1. REPRESENTATIONS OF QUIVERS

I follow here Crawley-Boevey lectures trying to give more details concerning extensions and exact sequences.

A quiver is an oriented graph. If $Q$ is a quiver, then we denote by $Q_0$ the set of vertices and by $Q_1$ the set of arrows. Usually we denote by $n$ the number of vertices. If $\gamma: j \leftarrow i$ is an arrow then $i = s(\gamma)$, $j = t(\gamma)$.

Fix an algebraically closed field $k$. A representation $V$ of a quiver is a collection of vector spaces $\{V_i\}_{i \in Q_0}$ and linear maps $\rho_\gamma: V_i \to V_j$ for each arrow $\gamma: i \to j$. For two representations of a quiver $Q$, $\rho$ in $V$ and $\sigma$ in $W$ define a homomorphism $\phi: V \to W$ as a set of linear maps $\phi_i: V_i \to W_i$ such that the diagram

\[
\begin{array}{ccc}
V_j & \xleftarrow{\rho_\gamma} & V_i \\
\downarrow \phi_j & & \downarrow \phi_i \\
W_j & \xleftarrow{\sigma_\gamma} & W_i
\end{array}
\]

is commutative. We say that two representations $V$ and $W$ are isomorphic if there is a homomorphism $\phi \in \text{Hom}_Q(V,W)$ such that each $\phi_i$ is an isomorphism. One can define a subrepresentation and a direct sum of representation of $Q$ in the natural way. A representation is irreducible if it does not have non-trivial proper subrepresentation and indecomposable if it is not a direct sum of non-trivial subrepresentations.

**Example 1.1.** Let $Q$ be the quiver $\bullet \to \bullet$. A representation of $Q$ is a pair of vector spaces $V$ and $W$ and a linear operator $\rho: V \to W$. Let $V_0 = \text{Ker} \rho$, $V_1$ is such that $V = V_0 \oplus V_1$, $W_0 = \text{Im} \rho$, and $W_1$ is such that $W = W_0 \oplus W_1$. Then $V_0 \to 0$, $V_1 \to W_0$ and $0 \to W_1$ are subrepresentations and $\rho$ is their direct sum. Furthermore, $V_0 \to 0$ is the direct sum of $\dim V_0$ copies of $k \to 0$, $V_1 \to W_0$ is the direct sum of $\dim V_1$ copies of $k \to k$ and finally $0 \to W_1$ is the direct sum of $\dim W_1$ copies of $0 \to k$. Thus, we see that there are exactly three isomorphism classes of indecomposable representations of $Q$, $0 \to k$, $k \to k$, $k \to 0$. The first and the last one are irreducible, $0 \to k$ is a subrepresentation of $k \to k$ and $k \to 0$ is a quotient of $k \to k$ by $0 \to k$.

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2. Path algebra

Given a quiver $Q$. A path $p$ is a sequence $\gamma_1 \ldots \gamma_k$ of arrows such that $s(\gamma_i) = t(\gamma_{i+1})$. Put $s(p) = s(\gamma_k)$, $t(p) = t(\gamma_1)$. Define a composition $p_1 p_2$ of two paths such that $s(p_1) = t(p_2)$ in the obvious way and we set $p_1 p_2 = 0$ if $s(p_1) \neq t(p_2)$. Introduce also elements $e_i$ for each vertex $i \in Q_0$ and define $e_i e_j = \delta_{ij} e_i$, $e_i p = p$ if $i = t(p)$ and 0 otherwise, $p e_i = p$ if $i = s(p)$ and 0 otherwise. The path algebra $k(Q)$ is the set of $k$-linear combinations of all paths and $e_i$ with composition extended by linearity from ones defined above.

One can easily check the following properties of a path algebra

1. $k(Q)$ is finite-dimensional if $Q$ does not have oriented cycles;
2. If $Q$ is a disjoint union of $Q_1$ and $Q_2$, then $k(Q) = k(Q_1) \times k(Q_2)$;
3. The algebra $k(Q)$ has a natural $\mathbb{Z}$-grading $\bigoplus_{n=0}^{\infty} k(Q)_n$ defined by $\deg e_i = 0$ and the degree of a path $p$ being the length of the path;
4. Elements $e_i$ are primitive idempotents of $k(Q)$, and hence $k(Q) e_i$ is an indecomposable projective $k(Q)$-module.

