WEYL AND CLIFFORD ALGEBRAS.

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0.1. **Weyl algebra.** Let k be a field and V be a finite-dimensional vector space over k. Let ω be a non-degenerate skew-symmetric bilinear form on V, i.e. $\omega(x,x)=0$ for any $x\in V$ and for $x\neq 0$ there exists y such that $\omega(x,y)\neq 0$.

Lemma 0.1. *The dimension of V is even and there exists a basis $x_1,\dots,x_n,y_1,\dots,y_n$ in V such that $\omega(x_i,x_j)=\omega(y_i,y_j)=0$ and $\omega(y_i,x_j)=\delta_{ij}$.*

Proof. By induction in the dimension of V. Pickup a non-zero $y_1\in V$. Since ω is non-degenerate, there exists $x_1\in V$ such that $\omega(y_1,x_1)=1$. Now let W be the subspace generated by x_1,y_1. For any $S\subset V$ let $S^\perp=\{x\in V\mid \omega(x,s)=0\forall s\in S\}$. Then $\dim x_1^\perp=\dim y_1^\perp=\dim V-1$, and $\dim W^\perp=\dim(x_1^\perp\cap y_1^\perp)=\dim V-2$. Now we have $V=W\oplus W^\perp$, the restriction of ω on $W^\perp$ is non-degenerate and by induction assumption we can choose a basis $x_2,\dots,x_n,y_2,\dots,y_n$ in $W^\perp$ satisfying the conditions of Lemma. Hence the statement. □

A *Weyl algebra* D_ω is by definition the quotient of the tensor algebra $T(V)$ by the ideal generated by $x\otimes y-y\otimes x-\omega(x,y)$ for all $x,y\in V$. It follows from Lemma 0.1 that D_ω actually depends only on $\dim V=2n$ and can be defined as an associative k-algebra with generators $x_1,\dots,x_n,y_1,\dots,y_n$ and relations

$$[x_i,x_j]=[y_i,y_j]=0,\ [y_i,x_j]=\delta_{ij}. \tag{0.1}$$

Lemma 0.2. *If V and W be finite-dimensional vector spaces with non-degenerate skew symmetric forms ω and ω' respectively. Then $D_{\omega\oplus\omega'}\simeq D_\omega\otimes D_{\omega'}$.*

Proof. The map $T(V\oplus W)\to D_\omega\otimes D_{\omega'}$ which maps (v,w) to $v\otimes 1+1\otimes w$ induces a homomorphism of associative algebras. Write $D_{\omega\oplus\omega'}=T(V\oplus W)/I$. By a straightforward check I lies in the kernel of this homomorphism, hence we have a homomorphism $D_{\omega\oplus\omega'}\to D_\omega\otimes D_{\omega'}$.

To construct the inverse map note that the embeddings $V,W\subset V\oplus W$ induce the homomorphisms $T(V)\to D_{\omega\oplus\omega'}$ and $T(W)\to D_{\omega\oplus\omega'}$, that can be pushed down to $D_\omega\to D_{\omega\oplus\omega'}$ and $D_{\omega'}\to D_{\omega\oplus\omega'}$. Since the images of D_ω and $D_{\omega'}$ commute we obtain a homomorphism $D_\omega\otimes D_{\omega'}\to D_{\omega\oplus\omega'}$ □

Lemma 0.3. *Let $x_1,\dots,x_n,y_1,\dots,y_n$ be a basis in V satisfying the conditions of Lemma 0.1. Then the set of all monomials of the form $x_1^{a_1}\cdots x_n^{a_n}y_1^{b_1}\cdots x_n^{b_n}$ for all non-negative integers $a_1,\dots,a_n,b_1,\dots,b_n$ form a basis of D_ω.*

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Proof. By Lemma 0.2 and Lemma 0.1 it is sufficient to prove the statement in the case $\dim V = 2$. This can be done by the diamond lemma using the lexicographical order on monomials $x^a y^b x^c y^d \dots$. ☐

Let $R = k[X_1, \dots, X_n]$. Let x_i denote the k-linear operator on R given by multiplication on X_i and $\partial_i = \frac{\partial}{\partial X_i}$. There exists a unique homomorphism $T(V) \to \mathrm{End}_k(R)$ that maps x_i to x_i and y_i to ∂_i. This homomorphism respects the relations (0.1) and therefore induces a homomorphsim $\rho : D_\omega \to \mathrm{End}_k(R)$. Thus, R is a D_ω-module.

Exercise 1. Prove that if characteristic of k is $p > 0$, then $\partial_i^p = 0$ and hence ρ is not injective. If characteristic of k is 0, then ρ is injective and M is a simple faithful D_ω-module.

