REPRESENTATION THEORY. LECTURE NOTES

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1. Some problems involving representation theory

Hungry knights. There are n hungry knights at a round table. Each of them has a plate with certain amount of food. Instead of eating every minute each knight takes one half of his neighbors servings. They start at 10 in the evening. What can you tell about food distribution in the morning?

Solution. Denote by x_i the amount of food on the plate of the i-th knight. The distribution of food at the table can be described by a vector $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$. Every minute a certain linear operator Φ is applied to a distribution x. Thus, we have to find $\lim \Phi^m$ as m approaches infinity. To find the limit we need to diagonalize Φ, and the easiest way to do this is to write

$$\Phi = \frac{T + T^{-1}}{2},$$

where T is the rotation operator:

$$T(x_1, \dots, x_n) = (x_n, x_1, \dots, x_{n-1}).$$

It is easy to see that the eigenvalues of T are the n-th roots of 1. Hence the eigenvalues of Φ are $\frac{\varepsilon^k + \varepsilon^{-k}}{2}$, where ε is a primitive root of 1, $k = 1, \dots, n$. The set of eigenvalues of Φ is

$$\left\{ \cos \frac{2\pi k}{n} \mid k = 1, \dots, n \right\}.$$

Let us chose a new basis $\{v_1, \dots, v_n\}$ in $\mathbb{C}^n$ such that

$$\Phi v_k = \cos \frac{2\pi k}{n} v_k.$$

For example, we can put $v_k = \left(\varepsilon^{-k}, \varepsilon^{-2k}, \dots, \varepsilon^{-nk}\right)$.

If n is odd all eigenvalues of Φ except 1 have the absolute value less than 1. Therefore if $x = a_1 v_1 + \dots + a_n v_n$, then

$$\lim_{m\to\infty} \Phi^m x = \lim_{m\to\infty} \sum_{k=1}^{n} a_k \left(\cos \frac{2\pi k}{n}\right)^m v_k = a_n v_n.$$

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But $v_n = (1, \dots, 1)$. Therefore eventually all knights will have the same amount of food equal to the average $\frac{x_1+\cdots+x_n}{n}$.

In case when n is even the situation is different, since there are two eigenvalues with absolute value 1, they are 1 and -1. Hence as $m \to \infty$,

$$\Phi^m x \to (-1)^m a_{n/2} v_{n/2} + a_n v_n.$$

Recall that $v_k = (-1, 1, -1, \dots, 1)$. Thus, eventually food alternates between even and odd knights, the amount on each plate is approximately $\frac{a_n \pm a_{n/2}}{2}$, where

$$a_n = \frac{x_1 + \cdots + x_n}{n},\ a_{n/2} = \frac{x_1 - x_2 + \cdots - x_n}{n}.$$

Slightly modifying this problem we will have more fun.

Breakfast at Mars. It is well known that marsians have four arms, a standard family has 6 persons and a breakfast table has a form of a cube with each person occupying a face on a cube. Do the analog of round table problem for the family of marsians.

Supper at Venus. They have five arms there, 12 persons in a family and sit on the faces of a dodecahedron (a regular polyhedron whose faces are pentagons).

Tomography problem. You have a solid in 3-dimensional space of unknown shape. You can measure the area of every plane cross-section which passes through the origin. Can you determine the shape of the solid? The answer is yes, if the solid in question is convex and centrally symmetric with respect to the origin.

In all four problems above the important ingredient is a *group of symmetries.* In the first case this is a cyclic group of rotations of the table, in the second one the group of rotations of a cube, in the Venus problem the group of rotations of a dodecahedron (can you describe these groups?). Finally, in the last problem the group of all rotations in $\mathbb{R}^3$ appears. In all cases the group acts on a vector space via linear operators, i.e. as we have a *linear representation of a group.* The main part of this course deals with representation of groups.

Linear algebra problems. Every standard course of linear algebra discusses the problem of classification of all matrices in a complex vector space up to equivalence. (Here A is equivalent to B if $A = XBX^{-1}$ for some invertible X.) Indeed, there exists some basis in $\mathbb{C}^n$, in which A has a canonical Jordan form. The following problem is less known.