The first three properties are trivial, let us check the last one. Suppose $e_i$ is not primitive, then one can find an idempotent $\varepsilon \in k(Q) e_i$. Let $\varepsilon = c_0 e_i + c_1 p_1 + \cdots + c_k p_k$, where $s(p_j) = i$ for all $j \leq k$. Then $\varepsilon^2 = \varepsilon$ implies $c_0 = 0$ or 1. Let $c_0 = 0, \varepsilon = \varepsilon_l + \cdots$, where $\deg \varepsilon_l = l$ and other terms have degree greater than $l$. But then $\varepsilon^2$ starts with degree greater than $2l$, hence $\varepsilon = 0$. If $c_0 = 1$, apply the same argument to the idempotent $(e_i - \varepsilon)$.

Given a representation $\rho$ of $Q$ one can construct a $k(Q)$-module

$$V = \bigoplus_{i \in Q_0} V_i, \quad e_i V_j = \delta_{ij} \text{Id}_{V_j}, \quad \gamma v = \rho_\gamma v \text{ if } v \in V_{s(\gamma)}, \gamma(v) = 0 \text{ otherwise.}$$

For any path $p = \gamma_1 \ldots \gamma_k$ and $v \in V$ put $p v = \rho_{\gamma_1} \circ \cdots \circ \rho_{\gamma_k} (v)$.

On the other hand, every $k(Q)$-module $V$ defines a representation $\rho$ of $Q$ if one puts $V_i = e_i V$.

The following theorem is straightforward.

**Theorem 2.1.** The category of representations of $Q$ and the category of $k(Q)$-modules are equivalent.

**Lemma 2.2.** The radical of $k(Q)$ is spanned by all paths $p$ satisfying the property that there is no return paths, i.e. back from $t(p)$ to $s(p)$.

**Proof.** It is easy to see that the paths with no return span a two-sided ideal $R$. Note that $R^n = 0$, where $n$ is the number of vertices. Thus, $R \subseteq \text{rad } k(Q)$. On the other hand, let $y \notin R$ and $p$ be a shortest path in decomposition of $y$ which has a return path. Choose a shortest path $s$ such that $\tau = sp$ is an oriented cycle. Consider the representation of $Q$ which has $k$ in each vertex of $\tau$ and 0 in all other vertices. Let $\rho_\gamma = \text{Id}$, if $\gamma$ is included in $\tau$ and $\rho_\gamma = 0$ otherwise. Let $V$ be the corresponding $k(Q)$-module. Then $V$ is simple, $s y (V) \neq 0$. Hence $y \notin \text{rad } k(Q)$. Contradiction. \(\square\)
Example 2.3. If $Q$ has one vertex and $n$ loops then $k(Q)$ is a free associative algebra with $n$ generators. If $Q$ does not have cycles, then $k(Q)$ is the subalgebra in $\text{Mat}_n(k)$ generated by elementary matrices $E_{ii}$ for each $i \in Q_0$ and $E_{ij}$ for each arrow $i \to j$.

3. Standard resolution

Theorem 3.1. Let $Q$ be a quiver, $A = k(Q)$ and $V$ be an $A$-module. Then the sequence

$$0 \to \bigoplus_{\gamma = (i \to j) \in Q_1} Ae_j \otimes V_i \xrightarrow{f} \bigoplus_{i \in Q_0} Ae_i \otimes V_i \xrightarrow{g} V \to 0,$$

where $f(ae_j \otimes v) = ae_j \gamma \otimes v - ae_j \otimes \gamma v$, $g(ae_i \otimes v) = av$ for any $v \in V_i$, is exact. It is a projective resolution.

