1. Reflection functors

Let $Q$ be a quiver. We say a vertex $i \in Q_0$ is $\pm$-admissible if all arrows containing $i$ have $i$ as a target. If all arrows containing $i$ have $i$ as a source, we call $i$ $-$-admissible. By $\sigma_i (Q)$ we denote the quiver obtained from $Q$ by inverting all arrows containing $i$.

Let $i$ be a $\pm$-admissible vertex and $Q' = \sigma_i (Q)$. Let us introduce the functor $F_i^+ : \text{Rep}_Q \to \text{Rep}_{Q'}$. Let $X$ be a representation of $Q$. Define $X' = F_i^+ X$ as follows. If $j \neq i$, then $X'_j = X_j$. Put $X'_i = \ker h$, where

$$h = \sum_{\gamma = (j \to i) \in Q_1} \rho_\gamma : \bigoplus X_j \to X_i,$$

for each $\gamma = (i \to j) \in Q'$ define $\rho'_\gamma : X'_i \to X_j = X'_j$ as the natural projection on the component $X_j \in \bigoplus X_j$.

If $i$ is a $-$-admissible vertex and $Q' = \sigma_i (Q)$ one can define the functor $F_i^- : \text{Rep}_Q \to \text{Rep}_{Q'}$ as follows. Let $X' = F_i^- (X)$, where $X'_j = X_j$ for $i \neq j$, and $X'_i = \coker \tilde{h}$, where

$$\tilde{h} = \sum_{\gamma = (i \to j) \in Q_1} \rho_\gamma : X_i \to \bigoplus X_j,$$

and for each $\gamma = (j \to i) \in Q'$ define $\rho'_\gamma : X_j = X'_j \to X'_i$ by restriction of the projection $\bigoplus X_j \to \coker \tilde{h}$ to $X_j$.

Example. Let $Q$ be the quiver $1 \to 2$, and $X$ is the representation $k \to 0$, then $F_1^- (X) = 0$ and $F_2^+ (X)$ is $k \leftarrow k$.

It is easy to check that $F_i^+$ is left-exact (maps an injection to an injection) and $F_i^-$ is right exact (maps a surjection to a surjection). Let $L_i$ denote the representation of $Q$ which has $k$ in the vertex $i$ and zero in all other vertices. Then $F_i^+ (L_i) = 0$ and $F_i^- (L_i) = 0$.

Theorem 1.1. Let $X$ be an indecomposable representation of $Q$ and $i$ be a $\pm$-admissible vertex. Then $F_i^+ (X) = 0$ iff $X \cong L_i$. Otherwise $X' = F_i^+ (X)$ is indecomposable,

$$\dim X'_i = - \dim X_i + \sum_{j \to i} \dim X_j$$

and $F_i^- F_i^+ (X) \cong X$. 

Date: April 27, 2011.
Lemma 1.2. Let $Q$ be a connected graph without cycles, $Q$ and $Q'$ be two quivers on the same graph. Then there exists an enumeration of vertices such that $Q' = \sigma_k \circ \cdots \circ \sigma_1 (Q)$ and $i$ is a $-$-admissible vertex for $\sigma_{i-1} \circ \cdots \circ \sigma_1 (Q)$. Therefore, one can define the natural injection $h$ restricted to Coker $h$. In the similar way one can define the natural surjection $\psi : X \to F^{-i} F^{-i} X$ if $i$ is a $-$-admissible vertex.