Kronecker problem. Let V and W be finite-dimensional vector spaces over algebraically closed field k, A and $B : V \to W$ be two linear operators. Classify all pairs (A, B) up to the change of bases in V and W. In other words we have to classify pairs of matrices up to the following equivalence relation: (A, B) is equivalent to (C, D) if there are invertible square matrices X and Y such that

$$C = XAY,\ D = XBY.$$

Theorem 1.1. *There exist decompositions*

$$V = V_1 \oplus \cdots \oplus V_k,\ W = W_1 \oplus \cdots \oplus W_k,$$

such that $A(V_i) \subset W_i$, $B(V_i) \subset W_i$ *and for each* $i \leq k$ *there exist bases in* V_i *and* W_i *such that the matrices for* A *and* B *have one of the following forms (here* 1_n *denotes the identity matrix of size* n, J_n *the nilpotent Jordan block of size* n *):*

$$A = \binom{1_n}{0},\ B = \binom{0}{1_n},\ n \geq 0;$$
$$A = (1_n, 0),\ B = (0, 1_n),\ n \geq 0;$$
$$A = 1_n,\ B = J_n,\ n \geq 1;$$
$$A = t1_n + J_n,\ B = 1_n,\ n \geq 1,\ t \in k.$$

2. Representations of groups. Definition and examples.

Let k denote a field, V be a vector space over k. By $\mathrm{GL}(V)$ we denote the group of all invertible linear operators in V. If $\dim V = n$, then $\mathrm{GL}(V)$ is isomorphic to the group of invertible $n \times n$ matrices with entries in k.

A *(linear) representation* of a group G in V is a homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

The dimension of V is called the *degree* or the *dimension* of a representation ρ and it may be infinite. For any $s \in G$ we denote by ρ_s the image of s in $\mathrm{GL}(V)$ and for any $v \in V$ we denote by $\rho_s v$ the image of v under the action of ρ_s. The following properties are obvious:

$$\rho_s \rho_t = \rho_{st},\ \rho_1 = \mathrm{Id},\ \rho_s^{-1} = \rho_{s^{-1}},\ \rho_s(xv + yw) = x\rho_s v + y\rho_s w.$$

Examples.

1. Let $G = \mathbb{Z}$ with operation $+$, $V = \mathbb{R}^2$, ρ_n is given by the matrix

$$\begin{matrix} 1 & n \\ 0 & 1 \end{matrix}$$

for $n \in \mathbb{Z}$.

2. *Permutation representation.* Let $G = S_n$, $V = k^n$. For each $s \in S_n$ put

$$\rho_s(x_1, \dots, x_n) = \left(x_{s(1)}, \dots, x_{s(n)}\right).$$

3. *Trivial representation.* For any group G the trivial representation is the homomorphism $\rho : G \to k^*$ such that $\rho_s = 1$ for all $s \in G$.

4. Let G be a group and

$$\mathcal{F}(G) = \{f : G \to k\}$$

be the space of functions on G with values in k. For any $s \in G$, $f \in \mathcal{F}(G)$ and $t \in G$ let

$$\rho_s f(t) = f(ts).$$

Then $\rho : G \to \mathrm{GL}(\mathcal{F}(G))$ is a linear representation.

5. *Regular representation.* Recall that the *group algebra* $k(G)$ is the vector space of all finite linear combinations $\sum c_g g$, $c_g \in k$ with natural multiplication. The regular representation $R : G \to \mathrm{GL}(k(G))$ is defined in the following way

$$R_s\left(\sum c_g g\right) = \sum c_g sg.$$

Two representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are *equivalent* (or *isomorphic*) if there exists an isomorphism $T : V \to W$ such that for all $s \in G$

$$T \circ \rho_s = \sigma_s \circ T.$$

Example. If G is finite then the representations in examples 4 and 5 are equivalent. Indeed, define $T : \mathcal{F}(G) \to k(G)$ by the formula

$$T(f) = \sum_{g \in G} f(g)\, g^{-1}.$$

Then for any $f \in \mathcal{F}(G)$ we have

$$T(\rho_s f) = \sum_{g\in G} \rho_s f(g)\, g^{-1} = \sum_{g \in G} f(gs)\, g^{-1} = \sum_{h \in G} f(h)\, sh^{-1} = R_s(Tf).$$

3. Operations with representations

Restriction on a subgroup: Let H be a subgroup of G. For any $\rho : G \to \mathrm{GL}(V)$ we denote by $\mathrm{Res}_H\, \rho$ the restriction of ρ on H.