Proof. First, check that $g \circ f = 0$. Indeed,

$$g(f(ae_j \otimes v)) = g(ae_j \gamma \otimes v - ae_j \otimes \gamma v) = ae_j \gamma v - ae_j \gamma v = 0.$$ 

Since $V = \bigoplus e_i V_i$, $g$ is surjective. To check that $f$ is injective, introduce the grading on $A \otimes V$ using $\deg V = 0$. By $grf$ denote the homogeneous part of highest degree for $f$. Note that the $grf$ increases the degree by one and

$$grf = \bigoplus_{\gamma \in Q_1} f_\gamma,$$

where $f_\gamma : Ae_j \otimes V_i \to A \gamma \otimes V_i$ is defined by

$$f_\gamma(ae_j \otimes v_i) = ae_j \gamma \otimes v_i,$$

for $\gamma : i \to j$. One can see from this formula that $f_\gamma$ is injective, therefore $grf$ is injective and hence $f$ is injective.

To prove that $\text{Im} f = \text{Ker} g$ note that

$$ae_j \gamma \otimes v \equiv ae_j \otimes \gamma v \mod \text{Im} f,$$

therefore for any $x \in \bigoplus_{i \in Q_0} Ae_i \otimes V_i$

$$x \equiv x_0 \mod \text{Im} f$$

for some $x_0$ of degree $0$. In other words $x_0 \in \bigoplus_{i \in Q_0} ke_i \otimes V_i$. If $g(x) = 0$, then $g(x_0) = 0$, and if $g(x_0) = 0$, then obviously $x_0 = 0$. Hence $x \equiv 0 \mod \text{Im} f$. □

Theorem 3.1 implies that $\text{Ext}^1(X, Y)$ can be calculated as $\text{coker} d$ of the following complex

$$(3.1) \quad 0 \to \bigoplus_{i \in Q_0} \text{Hom}_k(X_i, Y_i) \xrightarrow{d} \bigoplus_{\gamma = (i \to j) \in Q_1} \text{Hom}_k(X_i, Y_j) \to 0,$$

where

$$(3.2) \quad d\phi(x) = \phi(\gamma x) - \gamma \phi(x)$$

for any $x \in X_i$, $\gamma = (i \to j)$. 

Lemma 3.2. Every $\psi \in \text{Ext}^1(X,Y)$ induces a non-split exact sequence
$$0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0.$$ 
If $\text{Ext}^1(X,Y) = 0$, then every exacts sequence as above splits.

Proof. Let
$$0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$$
be an exact sequence of representations of $Q$. Then $Z_i$ can be identified with $X_i \oplus Y_i$ for every $i$. For every arrow $\gamma: i \rightarrow j$ the action on $Z$ is defined by
$$\gamma(x,y) = (\gamma x, \gamma y + \psi_{\gamma}(x)),$$
for some $\psi_{\gamma} \in \text{Hom}_k(X_i,Y_j)$. Thus, $\psi$ can be considered as an element in the second non-zero term of (3.1). If the exact sequence splits, then there is $\eta \in \text{Hom}_Q(X,Z)$ such that for each $i \in Q_0$, $x \in X_i$
$$\eta(x) = (x, \phi_i(x)),$$
for some $\phi_i \in \text{Hom}_k(X_i,Y_i)$. Furthermore, $\eta \in \text{Hom}_Q(X,Z)$ iff for each $\gamma: i \rightarrow j$
$$\gamma(x, \phi_i(x)) = (\gamma x, \gamma \phi_i(x) + \psi_{\gamma}(x)) = (\gamma x, \phi_j(\gamma x)),$$
which implies
$$\psi_{\gamma}(x) = \phi_j(\gamma x) - \gamma \phi_i(x).$$
In other words, $\psi = d\phi$. Thus, $\text{Ext}^1(X,Y)$ parameterizes the set of non-split exact sequences
$$0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0.$$ 
\qed

Corollary 3.3. In the category of representations of $Q$, $\text{Ext}^i(X,Y) = 0$ for $i \geq 2$.

Corollary 3.4. Let
$$0 \rightarrow Y \rightarrow Z \rightarrow X \rightarrow 0$$
be a short exact sequence of representations of $Q$, then
$$\text{Ext}^1(V,Z) \rightarrow \text{Ext}^1(V,X), \; \text{Ext}^1(Z,V) \rightarrow \text{Ext}^1(Y,V)$$
are surjective.