Finally let $Q' = \sigma_i (Q)$, $X$ be a representation of $Q'$ and $Y$ be a representation of $Q$, $X' = F^{-i} X$ and $Y' = F^{-i} Y$. Let $\eta \in \text{Hom}_Q (X, Y')$, define $\chi \in \text{Hom}_{Q'} (X', Y)$ by putting $\chi_j = 1$ for $j \neq i$ and obtaining $\chi_i$ from following commutative diagram

\[
\begin{array}{c}
0 \to X' \xrightarrow{\tilde{h}} \bigoplus_{(j \to i) \in Q_1} X_j \xrightarrow{h} X_i \to 0 \\
0 \to Y' \xrightarrow{\tilde{h}} \bigoplus_{(j \to i) \in Q_1} Y_j \xrightarrow{h} Y_i \\
\text{(1.2)}
\end{array}
\]

Note for an arbitrary $X$ the sequence (1.2) is not exact but $\tilde{h}$ is injective and $h \circ \tilde{h} = 0$. Therefore one can define a natural injection $\phi : F^{-i} F^{-i} X \to X$, where $\phi_j = 1$ for all $j \neq i$ and $\phi_i$ coincides with $h$ restricted to Coker $\tilde{h}$. In the similar way one can define the natural surjection $\psi : X \to F^{-i} F^{-i} X$ if $i$ is a $-$-admissible vertex.

Proof. Note that if $X \not\cong \sigma_i (Q)$, then $h$ must be surjective because of indecomposability of $X$, hence the formula (1.1) holds. Furthermore, we have the following exact sequence

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Proof. It is sufficient to prove the statement for two quivers $Q$ and $Q'$ different at one arrow. So let $\gamma \in Q_1$. After removing $\gamma$, $Q$ splits in two connected components; let $Q''$ be the component which contains $t(\gamma)$. Enumerate vertices of $Q''$ in such a way that if $i \to j \in Q''$, then $i > j$. This is possible since $Q''$ does not have cycles. Check that $Q' = \sigma_k \circ \cdots \circ \sigma_1 (Q)$ (here $k$ is the number of all vertices in $Q''$) and $i$ is a $+$-admissible vertex for $\sigma_{i-1} \circ \cdots \circ \sigma_1 (Q)$. □

Theorem 2.2. Let $i$ be a $+$-admissible vertex for $Q$ and $Q' = \sigma_i (Q)$. Then $F_i^+$ and $F_i^-$ establish a bijection between indecomposable representations of $Q$ (non-isomorphic to $L_i$) and indecomposable representations of $Q'$ (non-isomorphic to $L_i$).

Theorem 2.2 follows from 1.1. Together with Lemma 2.1 it allows to change an orientation on a quiver if the quiver does not have cycles.

3. Weyl group and reflection functors.

Given any graph $\Gamma$, one can associate with it a certain linear group, which is called a Weyl group of $\Gamma$. We denote by $\alpha_1, \ldots, \alpha_n$ vectors in the standard basis of $\mathbb{Z}^\Gamma_0 = \mathbb{Z}^n$, $\alpha_i$ corresponds to the vertex $i$. These vectors are called simple roots. For each simple root $\alpha_i$ put

$$r_i(x) = x - 2 \frac{(x, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i.$$  

One can check that $r_i$ preserves the scalar product and $r_i^2 = id$. The linear transformation $r_i$ is called a simple reflection. If $\Gamma$ has no loops, $r_i$ also preserves the lattice generated by simple roots. Hence $r_i$ maps roots to roots. If $\Gamma$ is Dynkin, the scalar product is positive-definite, and $r_i$ is a reflection in the hyperplane orthogonal to $\alpha_i$.

The Weyl group $W$ is a group generated by $r_1, \ldots, r_n$. For a Dynkin diagram $W$ is finite (since the number of roots is finite).

Example. Let $\Gamma = A_n$. Let $\varepsilon_1, \ldots, \varepsilon_{n+1}$ be an orthonormal basis in $\mathbb{R}^{n+1}$. Then one can take the roots of $\Gamma$ to be $\varepsilon_i - \varepsilon_j$, simple roots to be $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_n - \varepsilon_{n+1}$, $r_i(\varepsilon_j) = 0$ if $j \neq i, i+1$, and $r_i(\varepsilon_i) = \varepsilon_{i+1}$. Therefore $W$ is isomorphic to the permutation group $S_{n+1}$.