Lift. Let $p : G \to H$ be a homomorphism of groups. For every representation $\rho : H \to \mathrm{GL}(V)$, $\rho \circ p : G \to \mathrm{GL}(V)$ is also a representation. We often use this construction in case when $H = G/N$ is a quotient group and p is the natural projection.

Direct sum. If we have two representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$, then we can define $\rho \oplus \sigma : G \to \mathrm{GL}(V \oplus W)$ by the formula

$$(\rho \oplus \sigma)_s (v, w) = (\rho_s v, \sigma_s w).$$

Tensor product. The tensor product of $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ is defined by the formula

$$(\rho \otimes \sigma)_s\, v \otimes w = \rho_s v \otimes \sigma_s w.$$

Exterior tensor product. Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : H \to \mathrm{GL}(W)$ be representations of two different groups, then their exterior product $\rho \boxtimes \sigma : G \times H \to \mathrm{GL}(V \otimes W)$ is defined by

$$(\rho \boxtimes \sigma)_{(s,t)}\, v \otimes w = \rho_s v \otimes \sigma_t w.$$

If $\delta : G \to G \times G$ is the diagonal embedding, then

$$\rho \otimes \sigma = (\rho \boxtimes \sigma) \circ \delta.$$

Dual representation. For any representation $\rho : G \to \mathrm{GL}(V)$ one can define the dual representation $\rho^* : G \to \mathrm{GL}(V^*)$ by the formula

$$< \rho_s^* \varphi, v >=< \varphi, \rho_s^{-1} v >$$

for any $v \in V, \varphi \in V^*$. Here $<,>$ denotes the natural pairing between V and V^*.

More generally, if $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are two representations, then one can naturally define the representation τ of G in $\mathrm{Hom}_k(V, W)$ by the formula

$$\tau_s \varphi = \sigma_s \circ \varphi \circ \rho_s^{-1},\ s \in G,\ \varphi \in \mathrm{Hom}_k(V, W).$$

4. Invariant subspaces and irreducibility

Given a representation $\rho : G \to \mathrm{GL}(V)$. A subspace $W \subset V$ is called *invariant* if $\rho_s(W) \subset W$ for any $s \in G$. One can define naturally the subrepresentation

$$\rho^W : G \to \mathrm{GL}(W)$$

and the quotient representation

$$\sigma : G \to \mathrm{GL}(V/W).$$

Example. Let $\rho : S_n \to \mathrm{GL}(k^n)$ be the permutation representation, then

$$W = \{(x_1, \dots, x_n) \mid x_1 = x_2 = \dots = x_n\}$$

and

$$W' = \{(x_1, \dots, x_n) \mid x_1 + x_2 + \dots + x_n = 0\}$$

are invariant subspaces.

Theorem 4.1. *(Maschke) Let G be a finite group and* $\mathrm{char}\, k$ *do not divide $|G|$. Let $\rho : G \to \mathrm{GL}(V)$ be a representation and W be an invariant subspace. Then there exists another invariant subspace W' such that $V = W \oplus W'$.*

Proof. Let W'' be a subspace (not invariant) such that $V = W \oplus W''$. Let $P : V \to V$ be the linear operator such that $P_{|W} = \mathrm{Id}$ and $P(W'') = 0$. Then $P^2 = P$. Such operator is called a *projector*. Let

$$\bar{P} = \frac{1}{|G|} \sum_{g \in G} \rho_g \circ P \circ \rho_g^{-1}.$$

Check that $\rho_s \circ \bar{P} \circ \rho_s^{-1} = \bar{P}$, and hence $\rho_s \circ \bar{P} = \bar{P} \circ \rho_s$ for any $s \in G$. Check also that $\bar{P}_{|W} = \mathrm{Id}$ and $\mathrm{Im}\, \bar{P} = W$. Hence $\bar{P}^2 = \bar{P}$.