Lemma 3.5. If $X$ and $Y$ are indecomposable and $\text{Ext}^1(Y,X) = 0$, then every non-zero $\varphi \in \text{Hom}_Q(X,Y)$ is either surjective or injective.

Proof. Use the exact sequences
$$0 \rightarrow \text{Ker} \varphi \rightarrow X \rightarrow \text{Im} \varphi \rightarrow 0,$$
(3.3) $$0 \rightarrow \text{Im} \varphi \rightarrow Y \rightarrow S \cong Y/\text{Im} \varphi \rightarrow 0.$$
The exact sequence (3.3) can be considered as an element \( \psi \in \text{Ext}^1 (S, \text{Im} \varphi) \) by use of Lemma 3.2. By Corollary 3.3 we have an isomorphism \( g: \text{Ext}^1 (S, \text{Im} \varphi) \cong \text{Ext}^1 (S, X) \). Then \( g(\psi) \) induces the exact sequence

\[
0 \to X \to Z \to S \to 0,
\]

and this exact sequence together with (3.3) form the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & X \\
\downarrow{\beta} & & \downarrow{\gamma} \\
0 & \to & \text{Im} \varphi \\
\end{array}
\]

\[
\begin{array}{ccc}
& Z & \to S \\
\downarrow{\gamma} & & \downarrow{\gamma} \\
& Y & \to S \to 0 \\
\end{array}
\]

here \( \beta \) and \( \gamma \) are surjective. We claim that the sequence

\[
0 \to X \xrightarrow{\alpha + \beta} Z \oplus \text{Im} \varphi \xrightarrow{\gamma - \delta} Y \to 0
\]

is exact. Indeed, \( \alpha + \beta \) is obviously injective and \( \gamma - \delta \) is surjective. Finally, \( \dim Z = \dim X + \dim S, \dim \text{Im} \varphi = \dim Y - \dim S \). Therefore,

\[
\dim (Z \oplus \text{Im} \varphi) = \dim X + \dim Y,
\]

and therefore \( \text{Ker} (\gamma - \delta) = \text{Im} (\alpha + \beta) \).

But \( \text{Ext}^1 (Y, X) = 0 \). Hence the last exact sequence splits, \( Z \oplus \text{Im} \varphi \cong X \oplus Y \) and by Krull-Schmidt theorem either \( X \cong \text{Im} \varphi \) or \( Y \cong \text{Im} \varphi \). \( \square \)

Introduce \( \dim X \) as a vector \( x = (x_1, \ldots, x_n) \in \mathbb{Z}^n \) where \( n \) is the number of vertices and \( x_i = \dim X_i \). Define the bilinear form

\[
\langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{(i-j) \in Q_1} x_i y_j = \dim \text{Hom}_Q (X, Y) - \dim \text{Ext}^1 (X, Y)
\]

(the equality follows from (3.1)). We also introduce the symmetric form

\[
(x, y) = \langle x, y \rangle + \langle y, x \rangle
\]

and the quadratic form

\[
q(x) = \langle x, x \rangle.
\]

4. Bricks

Here we discuss further properties of finite-dimensional representations of \( A = k(Q) \).

Recall that if \( X \) is indecomposable and has finite length, then \( \varphi \in \text{End}_Q (X) \) is either isomorphism or nilpotent. Since we assumed that \( k \) is algebraically closed, \( \varphi = \lambda \text{Id} \) for any invertible \( \varphi \in \text{End}_Q (X) \). A representation \( X \) is a brick, if \( \text{End}_Q (X) = k \). If \( X \) is a brick, then \( X \) is indecomposable. If \( X \) is indecomposable and \( \text{Ext}^1 (X, X) = 0 \), then \( X \) is a brick due to Lemma 3.5.
Example 4.1. Consider the quiver $\bullet \to \bullet$. Then every indecomposable is a brick. For the Kronecker quiver $\bullet \Rightarrow \bullet$ the representation $k^2 \Rightarrow k^2$ with $\alpha = \text{Id}$, $\beta = (01 \atop 00)$ is not a brick. Indeed, $\varphi = (\varphi_1, \varphi_2)$ where $\varphi_1, \varphi_2$ are matrices $(01 \atop 00)$, belongs to $\text{End}_Q(X)$.

