REPRESENTATION THEORY.
WEEKS 10 – 11

1. Representations of quivers

I follow here Crawley-Boevey lectures trying to give more details concerning extensions and exact sequences.

A *quiver* is an oriented graph. If Q is a quiver, then we denote by Q_0 the set of vertices and by Q_1 the set of arrows. Usually we denote by n the number of vertices. If $\gamma\colon j \leftarrow i$ is an arrow then $i = s(\gamma)$, $j = t(\gamma)$.

Fix an algebraically closed field k. A representation V of a quiver is a collection of vector spaces $\{V_i\}_{i \in Q_0}$ and linear maps $\rho_\gamma : V_i \to V_j$ for each arrow $\gamma\colon i \to j$. For two representations of a quiver Q, ρ in V and σ in W define a homomorphism $\phi\colon V \to W$ as a set of linear maps $\phi_i\colon V_i \to W_i$ such that the diagram

$$\begin{array}{ccc} V_j & \stackrel{\rho_\gamma}{\longleftarrow} & V_i \\ \downarrow \phi_j & & \downarrow \phi_i \\ W_j & \stackrel{\sigma_\gamma}{\longleftarrow} & W_i \end{array}$$

is commutative. We say that two representations V and W are isomorphic if there is a homomorphism $\phi \in \mathrm{Hom}_Q(V, W)$ such that each ϕ_i is an isomorphism. One can define a subrepresentation and a direct sum of representation of Q in the natural way. A representation is *irreducible* if it does not have non-trivial proper subrepresentation and *indecomposable* if it is not a direct sum of non-trivial subrepresentations.

Example 1.1. Let Q be the quiver $\bullet \to \bullet$. A representation of Q is a pair of vector spaces V and W and a linear operator $\rho : V \to W$. Let $V_0 = \mathrm{Ker}\,\rho$, V_1 is such that $V = V_0 \oplus V_1$, $W_0 = \mathrm{Im}\,\rho$, and W_1 is such that $W = W_0 \oplus W_1$. Then $V_0 \to 0$, $V_1 \to W_0$ and $0 \to W_1$ are subrepresentations and ρ is their direct sum. Furthermore, $V_0 \to 0$ is the direct sum of $\dim V_0$ copies of $k \to 0$, $V_1 \to W_0$ is the direct sum of $\dim V_1$ copies of $k \to k$ and finally $0 \to W_1$ is the direct sum of $\dim W_1$ copies of $0 \to k$. Thus, we see that there are exactly three isomorphism classes of indecomposable representations of $Q, 0 \to k$, $k \to k$, $k \to 0$. The first and the last one are irreducible, $0 \to k$ is a subrepresentaion of $k \to k$ and $k \to 0$ is a quotient of $k \to k$ by $0 \to k$.

Date: December 4, 2005.

2. Path algebra

Given a quiver Q. A *path* p is a sequence $\gamma_1 \dots \gamma_k$ of arrows such that $s(\gamma_i) = t(\gamma_{i+1})$. Put $s(p) = s(\gamma_k)$, $t(p) = t(\gamma_1)$. Define a composition $p_1 p_2$ of two paths such that $s(p_1) = t(p_2)$ in the obvious way and we set $p_1 p_2 = 0$ if $s(p_1) \neq t(p_2)$. Introduce also elements e_i for each vertex $i \in Q_0$ and define $e_i e_j = \delta_{ij} e_i$, $e_i p = p$ if $i = t(p)$ and 0 otherwise, $p e_i = p$ if $i = s(p)$ and 0 otherwise. The *path algebra* $k(Q)$ is the set of k-linear combinations of all paths and e_i with composition extended by linearity from ones defined above.

One can easily check the following properties of a path algebra

1. $k(Q)$ is finite-dimensional iff Q does not have oriented cycles;
2. If Q is a disjoint union of Q_1 and Q_2, then $k(Q) = k(Q_1) \times k(Q_2)$;
3. The algebra $k(Q)$ has a natural $\mathbb{Z}$-grading $\oplus_{n=0}^{\infty} k(Q)_n$ defined by $\deg e_i = 0$ and the degree of a path p being the length of the path;
4. Elements e_i are primitive idempotents of $k(Q)$, and hence $k(Q)e_i$ is an indecomposable projective $k(Q)$-module.