Let $W' = \mathrm{Ker}\, \bar{P}$. First, we claim that W' is invariant. Indeed, let $w \in W'$, then $\bar{P}(\rho_s w) = \rho_s \left(\bar{P} w\right) = 0$, hence $\rho_s w \in \mathrm{Ker}\, \bar{P} = W'$.

Now we prove that $V = W \oplus W'$. Indeed, $W \cap W' = 0$, since $\bar{P}_{|W} = Id$. On the other hand, for any $v \in V$, we have $w = \bar{P} v \in W$ and $w' = v - \bar{P} v \in W'$. Thus, $v = w + w'$, and therefore $V = W + W'$. $\square$

In the previous example $V = W \oplus W'$ if $\operatorname{char} k$ does not divide n. Otherwise, $W \subset W'$, and the theorem is not true.

If G is infinite group, theorem is not true. Consider the representation of $\mathbb{Z}$ in $\mathbb{R}^2$ from Example 1. It has the unique one-dimensional invariant subspace, therefore $\mathbb{R}^2$ does not split into a direct sum of two invariant subspaces.

A representation is called *irreducible* if it does not contain a proper non-zero invariant subspace.

Exercise. Show that if $\operatorname{char} k$ does not divide n, then the subrepresentation W' of the permutation representation is irreducible.

Lemma 4.2. *Let G be a finite group, $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation. Then $\dim V \leq |G|$.*

Proof. Take any non-zero $v \in V$, then the set $\{\rho_s v\}_{s \in G}$ spans an invariant subspace which must coincide with V. Hence $\dim V \leq |G|$. $\square$

Example. Let $\rho : G \to \mathrm{GL}(V)$. We claim that $\rho \otimes \rho$ is irreducible if and only if $\dim V = 1$. Indeed the subspaces $S^2 V, \Lambda^2 V \subset V \otimes V$ are invariant and $\Lambda^2 V = \{0\}$ only in case when $\dim V = 1$.

A representation is called *completely reducible* if it splits into a direct sum of irreducible subrepresentations.

Corollary 4.3. *Let G be a finite group and k be a field such that $\operatorname{char} k$ does not divide $|G|$. Then every finite-dimensional representation of G is completely reducible.*

Proof. By induction on $\dim V$. $\square$

5. Schur's Lemma

For any two representations $\rho : G \to \mathrm{GL}(V)$, $\sigma : G \to \mathrm{GL}(W)$ let

$$\mathrm{Hom}_G(V, W) = \{T \in \mathrm{Hom}_k(V, W) \mid \sigma_s \circ T = T \circ \rho_s, s \in G\}.$$

An operator $T \in \mathrm{Hom}_G(V, W)$ is called an *intertwining* operator. It is clear that $\mathrm{Hom}_G(V, W)$ is a vector space. Moreover, if $\rho = \sigma$, then $\mathrm{Hom}_G(V, V) = \mathrm{End}_G(V)$ is closed under operation of composition, and therefore it is a k-algebra.

Lemma 5.1. *Let $T \in \mathrm{Hom}_G(V, W)$, then $\operatorname{Ker} T$ and $\operatorname{Im} T$ are invariant subspaces.*

Proof. Let $v \in \operatorname{Ker} T$, then $T(\rho_s v) = \rho_s(Tv) = 0$, hence $\rho_s v \in \operatorname{Ker} T$.

Let $w \in \operatorname{Im} T$. Then $w = Tv$ for some $v \in V$ and $\rho_s w = \rho_s(Tv) = T(\rho_s v) \in \operatorname{Im} T$. $\square$

Corollary 5.2. *(Schur's lemma) Let $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ be irreducible representations of G, then any $T \in \mathrm{Hom}_G(V, W)$ is either isomorphism or zero.*

Proof. Since both V and W do not have proper invariant subspaces, then either $\operatorname{Im} T = W$, $\operatorname{Ker} T = \{0\}$ or $\operatorname{Im} T = \{0\}$. $\square$

Corollary 5.3. *If $\rho : G \to \mathrm{GL}(V)$ is irreducible then $\mathrm{End}_G(V)$ is a division ring. If the field k is algebraically closed and V is finite-dimensional, then $\mathrm{End}_G(V) = k\,\mathrm{Id}$.*

