arXiv:math.DG/9903080 v2 10 Jan 2000

WEBS, LENARD SCHEMES, AND THE LOCAL GEOMETRY OF BIHAMILTONIAN TODA AND LAX STRUCTURES

ISRAEL M. GELFAND AND ILYA ZAKHAREVICH

ABSTRACT. We introduce a criterion that a given bihamiltonian structure admits a local coordinate system where both brackets have constant coefficients. This criterion is applied to the bihamiltonian open Toda lattice in a generic point, which is shown to be locally isomorphic to a Kronecker odd-dimensional pair of brackets with constant coefficients. This shows that the open Toda lattice cannot be locally represented as a product of two bihamiltonian structures.

In a generic point the bihamiltonian periodic Toda lattice is shown to be isomorphic to a product of two open Toda lattices (one of which is a (trivial) structure of dimension 1).

While the above results might be obtained by more traditional methods, we use an approach based on general results on geometry of webs. This demonstrates a possibility to apply a geometric language to problems on bihamiltonian integrable systems, such a possibility may be no less important than the particular results proven in this paper.

Based on these geometric approaches, we conjecture that decompositions similar to the decomposition of the periodic Toda lattice exist in local geometry of the Volterra system, the complete Toda lattice, the multidimensional Euler top, and a regular bihamiltonian Lie coalgebra. We also state general conjectures about geometry of more general "homogeneous" finite-dimensional bihamiltonian structures.

The class of homogeneous structures is shown to coincide with the class of system integrable by Lenard scheme. The bihamiltonian structures which admit a non-degenerate Lax structure are shown to be locally isomorphic to the open Toda lattice.

CONTENTS

Date: March 1999 (Revision II: January 2000) Archived as math.DG/9903080 Printed: January 9, 2000.

0. INTRODUCTION

A *local-geometric approach* consists of considering a geometric structure (for the purpose of our discussion this is a collection of tensor fields) up to a local diffeomorphism, studying its local automorphisms, invariant tensor fields for these automorphisms, and a possibility to decompose the structure into direct products. When applied to integrable systems, this accounts to forgetting all the information related to the given coordinate system (say, whether the structure is polynomial *in this system*).

This approach cannot explain the phenomenon of integrability of a Hamiltonian system, when the initial geometric structure is a Poisson bracket and a function on a manifold. This local geometric structure has too large group of automorphism, and there is no additional invariant functions one could have used to integrate the system. One needs global (or non-invariant) data to integrate a Hamiltonian system.

There is an alternative *bihamiltonian* approach to dynamic systems in which integrability becomes meaningful on the local level already. In this approach one starts with two *compatible*[1] Poisson brackets $\{,\}_1$ and $\{,\}_2$ on M. Basing on these brackets one constructs a dynamical system which is Hamiltonian with respect to any one of these brackets (and in fact to any linear combination of the brackets). The construction of the dynamical system basing on the brackets is called *Lenard scheme*. It provides a family of functions in involution (w.r.t. any linear combination of the brackets). Considering any function of this family as a Hamiltonian w.r.t. any bracket of two one obtains many Hamiltonian flows. In most cases which appear in practice the above family of functions is large enough to make these dynamics integrable (compare with examples in Section 10 and statements of Section 11).

Lenard scheme was formalized in [21, 23, 12, 9], see also [19]. Most of these formalizations assume that at least one of the brackets is symplectic[2] (thus M is

[1]Two Poisson brackets $\{,\}_1$ and $\{,\}_2$ on M are *compatible* if the bracket $\lambda_1\{,\}_1+\lambda_2\{,\}_2$ is Poisson for any λ_1, λ_2.

[2]Any symplectic structure carries a Poisson bracket. We call such Poisson brackets *symplectic*.

even-dimensional). That time it was not realized how these formalizations relate to known applications of Lenard scheme, which consist of a recurrence relation, and of initial data for these relations. The above formalizations of [21, 23, 12, 9, 19] studied the recurrence relations only, ignoring the initial data.

When even-dimensional bihamiltonian structures were classified in [35, 24, 25, 26, 15], it became clear that there is exactly one case where the above "symplectic" formalizations are compatible with the initial data for recurrence. This case is in no way analogous to known examples (see Remark 11.9).

Later, when the analysis of [13, 16] had shown that the periodic KdV system should be considered as an odd-dimensional (though infinite-dimensional) bihamiltonian structure, an alternative approach to the Lenard scheme became necessary. The philosophy of [14] and [15] is that such a substitute is given by the local classification of bihamiltonian structures.

By this philosophy the mentioned above "symplectic" formalizations of Lenard scheme are substituted by the local descriptions of generic even-dimensional bihamiltonian structures in [35, 24, 25, 26, 15]. Indeed, these descriptions provide all the information contained in [23] and [12], and demystify the assumptions of the former papers.

