1. Let $G$ be a connected simply connected Lie group, $\mathfrak{g}$ be its Lie algebra. Check that the commutator $[G,G]$ is a connected closed subgroup with Lie algebra $[\mathfrak{g},\mathfrak{g}]$.

2. Let $\mathfrak{g}$ be a semisimple Lie algebra, $V$ be a faithful representation of $\mathfrak{g}$, $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ be bases in $\mathfrak{g}$ dual with respect to the form $B_V(X,Y) = \text{tr}_VXY$. The Casimir operator in $V$ is defined by

$$ C_V = \sum_{i=1}^{n} e_i f_i. $$

Define the operator $\gamma : \text{Hom}_k(\Lambda^j \mathfrak{g}, V) \to \text{Hom}_k(\Lambda^{j-1} \mathfrak{g}, V)$ by the formula

$$ \gamma c(x_1, \ldots, x_{j-1}) = \sum_{i=1}^{n} e_i c(f_i, x_1, \ldots, x_{j-1}). $$

Let $d : \text{Hom}_k(\Lambda^{j-1} \mathfrak{g}, V) \to \text{Hom}_k(\Lambda^j \mathfrak{g}, V)$ be the differential in the cohomology complex. Check that

$$ \gamma \circ d + d \circ \gamma = C_V. $$

3. Let $V$ be an irreducible non-trivial representation of a semisimple Lie algebra $\mathfrak{g}$. Prove that $H^j(\mathfrak{g}; V) = 0$ for all $j$. (You have to modify the statement of problem 2 if $V$ is not faithful).

4. Evaluate $H^j(\mathfrak{g}; k)$ for $\mathfrak{g} = \mathfrak{sl}_2(k)$.

5. Let $\mathfrak{g}$ be a simple Lie algebra over algebraically closed field. Check that two invariant forms on $\mathfrak{g}$ are proportional.

\textit{Date:} February 25, 2006.