One can check by direct calculation, that (1.1) implies

Lemma 3.1. If $X$ is an indecomposable representation of $Q$ and $\dim X = x \neq \alpha_i$, then $\dim F_i^\pm X = r_i(x)$.

An element $c = r_1 \ldots r_n \in W$ is called a Coxeter transformation. It depends on the enumeration of simple roots.

Example. In the case $\Gamma = A_n$ a Coxeter element is always a cycle of length $n + 1$.

Lemma 3.2. If $c(x) = x$, then $(x, \alpha_i) = 0$ for all $i$. In particular for a Dynkin graph $c(x) = x$ implies $x = 0$.

1We will denote by the same letter $L_i$ the representations of quivers with different orientation.
Proof. By definition,
\[ c(x) = x + a_1\alpha_1 + \cdots + a_n\alpha_n, \quad a_i = -\frac{2(\alpha_i, x + a_1\alpha_1 + \cdots + a_{i-1}\alpha_{i-1})}{(\alpha_i, \alpha_i)}. \]

The condition \( c(x) = x \) implies all \( a_i = 0 \). Hence \( (x, \alpha_i) = 0 \) for all \( i \).

## 4. Coxeter Functor.

Let \( Q \) be a graph without oriented cycles. We call an enumeration of vertices admissible if \( i > j \) for any arrow \( i \to j \). Such an enumeration always exists. One can easily see that every vertex \( i \) is a \(+\)-admissible for \( \sigma_{i-1} \circ \cdots \circ \sigma_1 (Q) \) and \(-\)-admissible for \( \sigma_{i+1} \circ \cdots \circ \sigma_n (Q) \). Furthermore,
\[ Q = \sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_1 (Q) = \sigma_1 \circ \cdots \circ \sigma_n (Q). \]

Define Coxeter functors
\[ \Phi^+ = F_n^+ \circ \cdots \circ F_2^+ \circ F_1^+, \quad \Phi^- = F_n^- \circ F_{n-1}^- \circ \cdots \circ F_1^- . \]

**Lemma 4.1.**
1. \( \text{Hom}_Q (\Phi^- X, Y) \cong \text{Hom}_Q (X, \Phi^+ Y) \);
2. If \( X \) is indecomposable and \( \Phi^+ X \neq 0 \), then \( \Phi^- \Phi^+ X \cong X \);
3. If \( X \) is indecomposable of dimension \( x \) and \( \Phi^+ X \neq 0 \), then \( \dim \Phi^+ X = c(x) \);
4. If \( Q \) is Dynkin, then for any indecomposable \( X \) there exists \( k \) such that 
\[ (\Phi^+)^k X = 0. \]

**Proof.** (1) follows from Lemma 1.2, (2) follows from Theorem 1.1, (3) follows from Lemma 3.1. Let us prove (4). Since \( W \) is finite, \( c \) has finite order \( h \). It is sufficient to show that for any \( x \) there exists \( k \) such that \( c^k (x) \) is not positive. Assume that this is not true. Then \( y = x + c(x) + \cdots + c^{h-1} (x) > 0 \) is \( c \) invariant. Contradiction with Lemma 3.2. \( \square \)

**Lemma 4.2.** \( \Phi^\pm \) does not depend on a choice of admissible enumeration.

**Proof.** Note that if \( i \) and \( j \) are disjoint and both \(+\,-\)-admissible, then 
\[ F_i^+ \circ F_j^+ = F_j^+ \circ F_i^+. \]
If a sequence \( i_1, \ldots, i_n \) gives another admissible enumeration of vertices, and \( i_k = 1 \), then \( 1 \) is disjoint with \( i_1, \ldots, i_{k-1} \), hence
\[ F_1^+ \circ F_{i_{k-1}}^+ \circ \cdots \circ F_{i_1}^+ = F_{i_{k-1}}^+ \circ \cdots \circ F_{i_1}^+ \circ F_1^+. \]