From the classification of even-dimensional bihamiltonian structures in general position, it turns out that this geometry is pretty rigid: on an open subset the structure may be canonically decomposed into a direct product of two-dimensional components, with one distinguished canonically defined coordinate on each of these components. (It is this rigidity which allows a local construction of a big family of commuting Hamiltonians.) However, as in the case of a Hamiltonian system, locally it has discrete parameters only (up to minor details the only parameter is dimension). The morale of this classification is that only 2-dimensional geometry is important, anything else can be combined from 2-dimensional building blocks.

The situation becomes very different in an odd-dimensional case: the structures in general position are indecomposable. In fact such structures are even *micro-indecomposable*, i.e., one cannot represent them as a product of two structures of smaller dimension—even if one restricts attention to one tangent space to a point of the manifold. For *analytical* structures in general position a local classification is also possible ([14, 15]), but it is equivalent to a (local) classification of *non-linear* 1-dimensional bundles over a rational curve, i.e., analytical surfaces which have a submanifold isomorphic to $\mathbb{P}^1$ and a fixed projection onto this curve[3]. This classification involves functional parameters (several functions of two complex variables).

The geometry of such bihamiltonian structures is also very rigid, thus basing on local geometric data one can canonically construct enough functions in involution, thus produce integrable systems. Out of this huge pool of micro-indecomposable integrable systems of the given odd dimension one can single out one particular *flat*

[3] Dimension of the initial bihamiltonian structure depends on the degree of the normal (line) bundle to this curve.

structure, with *both* Poisson structures having constant coefficients in the same coordinate system (any two flat odd-dimensional indecomposable structures are locally isomorphic, compare [13]).

However, after the heuristic of [13] that the KdV system is in fact an infinite-dimensional analogue of an odd-dimensional bihamiltonian structure, no other bihamiltonian structure were (explicitly) considered from the point of view of classification up to a diffeomorphism[4]. One of the targets of this paper is to investigate from this point of view the simplest classical bihamiltonian structures: the open and the periodic finite-dimensional Toda lattices.

While we proceed to this goal, we also provide generally-useful easy-to-check criteria of flatness, investigate Lenard scheme in context of odd-dimensional bihamiltonian geometry, and provide geometric description of systems which admit a Lax representation.

For a detailed overview of the presented results in Section 2 we need to introduce some notions which are going to be used throughout the paper. We do this in Section 1. Here we only list the principal steps of our presentation:

1. criteria of being homogeneous and being Kronecker of corank 1;
2. introduction of webs as a way to encode mutual positions of Casimir functions;
3. proof of the criteria;
4. examples of bihamiltonian structures which demonstrate purposes of different conditions of the criteria;
5. relation of Lenard integrability and homogeneous structures;
6. relation of Lax structures and flatness;
7. application of criteria to Toda lattices.

We also discuss geometric conjecture which might provide geometric description of many other finite-dimensional bihamiltonian structures.

Authors are indebted to A. S. Fokas, A. Givental, Y. Kosmann-Schwarzbach, F. Magri, H. McKean, T. Ratiu, N. Reshetikhin for fruitful discussions, and to A. Gorokhovsky, M. Braverman, B. Khesin, A. Panasyuk, and V. Serganova for the remarks which lead to improvements of this paper. Special thanks go to A. Panasyuk for letting us see the preprint of [29] before it went to print, and to M. Gekhtman for his suggestions on using known Bäcklund–Darboux transformation for Volterra systems.

Revisions: The second revision of this paper (January 2000) introduced references to new papers [36] and [39], expanded bibliography on "classical" bi-Hamiltonian systems, and minor stylistic corrections. Numbering of statements did not change. The archive name of this paper is math.DG/9903080.

[4] Since the geometry of many "classical" bihamiltonian structures is investigated up to minor details, a specialist could easily concoct an answer to such a question from the known results. The conjectural reason why this was not done before is that the answer would not fit into the fixed mindset of "everything is a product of 2-dimensional components", compare with discussion in Section 16.

1. Basic notions

All the geometric definitions which follow are applicable in C^∞ and analytic geometry. We state only the C^∞-variant, the analytic one can be obtained by substituting $\mathbb{R}$ by $\mathbb{C}$.

In what follows if f is a function or a tensor field on M, $f|_m$ denotes the value of f at $m \in M$.

Definition 1.1. A *bracket* on a manifold M is a $\mathbb{R}$-bilinear skewsymmetric mapping $f, g \mapsto \{f, g\}$ from pairs of smooth functions on M to smooth functions on M. This mapping should satisfy the Leibniz identity $\{f, gh\} = g\{f, h\} + h\{f, g\}$. A bracket is *Poisson* if it satisfies Jacobi identity too (thus defines a structure of a Lie algebra on functions on M).

A *Poisson structure* is a manifold M equipped with a Poisson bracket.

Remark 1.2. Leibniz identity implies $\{f, g\}|_m = 0$ if f has a zero of second order at $m \in M$. Thus a bracket is uniquely determined by describing functions $\{f_i, f_j\}$, here $\{f_i\}_{i\in I}$ is an arbitrary collection of smooth functions on M such that for any $m \in M$ the collection $\{df_i|_m\}_{i\in I}$ of vectors in $\mathcal{T}^*_m M$ generates $\mathcal{T}^*_m M$ as a vector space.