0.2. **Differential operators.** Let R be an arbitrary commutative k-algebra. Set $D_0(R) = \mathrm{End}_R(R) = R$ and define $D_i(R)$ for $i > 0$ inductively by

$$D_i(R) = \{d \in \mathrm{End}_k(R) \mid [d, D_0(R)] \subset D_{i-1}(R).$$

The identity $[f, d_1 d_2] = [f, d_1]d_2 + d_1[f, d_2]$ implies $D_i(R)D_j(R) \subset D_{i+j}(R)$. Therefore

$$D(R) = \bigcup_{i \geq 0} D_i(R)$$

is a subalgebra in $\mathrm{End}_k(R)$. It is called the *algebra of differential operators* in R and its elements are called differential operators. The *order* of a differential operator d is the minimal i such that $d \in D_i(R)$.

Lemma 0.4. *$[D_i(R), D_j(R)] \subset D_{i+j-1}$ and therefore $\mathrm{Gr}\, D(R) = \bigoplus D_i(R)/D_{i-1}(R)$ is commutative.*

Proof. Proceed by induction in i and j. Let $d_1 \in D_i(R)$, $d_2 \in D_j(R)$. For any $f \in D_0$ we have

$$[f, [d_1, d_2]] = [[f, d_1], d_2] + [d_1, [f, d_2]].$$

Since $[f, d_1] \in D_{i-1}(R)$ and $[f, d_2] \in D_{j-1}(R)$, we obtain $[f, [d_1, d_2]] \in D_{i+j-2}(R)$ by induction assumption. Therefore $[d_1, d_2] \in D_{i+j-1}(R)$. ☐

Theorem 0.5. *Let ω be a non-degenerate skew-symmetric form on $2n$-dimensional vector space V. If $\mathrm{char}\, k = 0$, then $D_\omega \simeq D(k[X_1, \dots, X_n])$.*

Proof. We have defined already a D_ω-module structure on $k[X_1, \dots, X_n]$. We set

$$D^p_\omega = \mathrm{span}\{k[x_1, \dots, x_n]\partial_1^{m_1} \cdots \partial_n^{m_n} \mid m_1 + \cdots + m_n = p\}.$$

Then it is easy to check that $[D^0_\omega, D^p_\omega] \subset D^{p-1}_\omega$. Since $D^0_\omega = k[x_1, \dots, x_n]$, we have $D^p_\omega \subset D_p(k[X_1, \dots, X_n])$. To prove equality proceed by induction. Assume $d \in D_p(k[X_1, \dots, X_n])$. Then $d_i = [d, x_i] \in D_{p-1}(k[X_1, \dots, X_n])$ for all $i \leq n$. Note that by induction assumption $d_i \in D^{p-1}_\omega$. Moreover, $[[d, x_i], x_j] = [[d, x_j], x_i]$ hence $[d_i, x_j] = [d_j, x_i]$. I leave it to you as an exercise (see home work 11.2) to check that there exists $d' \in D^p_\omega$ such that $d_i = [d', x_i]$. Then $[d - d', x_i] = 0$ for all $i \leq n$, hence $[d-d', f] = 0$ for all $f \in k[X_1, \dots, X_n]$. Therefore $d - d' \in k[x_1, \dots, x_n]$ and $d \in D^p_\omega$. □

0.3. **The Clifford algebra.** Now let $\operatorname{char} k \neq 2$, V be a finite-dimensional vector space over k and g be a symmetric bilinear form on V. A *Clifford algebra* C_g is the quotient of the tensor algebra $T(V)$ by the ideal generated by $x \otimes y + y \otimes x - g(x, y)$ for all $x, y \in V$. If $x_1, \dots, x_n$ is an orthogonal basis in V with respect to the form g, then C_g is generated by $x_1, \dots, x_n$ subject to relations

$$x_i x_j = -x_j x_i, \text{ if } i \neq j;\ 2x_i^2 = g(x_i, x_i).$$

Examples. If $g = 0$, then $C_g = \Lambda(V)$.

If $\dim V = 1$ and $g \neq 0$, then C_g is either $k \oplus k$ or some quadratic extension of the field k. In particular, if $k = \mathbb{R}$, then C_g is either $\mathbb{R} \oplus \mathbb{R}$ or $\mathbb{C}$.

If $k = \mathbb{R}$ and g is a negative definite form on a 2-dimensional space, then C_g is isomorphic to the ring $\mathbb{H}$ of quaternions. Indeed, for a suitable basis $\{i, j\}$ in V we have $i^2 = j^2 = -1$, $ij = -ji$. Then $1, i, j, k = ij$ form a basis of C_g over $\mathbb{R}$ and satisfy the quaternion's relations.