The first three properties are trivial, let us check the last one. Suppose e_i is not primitive, then one can find an idempotent $\varepsilon \in k(Q)e_i$. Let $\varepsilon = c_0 e_i + c_1 p_1 + \dots + c_k p_k$, where $s(p_j) = i$ for all $j \leq k$. Then $\varepsilon^2 = \varepsilon$ implies $c_0 = 0$ or 1. Let $c_0 = 0$, $\varepsilon = \varepsilon_l + \dots$, where $\deg \varepsilon_l = l$ and other terms have degree greater than l. But then ε^2 starts with degree greater than $2l$, hence $\varepsilon = 0$. If $c_0 = 1$, apply the same argument to the idempotent $(e_i - \varepsilon)$.

Given a representation ρ of Q one can construct a $k(Q)$-module

$$V = \bigoplus_{i \in Q_0} V_i,\ e_i V_j = \delta_{ij} \operatorname{Id}_{V_j},\ \gamma v = \rho_\gamma v \text{ if } v \in V_{s(\gamma)},\ \gamma(v) = 0 \text{ otherwise.}$$

For any path $p = \gamma_1 \dots \gamma_k$ and $v \in V$ put $pv = \rho_{\gamma_1} \circ \dots \circ \rho_{\gamma_k}(v)$.

On the other hand, every $k(Q)$-module V defines a representation ρ of Q if one puts $V_i = e_i V$.

The following theorem is straightforward.

Theorem 2.1. *The category of representations of Q and the category of $k(Q)$-modules are equivalent.*

Lemma 2.2. *The radical of $k(Q)$ is spanned by all paths p satisfying the property that there is no return paths, i.e. back from $t(p)$ to $s(p)$.*

Proof. It is easy to see that the paths with no return span a two-sided ideal R. Note that $R^n = 0$, where n is the number of vertices. Thus, $R \subset \operatorname{rad} k(Q)$. On the other hand, let $y \notin R$ and p be a shortest path in decomposition of y which has a return path. Choose a shortest path s such that $\tau = sp$ is an oriented cycle. Consider the representation of Q which has k in each vertex of τ and 0 in all other vertices. Let $\rho_\gamma = \operatorname{Id}$, if γ is included in τ and $\rho_\gamma = 0$ otherwise. Let V be the corresponding $k(Q)$-module. Then V is simple, $sy(V) \neq 0$. Hence $y \notin \operatorname{rad} k(Q)$. Contradiction. □

Example 2.3. If Q has one vertex and n loops then $k(Q)$ is a free associative algebra with n generators. If Q does not have cycles, then $k(Q)$ is the subalgebra in $\mathrm{Mat}_n(k)$ generated by elementary matrices E_{ii} for each $i \in Q_0$ and E_{ij} for each arrow $i \to j$.

3. Standard resolution

Theorem 3.1. *Let Q be a quiver, $A = k(Q)$ and V be an A-module. Then the sequence*

$$0 \to \bigoplus_{\gamma=(i\to j)\in Q_1} Ae_j \otimes V_i \xrightarrow{f} \bigoplus_{i\in Q_0} Ae_i \otimes V_i \xrightarrow{g} V \to 0,$$

where $f(ae_j \otimes v) = ae_j\gamma \otimes v - ae_j \otimes \gamma v$, $g(ae_i \otimes v) = av$ for any $v \in V_i$, is exact. It is a projective resolution.