Definition 1.3. Call two Poisson brackets $\{,\}_1$ and $\{,\}_2$ on M *compatible* if the bracket $\lambda_1\{,\}_1 + \lambda_2\{,\}_2$ is Poisson for any λ_1, λ_2.

A *bihamiltonian structure* is a manifold M with a pair of compatible Poisson brackets.

In fact it is possible to show that if *one* linear combination $\lambda_1\{,\}_1 + \lambda_2\{,\}_2$ of two Poisson brackets is Poisson and $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, then *any* linear combination $\lambda_1\{,\}_1 + \lambda_2\{,\}_2$ is Poisson. In the analytic situation the coefficients λ_1, λ_2 may be taken to be complex numbers.

If M is a C^∞-manifold with a bracket, we may consider the extension of the bracket to the $\mathbb{C}$-vector space of complex-valued functions on M. In this case $\lambda_1\{,\}_1 + \lambda_2\{,\}_2$ is well-defined even for complex values of λ_1, λ_2. By the above remarks, complex linear combinations of brackets of a bihamiltonian structure are also Poisson. In what follows we always consider brackets as acting on the spaces of complex-valued functions.

Definition 1.4. Given two brackets, $\{\}_M$ on M and $\{\}_N$ on N, the *direct product* of brackets $\{\}_M$ and $\{\}_N$ is the bracket on $M \times N$ defined by

$$\{f_M \times f_N, g_M \times g_N\}_{M\times N} \stackrel{\text{def}}{=} \{f_M, g_M\}_M \times (f_N g_N) + (f_M g_M) \times \{f_N, g_N\}_N .$$

Call a bihamiltonian structure *decomposable* if it isomorphic to a direct product of two bihamiltonian structures of positive dimension.

Obviously, a direct product of two Poisson structures is a Poisson structure, and a direct product of two bihamiltonian structures is a bihamiltonian structure.

Definition 1.5. Consider a bihamiltonian structure $(V, \{,\}_1, \{,\}_2)$, here V is a vector space. The bihamiltonian structure is *translation-invariant* if $\{\mathfrak{T}f, \mathfrak{T}g\}_a = \mathfrak{T}\{f, g\}_a$, $a = 1, 2$, for any parallel translation $\mathfrak{T}$ on V, any f, and any g.

Definition 1.6. A bihamiltonian structure on M is *flat* if it is locally isomorphic to a translation-invariant bihamiltonian structure, i.e., there is a collection of open subsets $M_i \subset M$ such that $M = \bigcup_{i\in I} M_i$, and for any $i \in I$ the restriction of the bihamiltonian structure on M to M_i is isomorphic to an open subset $\widetilde{M}_i \subset V_i$, here V_i is a vector space with a translation-invariant bihamiltonian structure.

A bihamiltonian structure on M is *generically flat* if it is flat on a dense open subset $U \subset M$.

Remark 1.7. Throughout the paper the phrase "*at generic points*" means "at points of an appropriate open dense subset". Similarly, a "*small open subset*" is used instead of "an appropriate neighborhood of any given point".

Remark 1.8. It is possible to give a complete classification of translation-invariant bihamiltonian structures and a complete local classification of flat bihamiltonian structures. (See Remark 4.3.) Classification of generically flat bihamiltonian structures is an interesting unsolved problem which we do not consider in this paper.

Remark 1.9. Any flat structure is generically flat, and any translation-invariant structure is flat, but the opposite is not true. To construct an example of non-translation-invariant flat structure one can take a quotient of a translation-invariant structure on V by an arbitrary discrete subgroup of V. Later we will construct many generically flat structures which are not flat. One of the simplest possible cases will be provided in Example 1.12, see also Theorems 12.4, 12.5.

Not every bihamiltonian structure is generically flat. Important examples of non-generically-flat structures will be constructed in Section 8.

Remark 1.10. The classification of Remark 4.3 shows that *indecomposable* flat bihamiltonian structures break into two types with principally different geometries: even-dimensional structures are modeled by Jordan blocks, and odd-dimensional ones are modeled by Kronecker blocks.

Consider an interesting example of a translation-invariant bihamiltonian structure. In fact it is going to be a key example of this paper: we are going to show that this example is a "building block" in decomposition of many "classical" examples of bihamiltonian structures.

Example 1.11. Consider a vector space V with coordinates $x_0, \dots, x_{2k-2}$ and the Poisson brackets of coordinates

$$\{x_{2l}, x_{2l+1}\}_1 = 1, \qquad \{x_{2l+1}, x_{2l+2}\}_2 = 1, \qquad 0 \leq l \leq k-2, \tag{1.1}$$

any other brackets of coordinate functions $x_0, \dots, x_{2k-2}$ vanishing. This pair of brackets is in fact a translation-invariant bihamiltonian structure.

The following example is the simplest of classical examples of bihamiltonian structures arising in theory of integrable systems.