Lemma 4.2. Let $X$ be indecomposable and not a brick, then $X$ contains a brick $W$ such that $\text{Ext}^1(W, W) \neq 0$.

Proof. Choose $\varphi \in \text{End}_Q(X)$, $\varphi \neq 0$ of minimal rank. Since $\text{rk} \varphi^2 < \text{rk} \varphi$, $\varphi^2 = 0$. Let $Y = \text{Im} \varphi$, $Z = \text{Ker} \varphi$. Let $Z = Z_1 \oplus \cdots \oplus Z_p$ be a sum of indecomposables. Let $p_i : Z \to Z_i$ be the projection. Choose $i$ so that $p_i(Y) \neq 0$ and let $\eta = p_i \circ \varphi \in \text{End}_Q(X)$ (well defined since $\text{Im} \varphi \in \text{Ker} \varphi$). Note that by our assumption $\text{rk} \eta = \text{rk} \varphi$, therefore $p_i : Y \to Z_i$ is an embedding. Let $Y_i = p_i(Y)$. Then $\text{Ker} \eta = Z$, $\text{Im} \eta = Y_i$.

We claim now that $\text{Ext}^1(Z_i, Z_i) \neq 0$. Indeed, $\text{Ext}^1(Y_i, Z) \neq 0$ by exact sequence

$$0 \to Z \to X \overset{\eta}{\to} Y_i \to 0$$

and indecomposability of $X$. Then the induced exact sequence

$$0 \to Z_i \to X_i \overset{\eta}{\to} Y_i \to 0$$

does not split also. (If it splits, then $Z_i$ is a direct summand of $X$, which is impossible). Therefore $\text{Ext}^1(Y_i, Z_i) \neq 0$. But $Y_i$ is a submodule of $Z_i$. By Corollary 3.4 we have the surjection

$$\text{Ext}^1(Z_i, Z_i) \to \text{Ext}^1(Y_i, Z_i).$$

If $Z_i$ is not a brick, we repeat the above construction for $Z_i$ e.t.c. Finally, we get a brick. □

Corollary 4.3. Assume that the quadratic form $q$ is positive definite. Then every indecomposable $X$ is a brick with trivial $\text{Ext}^1(X, X)$; moreover, if $x = \dim X$, then $q(x) = 1$.

Proof. Assume that $X$ is not a brick, then it contains a brick $Y$ such that $\text{Ext}^1(Y, Y) \neq 0$. Then

$$q(Y) = \dim \text{End}_Q(Y) - \dim \text{Ext}^1(Y, Y) = 1 - \dim \text{Ext}^1(Y, Y) \leq 0,$$

but this is impossible. Therefore $X$ is a brick. Now

$$q(x) = \dim \text{End}_Q(X) - \dim \text{Ext}^1(X, X) = 1 - \dim \text{Ext}^1(X, X) \geq 0,$$

hence $q(x) = 1$ and $\dim \text{Ext}^1(X, X) = 0$. □

5. Orbits in representation variety

Fix a quiver $Q$, recall that $n$ denotes the number of vertices. Let $x = (x_1, \ldots, x_n) \in \mathbb{Z}_{\geq 0}^n$. Define

$$\text{Rep}(x) = \prod_{(i \to j) \in Q_1} \text{Hom}_k(k^{x_i}, k^{x_j}).$$
It is clear that every representation of $Q$ of dimension $x$ is a point in $\text{Rep}(x)$. Let
\[ G = \prod_{i \in Q_0} \text{GL}(k^i) \, . \]

Then $G$ acts on $\text{Rep}(x)$ by the formula $g \varphi_{ij} = g_i \varphi_j g_{ij}^{-1}$, for each arrow $i \to j$. Two representations of $Q$ are isomorphic iff they belong to the same orbit of $G$. For a representation $X$ we denote by $O_X$ the corresponding $G$-orbit in $\text{Rep}(x)$.

