*

$$0 \to X \to Z \to Y \to 0$$

is an exact sequence then Z is regular.

Proof. By Lemma 1.4 $\mathrm{Im}\,\varphi$ does not have preinjective summand and preprojective summand. By the same reason $\mathrm{Ker}\,\varphi$ does not have preinjective summand and def $(\mathrm{Ker}\,\varphi) = \mathrm{def}\,(X)\text{-def}\,(\mathrm{Im}\,\varphi) = 0$. Hence $\mathrm{Ker}\,\varphi$ is regular. Similarly, $\mathrm{Coker}\,\varphi$ is regular.

Finally, suppose Z has a preprojective direct summand Z_i. This is impossible by the long exact sequence

$$\mathrm{Hom}_Q(X, Z_i) = 0 \to \mathrm{Hom}_Q(Z, Z_i) \to \mathrm{Hom}_Q(Y, Z_i) = 0.$$

Similarly, Z could not have preinjective direct summand. $\square$

A regular representation X is called *regular simple* if X has no proper non-trivial regular subrepresentations. By Theorem 2.1 a regular simple representation is a brick, hence $q(x) \leq 1$ and x is a root.

Example 2.2. For Kronecker quiver a representation $k \rightrightarrows_{\lambda}^{\mu} k$ is simple for any $(\lambda, \mu) \neq (0, 0)$. One can see easily from classification of indecomposables that those are all regular simple representations.

Example 2.3. For the quiver $\widehat{D}_4$ an indecomposable representation

$$\begin{tikzcd}
 & k \arrow[d, "\tau_2"] & \\
k \arrow[r, "\tau_1"] & k^2 \arrow[r, "\tau_4"] & k \\
 & k \arrow[u, "\tau_3"] &
\end{tikzcd}$$

is regular simple iff $\mathrm{Im}\,\tau_i \neq \mathrm{Im}\,\tau_j$ for all $i \neq j$.

Let $\delta = a_0\alpha_0 + \dots + a_n\alpha_n$. Without loss of generality we may assume that $a_0 = 1$. Let $P = Ae_0$, $p = \dim P$ and R be the indecomposable preprojective representation of dimension $r = p + \delta$. There are the following identities

$$\langle p, \delta \rangle = \langle r, \delta \rangle = 1, \ \langle p, r \rangle = 2, \ \langle r, p \rangle = 0.$$

Since $\mathrm{Ext}^1(P, R) = 0$, $\mathrm{Hom}_Q(P, R) = k^2$.

Lemma 2.4. *Let $\theta \in \mathrm{Hom}_Q(P, R)$. If $\theta \neq 0$, then θ is injective. Let $\eta \in \mathrm{Hom}_Q(R, P)$. If $\eta \neq 0$ then η is surjective.*

Proof. Both $\operatorname{Ker}\theta$ and $\operatorname{Im}\theta$ are preprojective, because P and R are preprojective. Since $-1 = \operatorname{def}(P) = \operatorname{def}(\operatorname{Ker}\theta) + \operatorname{def}(\operatorname{Im}\theta)$, either $\operatorname{Ker}\theta$ or $\operatorname{Im}\theta$ is zero. The second statement is similar. ☐

Corollary 2.5. $\operatorname{Hom}_Q(R,P) = \operatorname{Ext}^1(R,P) = 0$.

Proof. Let $\eta \in \operatorname{Hom}_Q(R,P)$ be surjective, then $\eta = 0$ since P is projective and must split as a direct summand of R, but R is indecomposable. Hence $\operatorname{Hom}_Q(R,P) = 0$. Since $\langle r,p\rangle = 0$, $\operatorname{Ext}^1(R,P) = 0$. ☐

Corollary 2.6. *Let $\theta \in \operatorname{Hom}_Q(P,R)$, $\theta \neq 0$. Then $Z_\theta = \operatorname{Coker}\theta$ is indecomposable regular.*

Proof. Use the sequence

$$0 \to P \to R \to Z_\theta \to \ 0.$$

The long exact sequence

$$0 = \operatorname{Hom}_Q(R,P) \to \operatorname{Hom}_Q(R,R) \to \operatorname{Hom}_Q(R,Z_\theta) \to \operatorname{Ext}^1(R,P) = 0$$

implies $\operatorname{Hom}_Q(R,Z_\theta) = \operatorname{End}_Q(R) = k$. The long exact sequence

$$0 \to \operatorname{Hom}_Q(Z_\theta,Z_\theta) \to \operatorname{Hom}_Q(R,Z_\theta) = k \to \dots$$

implies $\operatorname{Hom}_Q(Z_\theta,Z_\theta) = k$. Hence Z_θ is indecomposable. Since $\operatorname{def}(Z_\theta) = 0$, Z_θ is regular. ☐