Now proceed by induction. Similarly for \( \Phi^- \). \( \square \)

**Corollary 4.3.** Let \( Q \) be a Dynkin quiver, \( X \) be an indecomposable representation of dimension \( x \), and \( k \) be the minimal number such that \( c^{k+1} (x) \) is not positive. There exists a unique vertex \( i \) such that
\[ x = c^{-k} r_1 \cdots r_{i-1} (\alpha_i), \quad X \cong (\Phi^-)^k \circ F_i^- \circ \cdots \circ F_{i-1}^- (L_i). \]
In particular, $x$ is a positive root and for each positive root $x$, there is a unique (up to an isomorphism) indecomposable representation of dimension $x$.

*Proof.* Follows from Theorem 1.1 and Lemma 3.1.

5. Further properties of Coxeter functors

Here we assume again that $Q$ is a quiver without oriented cycles and the enumeration of vertices is admissible. We discuss the properties of the bilinear form $\langle \cdot, \cdot \rangle$. Since we plan to change an orientation of $Q$ we use a subindex $\langle \cdot, \cdot \rangle_Q$, where it is needed to avoid ambiguity.

**Lemma 5.1.** Let $i$ be a $+$-admissible vertex, $Q' = \sigma_i(Q)$, and $\langle \cdot, \cdot \rangle_Q$, $\langle \cdot, \cdot \rangle_{Q'}$ the corresponding bilinear forms. Then

$$\langle r_i(x), y \rangle_{Q'} = \langle x, r_i(y) \rangle_Q.$$

*Proof.* It suffices to check the formula for a subquiver containing $i$ and all its neighbors. Let $x' = r_i(x)$ and $y' = r_i(y)$. Then

$$x_i' = -x_i + \sum_{i \neq j} x_j, \quad y_i' = -y_i + \sum_{i \neq j} y_j,$$

$$\langle x', y \rangle_{Q'} = x_i' y_i - x_i' \sum_{i \neq j} y_j + \sum_{i \neq j} x_j y_j = -x_i y_i' + \sum_{i \neq j} x_j y_j,$$

$$\langle x, y' \rangle_Q = x_i y_i' - y_i' \sum_{i \neq j} x_j = -x_i' y_i' + \sum_{i \neq j} x_j y_j.$$

*Corollary 5.2.* For a Coxeter element $c$ we have

$$\langle c^{-1}(x), y \rangle = \langle x, c(y) \rangle.$$  

If $\Phi^+(Y) \neq 0$, $\Phi^-(X) \neq 0$, then

$$\dim \text{Ext}^1(X, \Phi^+(Y)) = \dim \text{Ext}^1(\Phi^-(X), Y).$$

*Proof.* First statement follows directly from Lemma 5.1. The second statement follows from the first statement, Lemma 1.2 and the identity

$$\langle x, y \rangle_Q = \dim \text{Hom}_Q(X, Y) - \dim \text{Ext}^1(X, Y).$$

Let $A = k(Q)$ be the path algebra. Recall that any indecomposable projective module is isomorphic to $Ae_i$.

**Lemma 5.3.** $F_i^+ \circ \cdots \circ F_i^+ (Ae_i) = 0$, $F_i^{-1} \circ \cdots \circ F_1^+ (Ae_i) \cong L_i$. 


Proof. One can check by direct calculation that for each component \( e_j A e_i \), \( e_j A e_i = 0 \) for \( j > i \), and
\[
F^+_k \circ \cdots \circ F^+_1 (e_j A e_i) = e_j A e_i \text{ for } k < j, \quad F^+_j \circ \cdots \circ F^+_1 (e_j A e_i) = 0.
\]
\[\square\]

**Corollary 5.4.** \( \Phi^+ (P) = 0 \) for any projective module \( P \). For any indecomposable projective \( A e_i \) we have
\[
(5.1) \quad A e_i = F^-_1 \circ \cdots \circ F^-_{i-1} (L_i).
\]