Proof. First, check that $g \circ f = 0$. Indeed,

$$g(f(ae_j \otimes v)) = g(ae_j\gamma \otimes v - ae_j \otimes \gamma v) = ae_j\gamma v - ae_j\gamma v = 0.$$

Since $V = \oplus e_i V_i$, g is surjective. To check that f is injective, introduce the grading on $A \otimes V$ using $\deg V = 0$. By grf denote the homogeneous part of highest degree for f. Note that the grf increases the degree by one and

$$grf = \oplus_{\gamma\in Q_1} f_\gamma, \text{ where } f_\gamma \colon Ae_j \otimes V_i \to A\gamma \otimes V_i \text{ is defined by}$$

$$f_\gamma(ae_j \otimes v_i) = ae_j\gamma \otimes v_i,$$

for $\gamma : i \to j$. One can see from this formula that f_γ is injective, therefore grf is injective and hence f is injective.

To prove that $\mathrm{Im}\, f = \mathrm{Ker}\, g$ note that

$$ae_j\gamma \otimes v \equiv ae_j \otimes \gamma v \mod \mathrm{Im}\, f,$$

therefore for any $x \in \bigoplus_{i\in Q_0} Ae_i \otimes V_i$

$$x \equiv x_0 \mod \mathrm{Im}\, f$$

for some x_0 of degree 0. In other words $x_0 \in \bigoplus_{i\in Q_0} ke_i \otimes V_i$. If $g(x) = 0$, then $g(x_0) = 0$, and if $g(x_0) = 0$, then obviously $x_0 = 0$. Hence $x \equiv 0 \mod \mathrm{Im}\, f$. □

Theorem 3.1 implies that $\mathrm{Ext}^1(X, Y)$ can be calculated as coker d of the following complex

$$0 \to \bigoplus_{i\in Q_0} \mathrm{Hom}_k(X_i, Y_i) \xrightarrow{d} \bigoplus_{\gamma=(i\to j)\in Q_1} \mathrm{Hom}_k(X_i, Y_j) \to 0, \tag{3.1}$$

where

$$d\phi(x) = \phi(\gamma x) - \gamma\phi(x) \tag{3.2}$$

for any $x \in X_i$, $\gamma = (i \to j)$.

Lemma 3.2. *Every* $\psi \in \mathrm{Ext}^1(X,Y)$ *induces a non-split exact sequence*

$$0 \to Y \to Z \to X \to 0.$$

If $\mathrm{Ext}^1(X,Y) = 0$, *then every exacts sequence as above splits.*

Proof. Let

$$0 \to Y \to Z \to X \to 0$$

be an exact sequence of representations of Q. Then Z_i can be identified with $X_i \oplus Y_i$ for every i. For every arrow $\gamma\colon i \to j$ the action on Z is defined by

$$\gamma(x,y) = (\gamma x, \gamma y + \psi_\gamma(x)),$$

for some $\psi_\gamma \in \mathrm{Hom}_k(X_i, Y_j)$. Thus, ψ can be considered as an element in the second non-zero term of (3.1). If the exact sequence splits, then there is $\eta \in \mathrm{Hom}_Q(X,Z)$ such that for each $i \in Q_0$, $x \in X_i$

$$\eta(x) = (x, \phi_i(x)),$$

for some $\phi_i \in \mathrm{Hom}_k(X_i, Y_i)$. Furthermore, $\eta \in \mathrm{Hom}_Q(X,Z)$ iff for each $\gamma\colon i \to j$

$$\gamma(x, \phi_i(x)) = (\gamma x, \gamma\phi_i(x) + \psi_\gamma(x)) = (\gamma x, \phi_j(\gamma x)),$$

which implies

$$\psi_\gamma(x) = \phi_j(\gamma x) - \gamma\phi_i(x).$$

In other words, $\psi = d\phi$. Thus, $\mathrm{Ext}^1(X,Y)$ parameterizes the set of non-split exact sequences

$$0 \to Y \to Z \to X \to 0.$$

□

Corollary 3.3. *In the category of representations of* Q, $\mathrm{Ext}^i(X,Y) = 0$ *for* $i \geq 2$.

Corollary 3.4. *Let*

$$0 \to Y \to Z \to X \to 0$$

be a short exact sequence of representations of Q, *then*

$$\mathrm{Ext}^1(V,Z) \to \mathrm{Ext}^1(V,X), \ \mathrm{Ext}^1(Z,V) \to \mathrm{Ext}^1(Y,V)$$

are surjective.