Lemma 2.7. *Let X be regular indecomposable and $x_0 \neq 0$, where $x_0 = \dim X_0$. Then there exists $\theta \in \operatorname{Hom}_Q(P,R)$ such that $\operatorname{Hom}_Q(Z_\theta,X) \neq 0$.*

Proof. First note that

$$\dim \operatorname{Hom}_Q(P,X) = x_0 = \dim(Q,X)$$

since $\langle p,x\rangle = \langle q,x\rangle = x_0$.

Any $\theta \in \operatorname{Hom}_Q(P,R)$ defines the linear map

$$\theta^*\colon \operatorname{Hom}_Q(R,X) \to \operatorname{Hom}_Q(P,X).$$

Since $\dim \operatorname{Hom}_Q(P,R) = 2$, one can find $\theta \in \operatorname{Hom}_Q(P,R)$ such that θ^* is not invertible. Then there is $\varphi \in \operatorname{Hom}_Q(R,X)$ such that $\theta^*(\varphi) = \varphi \circ \theta = 0$. Then $\varphi(\theta(P)) = 0$, and φ is well defined homomorphism $Z_\theta \to X$. ☐

Corollary 2.8. *Let X be regular simple, then $x \leq \delta$.*

Proof. We already know that x is a root. If $x_0 \neq 0$, then $\operatorname{Hom}_Q(Z_\theta,X) \neq 0$ for some θ and therefore X is a quotient of Z_θ, hence $x \leq \delta$. If $x_0 = 0$, then $x < \delta$. ☐

Example 2.9. In case of $\widehat{D}_4$ the regular simples have dimensions $\alpha_1 + \alpha_2 + \alpha_3$, $\alpha_1 + \alpha_2 + \alpha_4$, $\alpha_1 + \alpha_2 + \alpha_3$, $\alpha_3 + \alpha_2 + \alpha_4$, $\alpha_3 + \alpha_2 + \alpha_5$, $\alpha_4 + \alpha_2 + \alpha_5$ or δ. There is exactly one simple for each real root and one-parameter family for δ.

Our next step is to describe extensions between regular simple representations.

Lemma 2.10. *Let X and Y be two regular simples, then* $\mathrm{Hom}_Q(X,Y)=k$ *iff* $X\cong Y$ *and* $\mathrm{Ext}^1(X,Y)=k$ *iff* $Y\cong\Phi^+X$. *Otherwise* $\mathrm{Ext}^1(X,Y)=\mathrm{Hom}_Q(X,Y)=0$.

Proof. The statement about Hom is trivial since any nonzero $\varphi\in\mathrm{Hom}_Q(X,Y)$ is an isomorphism . To prove the statement about Ext^1, use (5.5) from lecture notes week 12. First, assume that $Y\ncong\Phi^+X, X$, then

$$\langle x,y\rangle=\dim\mathrm{Hom}_Q(X,Y)-\dim\mathrm{Ext}^1(X,Y)\leq 0,$$
$$\langle y,c(x)\rangle=\dim\mathrm{Hom}_Q\left(Y,\Phi^+X\right)-\dim\mathrm{Ext}^1\left(Y,\Phi^+X\right)\leq 0.$$

Since $\langle x,y\rangle+\langle y,c(x)\rangle=0$, $\mathrm{Ext}^1(X,Y)=0$.

Now assume that $X\cong Y$. If x is a real root, then $\mathrm{Ext}^1(X,X)=0$, and $\langle x,x\rangle=1$. Then $\langle x,c(x)\rangle=-1$, which implies $\mathrm{Ext}^1(X,\Phi^+X)=k$. If $x=\delta$, then

$$\mathrm{Hom}_Q(X,X)=\mathrm{Ext}^1(X,X)=k.$$

□

Corollary 2.11. *If X is regular simple and $x<\delta$, then $(\Phi^+)^s X\cong X$ for some s. If $x=\delta$, then $\Phi^+X\cong X$.*

The minimal number s such that $(\Phi^+)^s X\cong X$ is called the *period* of X. Regular simples can be divided in orbits under action of Φ^+.

In the category of regular representations one can define Jordan-Hölder series and regular length, and regular series is again unique up to permutation.

The following theorem gives a complete description of indecomposable regular representations.