**Proof.** The first statement follows from Lemma 5.3 immediately. For the second use Theorem 1.1 and Lemma 5.3. \[\square\]

An injective module is a module \( I \) such that for any injective homomorphism \( i : X \to Y \) and any homomorphism \( \varphi : X \to I \), there exists a homomorphism \( \psi : Y \to I \) such that \( \varphi = \psi \circ i \). A module \( I \) is injective iff \( \text{Ext}^1 (X, I) = 0 \) for any \( X \). One can see analogy with projective modules, however in general there is no nice description of injective (like a summand of a free module).

**Exercise.** Check that \( \mathbb{Q} \) is an injective \( \mathbb{Z} \)-module.

In case when \( A \) is a finite-dimensional algebra, injective modules are easy to describe. Indeed, the functor \( D : A \to \text{mod} \to \text{mod} \to A \) such that \( D(X) = X^* \) maps left projective modules to right injective and vice versa. Therefore any indecomposable injective module is isomorphic to \( (e_j A)^* \). Since \( D \circ \Phi^+ = \Phi^- \circ D \), one can see easily that \( \Phi^- (I) = 0 \) for any injective module \( I \). Moreover, one can prove similarly to the projective case that
\[
(e_j A)^* \cong F^+_n \circ \cdots \circ F^+_{j+1} (L_j).
\]

Let \( P (j) = A e_j \) and \( I (j) = (e_j A)^* \) and \( p (j) = \dim P (j), i (j) = \dim I (j) \). Then
\[
(5.2) \quad c (p (j)) = r_n \cdots r_1 (p (j)) = r_n \cdots r_{j+1} (-\alpha_j) = -i (j).
\]

Note that \( \text{Ext}^1 (A e_j, X) = 0 \) for any \( X \) and \( \dim \text{Hom}_Q (A e_j, X) = x_j \). Hence
\[
(5.3) \quad \langle p (j), x \rangle = x_j.
\]

On the other hand, \( \text{Ext}^1 (X, (e_j A)^*) = 0 \) and
\[
\text{Hom}_Q (X, (e_j A)^*) \cong \text{Hom}_Q (e_j A, X^*),
\]
which implies \( \dim \text{Hom}_Q (X, (e_j A)^*) = x_j \). Thus, we obtain
\[
(5.4) \quad \langle x, i (j) \rangle = x_j.
\]

Combine together (5.2), (5.3), (5.4) and get
\[
\langle p (j), x \rangle + \langle x, c (p (j)) \rangle = 0.
\]
Since \( p(1), \ldots, p(n) \) form a basis, the last equation implies that for arbitrary \( x \) and \( y \)
\[
(y, x) + (x, c(y)) = 0.
\]

6. Affine root system

Let \( \Gamma \) be an affine Dynkin graph. Then the kernel of bilinear symmetric form in \( \mathbb{Z}^n \) is one-dimensional and generated by
\[
\delta = a_0\alpha_0 + a_1\delta_1 + \cdots + a_n\delta_n.
\]
We assume without loss of generality that the vertex \( \alpha_0 \) is such that \( a_0 = 1 \). By removing 0 from \( \Gamma \) we get a Dynkin graph which we denote by \( \Gamma^0 \). In affine case roots can be of two kinds: real, if \( q(\alpha) = 1 \), or imaginary, \( q(\alpha) = 0 \).

Lemma 6.1. Imaginary roots are all proportional to \( \delta \), real roots can be written as \( \alpha + m\delta \) for some root \( \delta \) of \( \Gamma^0 \). Every real root can be obtained from a simple root by the action of the Weyl group \( W \).