Lemma 3.5. *If* X *and* Y *are indecomposable and* $\mathrm{Ext}^1(Y,X) = 0$, *then every non-zero* $\varphi \in \mathrm{Hom}_Q(X,Y)$ *is either surjective or injective.*

Proof. Use the exact sequences

$$0 \to \mathrm{Ker}\,\varphi \to X \to \mathrm{Im}\,\varphi \to 0,$$

$$0 \to \mathrm{Im}\,\varphi \to Y \to S \cong Y/\mathrm{Im}\,\varphi \to 0. \tag{3.3}$$

The exact sequence (3.3) can be considered as an element $\psi \in \operatorname{Ext}^1(S, \operatorname{Im}\varphi)$ by use of Lemma 3.2. By Corollary 3.3 we have an isomorphism $g\colon \operatorname{Ext}^1(S, \operatorname{Im}\varphi) \cong \operatorname{Ext}^1(S, X)$. Then $g(\psi)$ induces the exact sequence

$$0 \to X \to Z \to S \to 0,$$

and this exact sequence together with (3.3) form the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X & \xrightarrow{\alpha} & Z & \to & S & \to & 0 \\ & & \downarrow^{\beta} & & \downarrow^{\gamma} & & \downarrow & & \\ 0 & \to & \operatorname{Im}\varphi & \xrightarrow{\delta} & Y & \to & S & \to & 0 \end{array}$$

here β and γ are surjective. We claim that the sequence

$$0 \to X \xrightarrow{\alpha+\beta} Z \oplus \operatorname{Im}\varphi \xrightarrow{\gamma-\delta} Y \to 0$$

is exact. Indeed, $\alpha+\beta$ is obviously injective and $\gamma-\delta$ is surjecitve. Finally, $\dim Z = \dim X + \dim S$, $\dim \operatorname{Im}\varphi = \dim Y - \dim S$. Therefore,

$$\dim(Z \oplus \operatorname{Im}\varphi) = \dim X + \dim Y,$$

and therefore $\operatorname{Ker}(\gamma-\delta) = \operatorname{Im}(\alpha+\beta)$.

But $\operatorname{Ext}^1(Y, X) = 0$. Hence the last exact sequence splits, $Z \oplus \operatorname{Im}\varphi \cong X \oplus \mathrm{Yand}$ by Krull-Schmidt theorem either $X \cong \operatorname{Im}\varphi$ or $Y \cong \operatorname{Im}\varphi$. □

Intoduce $\dim X$ as a vector $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$ where n is the number of vertices and $x_i = \dim X_i$. Define the bilinear form

$$\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{(i \to j) \in Q_1} x_i y_j = \dim \operatorname{Hom}_Q(X, Y) - \dim \operatorname{Ext}^1(X, Y)$$

(the equality follows from (3.1)). We also introduce the symmetric form

$$(x, y) = \langle x, y \rangle + \langle y, x \rangle$$

and the quadratic form

$$q(x) = \langle x, x \rangle.$$

4. Bricks

Here we discuss further properties of finite-dimensional representations of $A = k(Q)$.

Recall that if X is indecomposable and has finite length, then $\varphi \in \operatorname{End}_Q(X)$ is either isomorphism or nilpotent. Since we assumed that k is algebraically closed, $\varphi = \lambda \operatorname{Id}$ for any invertible $\varphi \in \operatorname{End}_Q(X)$. A representation X is a *brick*, if $\operatorname{End}_Q(X) = k$. If X is a brick, then X is indecomposable. If X is indecomposable and $\operatorname{Ext}^1(X, X) = 0$, then X is a brick due to Lemma 3.5.

Example 4.1. Consider the quiver $\bullet \to \bullet$. Then every indecomposable is a brick. For the Kronecker quiver $\bullet \rightrightarrows \bullet$ the representation $k^2 \rightrightarrows_{\beta}^{\alpha} k^2$ with $\alpha = \mathrm{Id}$, $\beta = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ is not a brick. Indeed, $\varphi = (\varphi_1, \varphi_2)$ where φ_1, φ_2 are matrices $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, belongs to $\mathrm{End}_Q(X)$.