Theorem 2.12. *Let X be regular indecomposable then there is a unique filtration*

$$0\subset X_1\subset X_2\subset\cdots\subset X_r=X \tag{2.1}$$

such that $Y_i\cong X_i/X_{i-1}$ is regular simple and

$$Y_{i-1}\cong\Phi^+(Y_i),\ \mathrm{Ext}^1(Y_r,X_{r-1})\cong\mathrm{Ext}^1(Y_r,Y_{r-1})\cong k.$$

Moreover, $\mathrm{Ext}^1(Z,X_{r-1})=0$ *for any regular simple Z not isomorphic to Y_r.*

Proof. We prove Theorem by induction on the regular length of X. Check yourself case $r=2$. If X has length r then it has a filtration (2.1), although it might be not unique. Assume first that X_{r-1} is indecomposable. Then it satisfies all the statements of Theorem by induction assumption. Consider the exact sequence

$$0\to X_{r-2}\to X_{r-1}\to Y_{r-1}\to 0$$

and induced long exact sequence

$$\mathrm{Hom}_Q(Y_r,X_{r-1})\xrightarrow{a}\mathrm{Hom}_Q(Y_r,Y_{r-1})\xrightarrow{b}\mathrm{Ext}^1(Y_r,X_{r-2})\xrightarrow{c}\mathrm{Ext}^1(Y_r,X_{r-1})\xrightarrow{d}\mathrm{Ext}^1(Y_r,Y_{r-1})\to 0.$$

We claim that d is an isomorphism. If $\mathrm{Ext}^1(Y_r,X_{r-2})=0$, then it is trivial. Assume that $\mathrm{Ext}^1(Y_r,X_{r-2})\neq 0$. By induction assumption $Y_r\cong\Phi^+Y_{r-2}\cong Y_{r-1}$. Then $a=0$

by uniqueness of filtration for X_{r-1}, b must be an isomorphism, c forced to be zero, and therefore d is an isomorphism.

Now we prove that X_{r-1} is indecomposable. Assume the opposite. Then $X_{r-1} = Z_1 \oplus \cdots \oplus Z_s$ where each Z_i is indecomposable and satisfies the statement of Theorem. Assume that Z_1 has maximal length among Z_i. Then there is a surjective homomorphism $p_i \colon Z_1 \to Z_i$ for each i, and this homomorphism induces the isomorphism

$$p_{i*} \colon \operatorname{Ext}^1(Y_r, Z_1) \to \operatorname{Ext}^1(Y_r, Z_i) \cong k.$$

Consider the exact sequence

$$0 \to \bigoplus Z_i \to X \to Y_r \to 0. \tag{2.2}$$

It is induced by some $\psi \in \operatorname{Ext}^1\left(Y_r, \bigoplus Z_i\right)$. But $\psi = \psi_1 + \cdots + \psi_s$, $\psi_i \in \operatorname{Ext}^1(Y_r, Z_i)$. Hence each $\psi_i = c_i p_{i*}(\psi_1)$. Let

$$Z' = \left\{ (z_1, \dots, z_s) \in \bigoplus Z_i \mid z_i = c_i p_i(z_1) \right\},$$

then one can find $X' \subset X$ such that

$$0 \to Z' \to X' \to Y_r \to 0,$$

is a subsequence of (2.2). Then X' splits as a summand in X. Contradiction.

Check now that X has a unique regular maximal submodule X_{r-1} (that implies the uniqueness of filtration). Consider the exact sequence

$$0 \to X_{r-1} \to X \xrightarrow{f} Y_r \to 0.$$

Let X' be another maximal submodule, then $f(X') = Y_r$ and we have an an exact sequence

$$0 \to X_{r-1} \cap X' \to X' \to Y_r \to 0.$$

However, the regular length of X' is $r-1$, hence $X_{r-1} \cap X' = X_{r-2}$. Therefore $X/X_{r-2} \cong Y_r \oplus Y_{r-1}$. But the sequence

$$0 \to Y_{r-1} \to X/X_{r-2} \to Y_r \to 0$$

does not split since it is induced by a non-zero element in $\operatorname{Ext}^1(Y_r, X_{r-1}) \cong \operatorname{Ext}^1(Y_r, Y_{r-1})$. Contradiction. □

Corollary 2.13. *Let $Y_1, \dots, Y_r$ be a sequnce of simple regular representations such that $Y_{i-1} \cong \Phi^+ Y_i$. Then there exists a unique up to an isomorphism regular indecomposable X with filtration $0 \subset X_1 \subset X_2 \subset \cdots \subset X_r = X$ such that $X_i/X_{i-1} \cong Y_i$.*

Proof. Construct X inductively using the isomorphism

$$\operatorname{Ext}^1(Y_{t+1}, X_t) \cong \operatorname{Ext}^1(Y_{t+1}, Y_t) \cong k.$$

□

As follows from Theorem 2.12 simple regular subquotients of an indecomposable regular representation belong to one orbit of Φ^+. Thus, each orbit of Φ^+ in the set of simple regular representations defines a family of indecomposables called a *tube*.