Example 1.12. Given a Lie algebra $\mathfrak{g}$ and an element $\alpha \in \mathfrak{g}^*$, define a bihamiltonian structure on $\mathfrak{g}^*$ as in [2]. An element $X \in \mathfrak{g}$ defines a linear function f_X on $\mathfrak{g}^*$. Due to Remark 1.2, to define a bihamiltonian structure on $\mathfrak{g}^*$ it is enough to describe brackets $\{f_X, f_Y\}_a$, $a = 1, 2$, $X, Y \in \mathfrak{g}$.

Let $\{f_X, f_Y\}_1$ be a constant function on $\mathfrak{g}^*$ and $\{f_X, f_Y\}_2$ be a linear function on $\mathfrak{g}^*$ given by the formulae

$$\{f_X, f_Y\}_1 \equiv c(X, Y) \stackrel{\text{def}}{=} f_{[X,Y]}(\alpha), \qquad \{f_X, f_Y\}_2 = f_{[X,Y]}.$$

The bracket $\{,\}_2$ is the natural Lie–Kirillov–Kostant–Souriau Poisson bracket on $\mathfrak{g}^*$. The bracket $\{,\}_1$ is translation-invariant. The bracket $\{,\}_2$ is translation-invariant only if $\mathfrak{g}$ is abelian.

Call this bihamiltonian structure *regular* if $\mathfrak{g}$ is semisimple and α is regular semisimple. In such a case Conjecture 16.2 states that this structure is in fact generically flat (compare with [29], where a weaker property is proven[5]). In the case $\mathfrak{g} = \mathfrak{sl}_2$ the conjecture follows from Theorem 3.2. This provides an example of generically flat, but not flat and not translation-invariant structure.

In the case $\mathfrak{g} = \mathfrak{sl}_2$ it is easy to see that this structure is not flat. Indeed, $\{f, g\}_2 |_0 = 0$ for any f, g. If the structure were flat, this would imply $\{f, g\}_2 = 0$ for any f, g, which is obviously false.

By its definition, any flat bihamiltonian structure is locally isomorphic to a direct product of several translation-invariant indecomposable bihamiltonian structures. Introduce a special class of bihamiltonian structures by allowing only special class of factors in the above direct product.

Definition 1.13. A bihamiltonian structure is a *Kronecker* structure if it is locally isomorphic to a direct product of several translation-invariant odd-dimensional indecomposable structures. A *type* of a Kronecker structure is the sequence of dimensions of factors in the above direct product. The Kronecker structure is *indecomposable* if the above product consists of one factor only.

A structure is *generically Kronecker* if it is Kronecker on an open dense subset.

Note that a direct product of translation-invariant structures is translation-invariant. In Section 4 we will see that components of a product of translation-invariant structures are uniquely determined by the product. Thus Kroneker structures are flat structures open subsets of which have *no even-dimensional* indecomposable components.

Remark 1.14. The restriction of having no even-dimensional factors looks very artificial. Moreover, one may think that bihamiltonian structures which have *only* Jordan blocks should be the common case. Say, the classification of even-dimensional bihamiltonian structures in general position ([35, 24, 25, 26, 15]) shows that on an open dense subset such pairs are isomorphic to direct product of 2-dimensional bihamiltonian factors (thus have Jordan blocks only in their decompositions). However, as we show later, some "classical" bihamiltonian systems are in fact generically Kronecker, and we conjecture that many more such examples exist.

The condition of having no Jordan blocks is equivalent to the condition of *completeness* of [2]. Note that the idea of the last condition is to be one of possible *integrability criteria*: bihamiltonian structures which are complete deserve to be called integrable.

[5] Paper [39] contains a proof of generic flatness of this structure.

By Remark 4.3, flat bihamiltonian structures are essentially pairs of skewsymmetric pairings on vector spaces, thus objects of linear algebra. These objects of linear algebra have a classification, but the building blocks of this classification are not only Jordan blocks, but also some new blocks, constructed by Kronecker one year after Jordan. This was the reason for our choice of the name.

Remark 1.15. As Remark 4.3 will show, indecomposable odd-dimensional flat bihamiltonian structures are locally isomorphic to the structure given by (1.1). Thus the local geometry of a Kronecker structure is uniquely determined by its type.

Definition 1.16. Consider a bracket $\{,\}$ on a manifold M. The *associated bivector*[6] *field* η is the section of $\Lambda^2 \mathcal{T} M$ given by $\{f, g\} |_m = \langle \eta|_m, df \wedge dg|_m \rangle$, $m \in M$, here $\langle,\rangle$ denotes the canonical pairing between $\Lambda^2 \mathcal{T}_m M$ and $\Omega^2_m M$.

Definition 1.17. Consider a bracket $\{,\}$ on M and $m_0 \in M$. The *associated pairing* $(,)$ in $\mathcal{T}^*_{m_0} M$ is defined as $(\alpha, \beta) = \{f, g\} |_{m_0}$ if $\alpha = df|_{m_0}$, $\beta = dg|_{m_0}$.