Lemma 4.2. *Let X be indecomposable and not a brick, then X contains a brick W such that* $\mathrm{Ext}^1(W, W) \neq 0$.

Proof. Choose $\varphi \in \mathrm{End}_Q(X)$, $\varphi \neq 0$ of minimal rank. Since $\mathrm{rk}\,\varphi^2 < \mathrm{rk}\,\varphi$, $\varphi^2 = 0$. Let $Y = \mathrm{Im}\,\varphi$, $Z = \mathrm{Ker}\,\varphi$. Let $Z = Z_1 \oplus \cdots \oplus Z_p$ be a sum of indecomposables. Let $p_i : Z \to Z_i$ be the projection. Choose i so that $p_i(Y) \neq 0$ and let $\eta = p_i \circ \varphi \in \mathrm{End}_Q(X)$ (well defined since $\mathrm{Im}\,\varphi \in \mathrm{Ker}\,\varphi$). Note that by our assumption $\mathrm{rk}\,\eta = \mathrm{rk}\,\varphi$, therefore $p_i : Y \to Z_i$ is an embedding. Let $Y_i = p_i(Y)$. Then $\mathrm{Ker}\,\eta = Z$, $\mathrm{Im}\,\eta = Y_i$.

We claim now that $\mathrm{Ext}^1(Z_i, Z_i) \neq 0$. Indeed, $\mathrm{Ext}^1(Y_i, Z) \neq 0$ by exact sequence

$$0 \to Z \to X \xrightarrow{\eta} Y_i \to 0$$

and indecomposability of X. Then the induced exact sequence

$$0 \to Z_i \to X_i \xrightarrow{\eta} Y_i \to 0$$

does not split also. (If it splits, then Z_i is a direct summand of X, which is impossible). Therefore $\mathrm{Ext}^1(Y_i, Z_i) \neq 0$. But Y_i is a submodule of Z_i. By Corollary 3.4 we have the surjection

$$\mathrm{Ext}^1(Z_i, Z_i) \to \mathrm{Ext}^1(Y_i, Z_i).$$

If Z_i is not a brick, we repeat the above construction for Z_i e.t.c. Finally, we get a brick. □

Corollary 4.3. *Assume that the quadratic form q is positive definite. Then every indecomposable X is a brick with trivial* $\mathrm{Ext}^1(X, X)$*; moreover, if $x = \dim X$, then* $q(x) = 1$.

Proof. Assume that X is not a brick, then it contains a brick Y such that $\mathrm{Ext}^1(Y, Y) \neq 0$. Then

$$q(Y) = \dim \mathrm{End}_Q(Y) - \dim \mathrm{Ext}^1(Y, Y) = 1 - \dim \mathrm{Ext}^1(Y, Y) \leq 0,$$

but this is impossible. Therefore X is a brick. Now

$$q(x) = \dim \mathrm{End}_Q(X) - \dim \mathrm{Ext}^1(X, X) = 1 - \dim \mathrm{Ext}^1(X, X) \geq 0,$$

hence $q(x) = 1$ and $\dim \mathrm{Ext}^1(X, X) = 0$. □

5. Orbits in representation variety

Fix a quiver Q, recall that n denotes the number of vertices. Let $x = (x_1, \dots, x_n) \in \mathbb{Z}_{\geq 0}^n$. Define

$$\mathrm{Rep}(x) = \prod_{(i \to j) \in Q_1} \mathrm{Hom}_k\left(k^{x_i}, k^{x_j}\right).$$

It is clear that every representation of Q of dimension x is a point in $\mathrm{Rep}\,(x)$. Let

$$G = \prod_{i \in Q_0} \mathrm{GL}\left(k^i\right).$$

Then G acts on $\mathrm{Rep}\,(x)$ by the formula $g\varphi_{ij} = g_i \varphi g_j^{-1}$, for each arrow $i \to j$. Two representations of Q are isomorphic iff they belong to the same orbit of G. For a representation X we denote by O_X the corresponding G-orbit in $\mathrm{Rep}\,(x)$.