Note that
\[ \dim \text{Rep}(x) = \sum_{(i \to j) \in Q_1} x_i x_j, \quad \dim G = \sum_{i \in Q_0} x_i^2, \]
therefore
\[ (5.1) \quad \dim \text{Rep}(x) - \dim G = -q(x) \, . \]

Since $G$ is an affine algebraic group acting on an affine algebraic variety, we can work in Zariski topology. Then each orbit is open in its closure, if $O$ and $O'$ are two orbits and $O' \subset O$, $O \neq O'$, then $\dim O' < \dim O$. Finally, we need the formula
\[ \dim O_X = \dim G - \dim \text{Stab}_X, \]
here $\text{Stab}_X$ stands for the stabilizer of $X$. Also note that in our case the group $G$ is connected, therefore each $G$-orbit is irreducible.

**Lemma 5.1.** $\dim \text{Stab}_X = \dim \text{Aut}_Q(X) = \dim \text{End}_Q(X)$.

**Proof.** The condition that $\phi \in \text{End}_Q(X)$ is not invertible is given by the polynomial equations $\det \phi_i = 0$. Since $\text{Aut}_Q(X)$ is not empty, we are done. $\square$

**Corollary 5.2.**
\[ \text{codim} O_X = \dim \text{Rep}(x) - \dim G + \dim \text{Stab}_X = -q(x) + \dim \text{End}_Q(X) = \dim \text{Ext}^1(X, X) \, . \]

**Lemma 5.3.** Let $Z$ be a nontrivial extension of $Y$ by $X$, i.e. there is a non-split exact sequence
\[ 0 \to X \to Z \to Y \to 0. \]
Then $O_{X \oplus Y} \subset \bar{O}_Z$ and $O_{X \oplus Y} \neq O_Z$.

**Proof.** Write each $Z_i$ as $X_i \oplus Y_i$ and define $g_i^\lambda|_{X_i} = \text{Id}, g_i^\lambda|_{Y_i} = \lambda \text{Id}$ for any $\lambda \neq 0$. Then obviously $X \oplus Y$ belongs to the closure of $g_i^\lambda(Z)$. It is left to check that $X \oplus Y \not\subset Z$. But the sequence is non-split, therefore
\[ \dim \text{Hom}_Q(Y, Z) < \dim \text{Hom}_Q(Y, X \oplus Y) \, . \]

**Corollary 5.4.** If $O_X$ is closed then $X$ is semisimple.
6. DYNKIN AND AFFINE GRAPHS

Let $\Gamma$ be a connected graph with $n$ vertices, then $\Gamma$ defines a symmetric bilinear form $(\cdot, \cdot)$ on $\mathbb{Z}^n$:
\[(x, y) = \sum_{i \in \Gamma_0} 2x_i y_i - \sum_{(i, j) \in \Gamma_1} x_i y_j.\]

If $\Gamma$ is equipped with orientation then the symmetric form coincides with the introduced earlier symmetric form of the corresponding quiver. The matrix of the form $(\cdot, \cdot)$ in the standard basis is called the \textit{Cartan matrix} of $\Gamma$.

\textbf{Example 6.1.} The Cartan matrix of $\bullet - \bullet$ is $\left( \begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix} \right)$.

\textbf{Theorem 6.2.} Given a connected graph $\Gamma$, exactly one of the following conditions holds:

1. The symmetric $(\cdot, \cdot)$ form is positive definite, then $\Gamma$ is called \textit{Dynkin graph}.
2. The symmetric form $(\cdot, \cdot)$ is positive semidefinite, there exist $\delta \in \mathbb{Z}^n_{\geq 0}$ such that $(\delta, x) = 0$ for any $x \in \mathbb{Z}^n$. The kernel of $(\cdot, \cdot)$ is $\mathbb{Z}\delta$. In this case $\Gamma$ is called \textit{affine} or \textit{Euclidean}.
3. There is $x \in \mathbb{Z}^n_{\geq 0}$ such that $(x, x) < 0$. Then $\Gamma$ is called of \textit{indefinite type}.

A Dynkin graphs is one of $A_n, D_n, E_6, E_7, E_8$. An affine graphs is one of $\tilde{A}_n, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$. Every affine graph is obtained from a Dynkin graph by adding one vertex.