Obviously, the associated bivector field uniquely determines the bracket and visa versa. The associated pairing is a skewsymmetric bilinear pairing.

Given a pair of brackets $\{,\}_1$ and $\{,\}_2$, one obtains two bivector fields η_1, η_2. Analogously, one obtains two skewsymmetric bilinear pairings $(,)_1$, $(,)_2$ on $\mathcal{T}^*_m M$, so that $(\alpha, \beta)_a = \{f, g\}_a |_m$ if $\alpha = df|_m$, $\beta = dg|_m$, $a = 1, 2$.

Definition 1.18. The *rank* of the bracket $\{,\}$ at $m \in M$ is r if the associated skewsymmetric bilinear pairing on $\mathcal{T}^*_m M$ has rank r. In this case the *corank* of the bracket is $\dim M - r$.

A bracket has a *constant (co)rank* if its rank does not depend on the point $m \in M$. A bracket is *symplectic* if the corank is constant and equal to 0.

Definition 1.19. Given a pair of vector spaces V^α and V^β, each equipped with a pair of skewsymmetric bilinear pairings, equip $V^\alpha \oplus V^\beta$ with two pairings $(,)_a \stackrel{\text{def}}{=} (,)^\alpha_a \oplus (,)^\beta_a$, $a = 1, 2$. If a pair is isomorphic to such a direct sum with $\dim V^i \neq 0$, $i = \alpha, \beta$, it is *decomposable*.

[6] A *bivector field* is a skewsymmetric contravariant tensor of valence 2.

It is possible to provide a complete description of indecomposable pairs of skewsymmetric pairings (we will do it in Theorem 4.1).

Definition 1.20. A bihamiltonian structure $(M, \{\}_1, \{\}_2)$ is *homogeneous*[7] of type $(2k_1 - 1, 2k_2 - 1, \dots, 2k_l - 1)$ if for any $m \in M$ the pair of bilinear pairings on $\mathcal{T}_m^* M$ decomposes into a direct sum of indecomposable blocks of dimensions $2k_1 - 1$, $2k_2 - 1$, $\dots$, $2k_l - 1$.

Such homogeneous system is *micro-indecomposable* if $l = 1$.

By uniqueness of decomposition into indecomposable blocks (Theorem 4.1), Kronecker structures are those bihamiltonian structures which are simultaneously homogeneous and flat. There exist important examples of homogeneous structures which are not flat (see Section 8).

What makes homogeneous structures important is the fact that the standard algorithm of "complete integration" (so-called *anchored Lenard scheme*) is applicable to these structures, and this algorithm provides enough functions in involution for these structures only. (See Section 11 for details.)

In fact Kronecker structures are a *very special* case of homogeneous structures:

Conjecture 1.21. Given a sequence $(2k_1 - 1, 2k_2 - 1, \dots, 2k_l - 1)$ there exist $N > 0$ and a natural ways to assign tensor fields $K_1, \dots, K_N$ to a homogeneous bihamiltonian structure such that the structure is Kronecker iff $K_i = 0$, $1 \leq i \leq N$.

In [14] we proved this conjecture in the case of micro-indecomposable structures of dimension 3. This generalized to the case of a general micro-indecomposable structure. In these cases $N = 1$, and the tensor field K_1 is in fact a 2-form of curvature of a connection on an appropriate line bundle (compare with [31]). This 2-form plays the same rôle for bihamiltonian structures as tensor of curvature plays for Riemannian structures.

In what follows we provide criteria of homogeneity and of being an indecomposable Kronecker structure. All these criteria are going to be expressed in the following terms:

Definition 1.22. Call a smooth function F on a manifold M with a Poisson bracket $\{,\}$ a *Casimir* function if $\{F, f\} = 0$ for any smooth function f on M.

Obviously, any function $\varphi(F_1, F_2, \dots, F_k)$ of several Casimir functions is again Casimir.

Definition 1.23. A collection of smooth functions $F_1, \dots, F_r$ on M is *dependent* if $\varphi(F_1, \dots F_r) \equiv 0$ for an appropriate smooth function $\varphi \not\equiv 0$.

We will use this definition when we want to pick up a small independent collection of Casimir function out of the set of all Casimir functions (possibly Casimir functions for several different brackets).

[7] A similar definition appears in [29].

2. Overview

One of the principal targets of this paper is to state three criteria which for a given bihamiltonian structure determine whether it is

1. homogeneous micro-indecomposable structure (Theorem 3.1);
2. indecomposable Kronecker structure (Theorem 3.2);
3. homogeneous structure (Amplification 4.9).

We will use the criterion of Theorem 3.2 to prove that open and periodic *Toda lattices* are generically Kronecker (in Theorems 12.4 and 12.5), and to show that so-called *Lax structures* are indecomposable Kronecker structures provided some conditions of general position hold (in Theorem 15.2.)

The most interesting feature of all these criteria is that they are stated in terms of *mutual position*[8] of Casimir functions for different linear combinations $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ of Poisson brackets of the bihamiltonian structure. We propose a way to encode these mutual positions in a geometric structure of a new type, which we call a *web*.