Note that

$$\dim \mathrm{Rep}\,(x) = \sum_{(i \to j) \in Q_1} x_i x_j, \ \dim G = \sum_{i \in Q_0} x_i^2,$$

therefore

$$\dim \mathrm{Rep}\,(x) - \dim G = -q\,(x)\,. \tag{5.1}$$

Since G is an affine algebraic group acting on an affine algebraic variety, we can work in Zariski topology. Then each orbit is open in its closure, if O and O' are two orbits and $O' \subset \bar{O}$, $O \neq O'$, then $\dim O' < \dim O$. Finally, we need the formula

$$\dim O_X = \dim G - \dim \mathrm{Stab}_X,$$

here Stab_X stands for the stabilizer of X. Also note that in our case the group G is connected, therefore each G-orbit is irreducible.

Lemma 5.1. $\dim \mathrm{Stab}_X = \dim \mathrm{Aut}_Q\,(X) = \dim \mathrm{End}_Q\,(X)$.

Proof. The condition that $\phi \in \mathrm{End}_Q\,(X)$ is not invertible is given by the polynomial equations $\det \phi_i = 0$. Since $\mathrm{Aut}_Q\,(X)$ is not empty, we are done. □

Corollary 5.2.

$$\operatorname{codim} O_X = \dim \mathrm{Rep}\,(x) - \dim G + \dim \mathrm{Stab}_X = -q\,(x) + \dim \mathrm{End}_Q\,(X) = \dim \mathrm{Ext}^1\,(X, X)\,.$$

Lemma 5.3. *Let Z be a nontrivial extension of Y by X, i.e. there is a non-split exact sequence*

$$0 \to X \to Z \to Y \to 0.$$

Then $O_{X \oplus Y} \subset \bar{O}_Z$ and $O_{X \oplus Y} \neq O_Z$.

Proof. Write each Z_i as $X_i \oplus Y_i$ and define $g_i^\lambda|_{X_i} = \mathrm{Id}$, $g_i^\lambda|_{Y_i} = \lambda\,\mathrm{Id}$ for any $\lambda \neq 0$. Then obviously $X \oplus Y$ belongs to the closure of $g_i^\lambda\,(Z)$. It is left to check that $X \oplus Y \not\cong Z$. But the sequence is non-split, therefore

$$\dim \mathrm{Hom}_Q\,(Y, Z) < \dim \mathrm{Hom}_Q\,(Y, X \oplus Y)\,.$$

□

Corollary 5.4. *If O_X is closed then X is semisimple.*

6. Dynkin and affine graphs

Let Γ be a connected graph with n vertices, then Γ defines a symmetric bilinear form $(\cdot,\cdot)$ on $\mathbb{Z}^n$

$$(x,y)=\sum_{i\in\Gamma_0}2x_iy_i-\sum_{(i,j)\in\Gamma_1}x_iy_j.$$

If Γ is equipped with orientation then the symmetric form coincides with the introduced earlier symmetric form of the corresponding quiver. The matrix of the form $(\cdot,\cdot)$ in the standard basis is called the *Cartan matrix* of Γ.

Example 6.1. The Cartan matrix of $\bullet - \bullet$ is $\binom{2\,-1}{-1\,2}$.

Theorem 6.2. *Given a connected graph Γ, exactly one of the following conditions holds*

1. *The symmetric $(\cdot,\cdot)$form is positive definite, then Γ is called Dynkin graph.*
2. *The symmetric form $(\cdot,\cdot)$is positive semidefinite, there exist $\delta\in\mathbb{Z}^n_{>0}$ such that $(\delta,x)=0$ for any $x\in\mathbb{Z}^n$. The kernel of $(\cdot,\cdot)$ is $\mathbb{Z}\delta$. In this case Γ is called affine or Euclidean.*
3. *There is $x\in\mathbb{Z}^n_{\geq 0}$ such that $(x,x)<0$. Then Γ is called of indefinite type.*

A Dynkin graphs is one of A_n, D_n, E_6, E_7, E_8. An affine graphs is one of $\hat{A}_n, \widehat{D}_n, \widehat{E}_6, \widehat{E}_7, \widehat{E}_8$. Every affine graph is obtained from a Dynkin graph by adding one vertex.