Proof. The first statement is obvious, the second follows from the fact that \( q(\alpha) = q(\alpha + m\delta) \), hence the projection on the hyperplane generated by \( \alpha_1, \ldots, \alpha_n \) maps a root to a root. To prove the last statement, note that \( r_i \) maps every positive root different from \( \alpha_i \) to a positive root. Let \( \alpha \) be a positive real root, \( \alpha = a_0\alpha_0 + \cdots + a_n\alpha_n \), and \( h(\alpha) = a_0 + a_1 + \cdots + a_n \). Then \( (\alpha, \alpha_i) > 0 \) at least for one \( i \). But then \( h(r_i(\alpha)) < h(\alpha) \). Thus, one can decrease \( h(\alpha) \) by application of simple reflection. In the end one can get a root of height 1, which is a simple root. Similarly for negative roots.

7. Kronecker quiver

In this section we use Coxeter functors to classify indecomposable representation of the quiver \( \hat{A}_1 = \bullet \Rightarrow \bullet \). The admissible enumeration of vertices is \( 1 \Rightarrow 0 \), \( \delta = \alpha_0 + \alpha_1 \). Positive real roots are
\[
ma_1 + (m + 1)\alpha_0 = -\alpha_1 + (m + 1)\delta, \quad (m + 1)\alpha_1 + ma_0 = \alpha_1 + m\delta, \quad m \geq 0.
\]

The Coxeter element \( c = r_1r_0 \) satisfies
\[
c(\alpha_1) = \alpha_1 + 2\delta, \quad c(\delta) = \delta.
\]

Let \( x = ma_1 + l\delta \). If \( m > 0 \) then \( c^{-s}(x) \) is not positive for sufficiently large \( s \). Hence if \( X \) is indecomposable of dimension \( x \), then \( (\Phi^-)^s X = 0 \). If \( m < 0 \), then \( (\Phi^+)^s X = 0 \). Thus if \( m \neq 0 \), then as in the case of Dynkin quiver, \( X \) can be obtained from some \( L_i \) by application of reflection functor. In particular, we obtain that the dimension of an indecomposable representation is always a root and if this root is real, then the indecomposable with this dimension is unique up to an isomorphism. Indeed, we have either
\[
k^m \Rightarrow_A k^{m+1},
\]
where $A = (1_m, 0)$, $B = (0, 1_m)$, or

$$k^{m+1} \Rightarrow \mathcal{C} \mathcal{D} k^m,$$

where $C = A^t, D = B^t$.

Classification of indecomposables of dimension $m\delta$ is equivalent to classification of pairs of linear operators $(A, B) : k^m \to k^m$ up to equivalence $(A, B) \sim (PAQ^{-1}, PBQ^{-1})$. Assume that $A$ is invertible, then one may assume that $A = \text{Id}$, and then classify $B$ up to conjugation. Indecomposability of the representation implies that $B$ is equivalent to the Jordan block with some eigenvalue $\mu$. Denote the corresponding representation by $\rho_\mu$. If $B$ is invertible, then $A$ is equivalent to a Jordan block. Denote such representation by $\sigma_\mu$. One can see that $\rho_\mu$ is isomorphic to $\sigma_{\mu-1}$ if $\mu \neq 0$. Now let us prove that at least one of $A$ and $B$ is invertible. Indeed, indecomposability implies that $\text{Ker} \ A \cap \text{Ker} \ B = 0$. Hence $A + tB$ is invertible for some $t$. Without loss of generality one can assume that $A + tB = \text{Id}$. But then either $A$ or $B$ must be invertible. Thus, we proved that indecomposable representation of dimension $(m, m) = \delta$ are parameterized by a projective line.

For other affine quivers, the situation is more complicated, as there are real roots which remain positive under Coxeter transformation. For example consider the quiver $\hat{D}_4$

```
  5
 / \  \\
2   1 \ 4
 \ /  \\
  3
```

Then $c(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_4 + \alpha_1 + \alpha_5$, $c^2(\alpha_1 + \alpha_2 + \alpha_3) = \alpha_1 + \alpha_2 + \alpha_3$. 