Recall that the traditional Liouville approach to complete integration of a dynamical system is to provide a system of so-called *action-angle variables*. It so happens that in typical examples the Casimir functions depend on action variables only. Moreover, the action variables are typically much easier to find than the angle variable. This indicates a fundamental asymmetry between action variables and angle variables.

The notion of web (Definition 5.2) amplifies this asymmetry by providing a way to remove angle variables from consideration whatsoever. Since the Casimir functions do not depend on angle variables, it is possible to study the mutual position of Casimir functions in terms of the geometry of the web which corresponds to the given bihamiltonian structure. Thus the conditions of the above criteria (of being homogeneous or Kronecker structures) may be reformulated in terms of webs.

The webs for micro-indecomposable bihamiltonian structures coincide with *Veronese webs* which were studied[9] in [14] and [15]. After the criterion of being a Kronecker structure is reformulated as a statement about webs, it becomes a direct corollary of results of [14]. The results of [14] we use here only scratch the surface of the beautiful theories of [14, 15, 29], in Section 6 we provide an independent formulation of these results, and prove the simplest of them. In Section 7 we deduce from these results the criterion 3.2 of being an indecomposable Kronecker structure.

Though the criteria 3.1 and 4.9 of being a homogeneous system may be formulated in terms of webs, in fact both the hypotheses and the conclusions of these statements

[8] Given several functions $\{F_i\}_{i \in I}$ on a manifold M and a point $m_0 \in M$, consider the directions of differentials $dF_i|_{m_0}$ of these functions at m_0. These directions can be considered as points of the projectivization $\mathcal{P}(\mathcal{T}_{m_0}^* M)$ of the vector space $\mathcal{T}_{m_0}^* M$. Thus we obtain a configuration of $|I|$ points in a projective space, and this configuration depends on $m_0 \in M$. The term "mutual position" refers to studying these configurations of points.

[9] A beginning of a similar study in the case of general homogeneous structures is done in [29].

may be stated in terms of individual cotangent spaces $\mathcal{T}_m^* M$ to the bihamiltonian structure M. Thus these statements may be reduced to appropriate statements of linear algebra. We do this reduction in Section 4.

The criterion 3.2 of being an indecomposable Kronecker structure is expressed in terms of several inequalities. In Sections 8 and 9 we provide examples of bihamiltonian structures which show that no inequality may be weakened without breaking the criterion. These examples are homogeneous bihamiltonian structures which are not flat. One of these examples shows that even a presence of a family of Casimir functions which depend *polynomially* on a parameter does not guarantee flatness.

Note that all the examples of Section 8 are completely integrable. Here we use this vague term in the following sense: the "anchored" Lenard scheme works for these examples, and provides enough functions in involution to construct action-angle variables. In Section 10 we describe the anchored Lenard scheme, and show its relations with Casimir functions (thus with webs).

In Section 11 we show that any homogeneous structure is completely integrable via the anchored Lenard scheme. Theorem 11.6 shows that in fact the class of bihamiltonian structures which may be completely integrated via the anchored Lenard scheme *coincides* with the class of homogeneous structures. This answers a long-standing question in the theory of integrable systems.

We finish the paper with applications of the criterion of flatness to classical examples of integrable systems. After recalling (in Section 12) definitions of *Toda lattices*, we show that the open and the periodic Toda lattices are in fact generically flat (Theorems 12.4 and 12.5).

In Section 15 we introduce a notion of a *Lax structure*. It is a natural modification of the notion of Lax operator from [19]. We show that under appropriate non-degeneracy conditions all the Lax structures (in generic points) are indecomposable Kronecker structures. In particular, two non-degenerate Lax structures of the same dimension become isomorphic when restricted to appropriate open subsets.

Section 16 contains conjectures which extend results of this paper to the case of homogeneous systems which are not micro-indecomposable.

3. The principal criteria

One of the key ideas of this paper (compare with Conjecture 16.2) is that many integrable systems admit a decomposition into a product of "simple" bihamiltonian structures given by (1.1). Theorem 3.2 will provide an easy-to-check criterion when an open subset of a given bihamiltonian structure is *isomorphic* to one given by (1.1). Note that to check the criterion all one needs to know are Casimir functions.

Note that a structure is locally isomorphic to one given by (1.1) iff it is an indecomposable Kronecker structure. In other words, it is simultaneously a micro-indecomposable homogeneous structure, and a flat structure. The following statement provides a criterion for the first part, being a micro-indecomposable homogeneous structure.