Lemma 2.14. *Let X be regular indecomposable, then $\dim X$ is a root.*

Proof. Lemma follows from Theorem 2.12 and the following fact. Let α, β be real roots. Then $(\alpha, \beta) = -1$ implies $\alpha + \beta$ is a real root, $(\alpha, \beta) = -2$ implies $\alpha + \beta$ is an imaginary root. □

Lemma 2.15. *Every tube contains exactly one indecomposable representation isomorphic to Z_θ.*

Proof. Let X be simple regular of period s, i.e. $(\Phi^+)^s X \cong X$. If $x = \dim X$, then

$$x + c(x) + \cdots + c^{s-1}(x) = m\delta. \tag{2.3}$$

Choose $y = c^i(x)$ such that $y_0 \neq 0$. Let $Y = (\Phi^+)^i X$. By Lemma 2.7 there exist $\theta \in \operatorname{Hom}_Q(P, R)$ and a non-zero homomorphism $\varphi : Z_\theta \to Y$. Then the indecomposable Z_θ has the filtration

$$0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z_s = Z_\theta$$

such that $Z_s / Z_{s-1} \cong Y$. Then Z_θ is in a tube. Moreover, one can see now that $m = 1$ in (2.3) and therefore Z_θ is unique. □

3. INDECOMPOSABLE REPRESENTATIONS OF AFFINE QUIVERS

In the next theorem we summarize our results about affine quivers.

Theorem 3.1. *Let Q be an affine quiver, then dimension of every indecomposable representation of Q is a root. If α is a real root, then there exists exactly one (up to an isomorphism) indecomposable representation of dimension α. If $\alpha = m\delta$, then there are infinitely many indecomposable representations of dimension α.*

Proof. Let α be the dimension of an indecomposable representation X. If $\langle \alpha, \delta \rangle \neq 0$, then X is preprojective or preinjective, and α is a real root by Corollary 1.3. If $\langle \alpha, \delta \rangle = 0$, then X is regular and α is a root by Lemma 2.14. The uniqueness of X follows from Theorem 2.12. We also have to prove that for each α there is an indecomposable of dimension α. If $\langle \alpha, \delta \rangle > 0$, choose the minimal i and s such that $r_i \dots r_1 c^s(\alpha) < 0$, then put $X = (\Phi^-)^s \circ F_1^- \circ \cdots \circ F_{i-1}^-(L_i)$. The case $\langle \alpha, \delta \rangle < 0$ is similar. Let $\langle \alpha, \delta \rangle = 0$. Assume first that $\alpha < \delta$. Construct X as an orbit of maximal dimension in $\operatorname{Rep}(\alpha)$. If $\alpha = \beta + m\delta$, for some $\beta < \delta$, construct an indecomposable Y of dimension β, and extend it using the description of a tube. □

Example 3.2. Consider the quiver $\hat{A}_n$. The indecomposable representations of real dimension and regular indecomposables of imaginary dimension with period greater than 1 are enumerated by counterclockwise walks around the quiver (ignoring the orientation). A basis $\{v_i\}$ in representation X is enumerated by vertices which appear in a walk. For each γ put $\rho_\gamma(v_i) = v_{i+1}$ if the orientation of γ is counterclockwise and $\rho_\gamma(v_{i+1}) = v_i$ if the orientation of γ is clockwise. The last vector in the walk goes to 0 if γ is counterclockwise oriented.

If X has imaginary dimension and X is in a tube of period 1, then X can be described by the following construction. Put $X_i \cong k^m$ for all i, $\rho_\gamma = \mathrm{Id}$ for all γ except one arrow σ (it does not matter which one you choose). Let ρ_σ be a Jordan block with non-zero eigenvalue.

Let Q and Q' two different quivers in the graph $\hat{\mathrm{A}}_n$. It is not always possible to obtain Q' from Q using reflection functors. If Q and Q' have the same number of clockwise (hence counterclockwise) arrows, one can obtain Q' from Q by a chain of reflections.

Check yourself that if Q has p counterclockwise arrows, then Q has one tube of period $p-1$ and one tube of period $n-p-1$.