Proof. The first assertion follows immediately from the Corollary 5.2. To prove the second, let $T \in \mathrm{End}_G(V)$ and $T \neq 0$. Then T is invertible. Let λ be an eigenvalue of T and $S = T - \lambda Id$. Since $S \in \mathrm{End}_G(V)$ and $\mathrm{Ker}\, S \neq \{0\}$, by Corollary 5.2, $S = 0$. Thus, $T = \lambda\,\mathrm{Id}$. □

Corollary 5.4. *Let G be an abelian group, $\rho : G \to \mathrm{GL}(V)$ be an irreducible finite-dimensional representation of G over algebraically closed field k. Then $\dim V = 1$.*

Irreducible representations of a finite cyclic group over $\mathbb{C}$. Let G be a cyclic group of order n and g be a generator. By Corollary 5.4 every irreducible representation of G is one-dimensional. Thus, we have to classify homomorphisms $\rho : G \to \mathbb{C}^*$. Let $\rho_g = \varepsilon$. Then clearly ε is an n-th root of 1. Therefore we have exactly n non-equivalent irreducible representations.

Irreducible representations of a finite abelian group over $\mathbb{C}$. Any finite abelian group is a direct product $G_1 \times \cdots \times G_k$ of cyclic groups. Let g_i be a generator of G_i. Then any irreducible $\rho : G \to \mathbb{C}^*$ is determined by its values $\rho_{g_i} = \varepsilon_i$, where $\varepsilon_i^{|G_i|} = 1$. Hence the number of isomorphism classes of irreducible representations of G equals $|G|$.

Remark 5.5. It is not difficult to see that the set of one-dimensional representations of G is a group with respect to the operation of tensor product. In case when G is finite and abelian and k is algebraically closed, all irreducible representations are one dimensional and form a group. We denote this group by $G^\vee$. As easily follows from above $G^\vee \cong G$ when $k = \mathbb{C}$, however this isomorphism is not canonical.

Here is another application of Schur's Lemma.

Theorem 5.6. *Let $\rho \cong \rho_1 \oplus \cdots \oplus \rho_k \cong \sigma_1 \oplus \cdots \oplus \sigma_m$, where ρ_i, σ_j are irreducible. Then $m = k$ and there exists $s \in S_k$ such that $\rho_j \cong \sigma_{s(j)}$.*

Proof. Let V be the space of a representation ρ. There are two decompositions of V into the direct sum of irreducible decomposable subspaces

$$V = V_1 \oplus \cdots \oplus V_k = W_1 \oplus \cdots \oplus W_m.$$

Let $p_i : V \to W_i$ be the projection which maps W_j to zero for $j \neq i$, $q_j : V_j \to V$ be the embedding. Then $p_i \in \mathrm{Hom}_G(V, W_i)$ and $q_j \in \mathrm{Hom}_G(V_j, W)$. The map

$$F = \sum_{i=1}^{m} \sum_{j=1}^{k} p_i \circ q_j : \oplus_{j=1}^{k} V_j \to \oplus_{i=1}^{m} W_i$$

is an isomorphism. There exists i such that $p_i \circ q_1 \neq 0$. (Otherwise $F(V_1) = 0$ which is impossible.) Put $s(1) = i$ and note that $p_i \circ q_1$ is an isomorphism by Schur's Lemma. We continue inductively. For each j there exists i such that $p_i \circ q_j$ is an isomorphism and $i \neq s(r)$ for any $r < j$. (Indeed, otherwise $F(V_1 \oplus \cdots \oplus V_j) \subset W_{s(1)} \oplus \cdots \oplus W_{s(j-1)}$ which is impossible because F is an isomorphism.) We put $i = s(j)$. Thus, we can construct an injective map

$$s : \{1, \ldots, k\} \to \{1, \ldots, m\}$$

such that $\rho_j \cong \sigma_{s(j)}$. In particular, $k \leq m$. But by exchanging ρ_i and σ_j we can prove that $m \leq k$. Hence $k = m$ and s is a permutation. $\square$