Theorem 3.1. *Consider a manifold M, $\dim M \neq 0$, with two compatible Poisson structures $\{,\}_1$ and $\{,\}_2$. Consider an open subset $\mathcal{U} \subset \mathbb{R}$ and a family of smooth functions F_λ, $\lambda \in \mathcal{U}$, on M. Suppose that for any $\lambda \in \mathcal{U}$ the function F_λ is Casimir w.r.t. the Poisson bracket $\lambda \{,\}_1 + \{,\}_2$, and that $dF_\lambda|_m \in \mathcal{T}_m^* M$ depends continuously on λ for any $m \in M$. For $m \in M$ denote by $W_1(m) \subset \mathcal{T}_m^* M$ the vector subspace spanned by the the differentials $dF_\lambda|_m$ for all possible $\lambda \in \mathcal{U}$. If*

1. *for one particular value $m_0 \in M$ one has $\dim W_1(m_0) \geq \frac{\dim M}{2}$;*
2. *for one particular value of $\lambda_1, \lambda_2 \in \mathbb{R}^2$ the Poisson structure $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ has at most one independent Casimir function on any open subset of M near m_0;*

then $\dim M$ is odd, and the bihamiltonian structure on M is homogeneous of type $(\dim M)$ on an open subset $U \subset M$ such that m_0 is in the closure of U.

The proof of this theorem is finished with the proof of Corollary 4.8 in Section 4. Note that this proof implies also that $\dim W_1(m_0) = \frac{\dim M + 1}{2}$. In fact the proof will show that if the Poisson bracket $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ is of constant corank 1, then one may require that $m_0 \in U$.

Amplification 4.9 provides a similar criterion of homogeneity with an arbitrary type.

The following statement shows what one needs to know about a micro-indecomposable homogeneous structure to ensure its flatness (thus it being Kronecker):

Theorem 3.2. *In addition to the conditions of Theorem 3.1 suppose that M is analytic, and $F_\lambda(m)$ depends polynomially on λ:*

$$F_\lambda(m) = \sum_{k=0}^{d} f_k(m) \lambda^k,$$

with analytic coefficients $f_k(m)$ and the degree d satisfying $d < \frac{\dim M}{2}$. Then the bihamiltonian structure on M is flat indecomposable of odd dimension on an open subset U the closure of which contains m_0.

The proof of this theorem takes up to Section 7. Note that this proof implies also that $d = \frac{\dim M - 1}{2}$. Note that Conjecture 16.5 may provide a similar criterion applicable to arbitrary (i.e., not necessarily indecomposable) Kronecker structures. The proof will actually show the following statement (which cannot be expressed in terms of Casimir functions only):

Amplification 3.3. *In the case when in addition to conditions of Theorem 3.2 the Poisson structure $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ is of constant corank 1, the open subset U is in fact a neighborhood of m_0.*

Remark 4.3 will show that all flat indecomposable structures of dimension $2k-1$ are locally isomorphic to each other, thus to the structure given by (1.1). It is easy to see that for the structure of (1.1) one has $\dim M = 2k - 1$, the vector space $W_1(m)$

is spanned by dx_0, $dx_2, \dots, dx_{2k-2}$, and the family $F_\lambda(x)$ of degree $k-1$ is given by (7.1).

Remark 3.4. Not all homogeneous bihamiltonian structures of type $(2k-1)$ are flat, as the examples of Section 8 show (already in the case $k=2$).

The example of $\{,\}_1 = \{,\}_2 \equiv 0$ shows that in Theorem 3.1 one cannot drop the restriction on the number of independent Casimir functions. Considering a direct product of M with any bihamiltonian structure shows the significance of the bound on $\dim W_1$. Moreover, Proposition 9.2 implies that one cannot weaken the bound $d < \frac{\dim M}{2}$ of Theorem 3.2.

Remark 3.5. As Theorem 12.4 will show, one can also consider Theorem 3.2 as a criterion that a given bihamiltonian structure is locally isomorphic to an open subset of the open Toda lattice.

Remark 3.6. Theorems 3.1 and 3.2 are almost immediate corollaries of results of [14] and [15]. However, since we will need many results of these papers anyway, the following three sections provide almost self-contained proof of these theorems. The only component of the proof which requires a reference to [14] is the last statement of Theorem 6.3. The proof of this statement is outside of the scope of this paper (compare with Remark 6.6).

4. Linear case and criterion of homogeneity

Recall the classification of pairs of skewsymmetric bilinear pairings from [13] (see also [14, 15]). For $k \in \mathbb{N}$ consider the identity $k \times k$ matrix I_k. For $\mu \in \mathbb{C}$ consider the Jordan block $J_{k,\mu}$ of size k and eigenvalue μ. The pair of matrices

$$\mathrm{H}_1^{(\mu)} = \begin{pmatrix} 0 & J_{k,\mu} \\ -J_{k,\mu}^t & 0 \end{pmatrix}, \qquad \mathrm{H}_2^{(\mu)} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}$$

defines a pair of skewsymmetric bilinear pairings on vector space $\mathbb{C}^{2k}$. The limit case of $\mu \to \infty$ may be deformed to

$$\mathrm{H}_1^{(\infty)} = \begin{pmatrix} 0 & I_k \\ -I_k & 0 \end{pmatrix}, \qquad \mathrm{H}_2^{(\infty)} = \begin{pmatrix} 0 & J_{k,0} \\ -J_{k,0}^t & 0 \end{pmatrix}.$$

Denote the pair $\left(\mathrm{H}_1^{(\mu)}, \mathrm{H}_2^{(\mu)}\right)$ of skewsymmetric bilinear pairings by $\mathcal{J}_{2k,\mu}$, $k \in \mathbb{N}$, $\mu \in \mathbb{CP}^1$.