It is easy to see that C_g is a superalgebra, i.e. C_g has a $\mathbb{Z}_2$-grading that is induced by the natural $\mathbb{Z}$-grading on $T(V)$. In this grading all $x \in V$ are odd. Recall the definition of super tensor product (Lang XVI.6). If A and B are two superalgebras, then $A \otimes_{su} B$ coincides with $A \otimes B$ as a vector space, and multiplication is defined by the formula

$$(a \otimes b)(a' \otimes b') = (-1)^{deg(b) deg(a')} aa' \otimes bb'.$$

Lemma 0.6. *Let V and W be vector spaces equipped with symmetric forms g and g' respectively. Then $C_{g \oplus g'} = C_g \otimes_{su} C_{g'}$*

The proof is similar to the proof of Lemma 0.2 and is left to the reader.

Since g has an orthonormal basis the above lemma implies

Corollary 0.7. *If $\dim V = n$, then $\dim C_g = 2^n$. If $x_1, \dots, x_n$ is an orthogonal basis in V, then the set $\{x_{i_1} \cdots x_{i_s} \mid i_1 < \cdots < i_s\}$ is a basis of C_g.*

Lemma 0.8. *Let k be algebraically closed, $W \subset V$, $\dim V = \dim W + 1$ and $\dim W$ is even. Let g be a non-degenerate symmetric form on V such that the restriction g' of g on W is also non-degenerate. Then $C_g \simeq C_{g'} \oplus C_{g'}$.*

Proof. One can choose an orthogonal basis $x_1, \dots, x_{2m+1}$ in V such that $x_1, \dots, x_{2m}$ is a basis of W. Then $\theta = x_1 \cdots x_{2n+1}$ is a central element in C_g since $x_i \theta = \theta x_i$ for all $i \leq 2m+1$. After a suitable normalization of x_i, we can obtain $\theta^2 = 1$. Then $e^{\pm} = \frac{1 \pm \theta}{2}$ are two central idempotents, $e^+ + e^- = 1$, and it follows by a straightforward check that $e^{\pm} C_g \simeq C_{g'}$. □

Assume now that $\dim V = 2m$ and there exists a basis $x_1, \dots, x_m, y_1, \dots, y_m$ in V such that $g(x_i, x_j) = g(y_i, y_j) = 0$, $g(x_i, y_j) = \delta_{ij}$. Let W be the subspace spanned by $x_1, \dots, x_m$ and W' be the subspace spanned by $y_1, \dots, y_m$. As in the case of the

Weyl algebra we can define a structure of a C_g-module on the exterior algebra $\Lambda(W)$. For any $x \in W$ we define $\sigma(x) \in \mathrm{End}_k(\Lambda(W))$ as the wedge multiplication by x on the left. If $y \in W'$ we define $\sigma(y)(x) = g(y, x)$ for any $x \in W$ and extend it to $\sigma(y) \in \mathrm{End}_k(\Lambda(W))$ by the following rule

$$y(\alpha \wedge \beta) = y(\alpha) \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge y(\beta).$$

Exercise 2. Prove that the above map defines an isomorphism $\sigma : C_g \to \mathrm{End}_k(\Lambda(W))$.

Remark. For an associative superalgebra A denote the commutator by the formula

$$[a, b] = ab - (-1)^{deg(a)deg(b)} ba.$$

In particular, a superalgebra is commutative if $[a, b] = 0$ for all $a, b \in A$. As in the usual case we can define the superalgebra of differential operators $D(A)$ for any commutative superalgebra A. Note that $\mathrm{End}_k(A)$ has a natural $\mathbb{Z}_2$ grading with even part $\mathrm{End}_k(A_{\bar 0}) \oplus \mathrm{End}_k(A_{\bar 1})$ and odd part $\mathrm{Hom}_k(A_{\bar 0}, A_{\bar 1}) \oplus \mathrm{Hom}_k(A_{\bar 1}, A_{\bar 0})$. The definition of the superalgebra of differential operators is the same as the defintion of the algebra of differential operators with understanding that $[\cdot, \cdot]$ defines the supercommutator. Then under above assumptions we have $C_g \simeq D(\Lambda(W))$.

Corollary 0.9. *Let k be algebraically closed and g be a non-degenerate symmetric bilinear form on V. If $\dim V = 2m$, then $C_g \simeq \mathrm{Mat}_{2^m}(k)$ and if $\dim V = 2m + 1$, then $C_g \simeq \mathrm{Mat}_{2^m}(k) \oplus \mathrm{Mat}_{2^m}(k)$.*