\textit{Proof.} First, we check that $A_n, D_n, E_6, E_7, E_8$ define a positive definite form using the Sylvester criterion and the fact that every subgraph of a Dynkin graph is Dynkin. One can calculate the determinant of the Cartan matrix inductively. It is $n + 1$ for $A_n$, 4 for $D_n$, 3 for $E_6$, 2 for $E_7$ and 1 for $E_8$. In the same way one can check that the Cartan matrices of affine graphs have determinant 0 and corank 1. The rows are linearly dependent with positive coefficients. Any other graph $\Gamma$ has an affine graph $\Gamma'$ as a subgraph, hence either $(\delta, \delta) < 0$ or $(2\delta + \alpha_i, 2\delta + \alpha_i) < 0$, if $\alpha_i$ is the basis vector corresponding to a vertex $i$ which does not belong to $\Gamma'$ but is connected to some vertex of $\Gamma'$.

A vector $\alpha \in \mathbb{Z}^n$ is called a \textit{root} if $q(\alpha) = \frac{(\alpha, \alpha)}{2} \leq 1$. It is clear that $\alpha_1, \ldots, \alpha_n$ are roots. They are called \textit{simple roots}.

\textbf{Lemma 6.3.} Let $\Gamma$ be Dynkin or affine. If $\alpha$ is a root and $\alpha = m_1 \alpha_1 + \cdots + m_n \alpha_n$, then either all $m_i \geq 0$ or all $m_i \leq 0$.

\textit{Proof.} Let $\alpha = \beta - \gamma$, where $\beta = \sum_{i \in I} m_i \alpha_i$, $\gamma = \sum_{j \notin I} m_j \alpha_j$ for some $m_i, m_j \geq 0$, then $q(\alpha) = q(\beta) + q(\gamma) - (\beta, \gamma)$. Since $\Gamma$ is Dynkin or affine, then $q(\beta) \geq 0$, $q(\gamma) \geq 0$. On the other hand $(\beta, \gamma) \leq 0$. Since $q(\alpha) \leq 1$, only one of three terms $q(\beta), q(\gamma), -(\beta, \gamma)$ can be positive, which is possible only if $\beta$ or $\gamma$ is zero. A root $\alpha$ is positive if $\alpha = m_1 \alpha_1 + \cdots + m_n \alpha_n$, $m_i \geq 0$ for all $i$.

A quiver has \textit{finite type} if there are finitely many isomorphism classes of indecomposable representations.
Theorem 6.4. (Gabriel) A connected quiver $Q$ has finite type iff the corresponding graph is Dynkin. For a Dynkin quiver there exists a bijection between positive roots and isomorphism classes of indecomposable representations.

Proof. If $Q$ is of finite type, then $\text{Rep}(x)$ has finitely many orbits for each $x \in \mathbb{Z}_{\geq 0}^n$. If $Q$ is not Dynkin, then there exists $x \in \mathbb{Z}_{\geq 0}^n$ such that $q(x) \leq 0$. If $Q$ has finite type, then $\text{Rep}(x)$ must have an open orbit $O_X$. By Corollary 5.2

\begin{equation}
\text{codim } O_X = \dim \text{End}_Q(X) - q(x) > 0.
\end{equation}

Contradiction.

Now suppose that $Q$ is Dynkin. Every indecomposable representation $X$ is a brick with trivial self-extensions by Corollary 4.3. Hence $q(x) = 1$, i.e. $x$ is a root. By (6.1) $O_X$ is the unique open orbit in $\text{Rep}(x)$. What remains is to show that for each root $x$ there exists an indecomposable representation of dimension $x$. Indeed, let $X$ be such that $\dim O_X$ in $\text{Rep}(x)$ is maximal. We claim that $X$ is indecomposable. Indeed, let $X = X_1 \oplus \cdots \oplus X_s$ be a sum of indecomposable bricks. Then by Lemma 5.3 $\text{Ext}^1(X_i, X_j) = 0$. Therefore $q(x) = s = 1$. Hence $X$ is indecomposable. \qed