Proof. First, we check that A_n, D_n, E_6, E_7, E_8 define a positive definite form using the Sylvester criterion and the fact that every subgraph of a Dynkin graph is Dynkin. One can calculate the determinant of the Cartan matrix inductively. It is $n+1$ for A_n, 4 for D_n, 3 for E_6, 2 for E_7 and 1 for E_8. In the same way one can check that the Cartan matrices of affine graphs have determinant 0 and corank 1. The rows are linearly dependent with positive coefficients. Any other graph Γ has an affine graph Γ' as a subgraph, hence either $(\delta,\delta)<0$ or $(2\delta+\alpha_i,2\delta+\alpha_i)<0$, if α_i is the basis vector corresponding to a vertex i which does not belong to Γ' but is connected to some vertex of Γ'. □

A vector $\alpha\in\mathbb{Z}^n$ is called a *root* if $q(\alpha)=\frac{(\alpha,\alpha)}{2}\leq 1$. It is clear that $\alpha_1,\dots,\alpha_n$ are roots. They are called *simple roots*.

Lemma 6.3. *Let Γ be Dynkin or affine. If α is a root and $\alpha=m_1\alpha_1+\cdots+m_n\alpha_n$, then either all $m_i\geq 0$ or all $m_i\leq 0$.*

Proof. Let $\alpha=\beta-\gamma$, where $\beta=\sum_{i\in I}m_i\alpha_i$, $\gamma=\sum_{j\notin I}m_j\alpha_j$ for some $m_i,m_j\geq 0$, then $q(\alpha)=q(\beta)+q(\gamma)-(\beta,\gamma)$. Since Γ is Dynkin or affine, then $q(\beta)\geq 0$, $q(\gamma)\geq 0$. On the other hand $(\beta,\gamma)\leq 0$. Since $q(\alpha)\leq 1$, only one of three terms $q(\beta)$, $q(\gamma)$, $-(\beta,\gamma)$ can be positive, which is possible only if β or γ is zero. □

A root α is positive if $\alpha=m_1\alpha_1+\cdots+m_n\alpha_n$, $m_i\geq 0$ for all i.

A quiver has *finite type* if there are finitely many isomorphism classes of indecomposable representations.

Theorem 6.4. *(Gabriel) A connected quiver Q has finite type iff the corresponding graph is Dynkin. For a Dynkin quiver there exists a bijection between positive roots and isomorphism classes of indecomposable representations.*

Proof. If Q is of finite type, then $\operatorname{Rep}(x)$ has finitely many orbits for each $x \in \mathbb{Z}^n_{\geq 0}$. If Q is not Dynkin, then there exists $x \in \mathbb{Z}^n_{\geq 0}$ such that $q(x) \leq 0$. If Q has finite type, then $\operatorname{Rep}(x)$ must have an open orbit O_X. By Corollary 5.2

$$\operatorname{codim} O_X = \dim \operatorname{End}_Q(X) - q(x) > 0. \tag{6.1}$$

Contradiction.

Now suppose that Q is Dynkin. Every indecomposable representation X is a brick with trivial self-extensions by Corollary 4.3. Hence $q(x) = 1$, i.e. x is a root. By (6.1) O_X is the unique open orbit in $\operatorname{Rep}(x)$. What remains is to show that for each root x there exists an indecomposable representation of dimension x. Indeed, let X be such that $\dim O_X$ in $\operatorname{Rep}(x)$ is maximal. We claim that X is indecomposable. Indeed, let $X = X_1 \oplus \cdots \oplus X_s$ be a sum of indecomposable bricks. Then by Lemma 5.3 $\operatorname{Ext}^1(X_i, X_j) = 0$. Therefore $q(x) = s = 1$. Hence X is indecomposable. □