Add to this list the so-called Kroneker pair $\mathcal{K}_{2k-1}$. This is a pair in a vector space $\mathbb{C}^{2k-1}$ with a basis $(\boldsymbol{w}_0, \boldsymbol{w}_1, \dots, \boldsymbol{w}_{2k-2})$. The only non-zero pairings are

$$(\boldsymbol{w}_{2l}, \boldsymbol{w}_{2l+1})_1 = 1, \qquad (\boldsymbol{w}_{2l+1}, \boldsymbol{w}_{2l+2})_2 = 1, \tag{4.1}$$

for $0 \leq l \leq k-2$. Obviously, different pairs from this list are not isomorphic.

Theorem 4.1. ([13, 33]) *Any pair of skewsymmetric bilinear pairings on a finite-dimensional complex vector space can be decomposed into a direct sum of pairs of the pairings isomorphic to $\mathcal{J}_{2k,\mu}$, $k \in \mathbb{N}$, $\mu \in \mathbb{P}^1$, and $\mathcal{K}_{2k-1}$, $k \in \mathbb{N}$. The types of the components of this decomposition are uniquely determined.*

Though this simple statement was known for a long time (say, the preprint of [33] existed in 1973), we do not know whether it was published before it was used in [13]. The discussions in [10] and [37] come very close, but do not state this result.

Remark 4.2. The papers [13, 16] described significance of Kronecker blocks in the spectral theory of pencils $A_\lambda = A + \lambda B$, $\lambda \in \mathbb{C}$, of differential operators. Though it is not used in this paper, let us highlight the details of this description.

The Jordan blocks which appear in spectral theory of pencils correspond to values of λ where the dimension of $\operatorname{Ker} A_\lambda$ jumps up. It so happens that due to special properties of the pencil A_λ (say, skew symmetry of operators) it may happen that $\operatorname{Ker} A_\lambda \neq 0$ for any λ (this is what actually happens in the pencil related to the periodic case of KdV equation). In such a case the direct sum of Jordan blocks has a non-trivial complement in the vector space where the pencil acts.

For so-called *finite gap potentials* this *defect space* happens to be exactly the Kronecker block $\mathcal{K}_{2k-1}$ (here k is the number of gaps), thus the situation is absolutely parallel to the finite-dimensional case discussed above. In the case of infinitely many gaps an appropriate infinite-dimensional analogue of Kronecker blocks may be described.

Note, however, that it is absolutely unclear how to translate this description of the linear situation (which is associated to one cotangent space to the phase space of KdV) to the nonlinear bihamiltonian geometry of KdV. While results and conjectures of this paper illuminate the bihamiltonian geometry of finite-dimensional systems in many details, they do not look applicable in infinite-dimensional situation.

The main obstruction is that while all the Kronecker blocks of the same dimension are isomorphic, infinite-dimensional Kronecker blocks acquire new invariants—*fuzzy eigenvalues*. Though fuzzy, these data in fact completely disambiguate points which may be distinguished by Casimir functions (at least for real-analytic potentials, for details see [13]).

One can see that the linearized geometry of periodic KdV is very similar to geometry on odd-dimensional manifolds—there is exactly one Kronecker block, the rest is Jordan blocks with $k=1$, and in generic points there is no Jordan block. But the non-linear geometry of KdV is in some regards also similar to even-dimensional geometry in the sense that the points $m_1, m_2 \in M$ which are separated by Casimir functions also have non-isomorphic pairings in $\mathcal{T}^*_{m_1}M$, $\mathcal{T}^*_{m_2}M$.

Remark 4.3. Given a skewsymmetric bilinear pairing $(,)$ on a vector space V^*, consider the bracket $\{,\}$ on the vector space V described by $\{f,g\}|_m = (df|_m, dg|_m)$. As it is easy to check, this bracket is translation-invariant and Poisson. Given a pair of such pairings $(,)_1$, $(,)_2$ on V^* one obtains a translation-invariant bihamiltonian structure on V. Obviously, any translation-invariant bihamiltonian structure may be obtained this way.

Similarly, any decomposable flat bihamiltonian structure is locally isomorphic to a product of two flat bihamiltonian structures. Indeed, it is enough to show that if an open subset U of the above bihamiltonian structure on V is decomposable, then the pair of pairings on V^{*} is decomposable, which is obvious.

Thus Theorem 4.1 gives also a complete classification of translation-invariant bihamiltonian structures, a complete local classification of flat bihamiltonian structures, and a description of indecomposable flat structures.

For the topics we discuss here it is not necessary to answer the following question, but it is interesting nevertheless:

Conjecture 4.4. Consider two bihamiltonian structures on M_1 and M_2. Suppose that $M_1 \times M_2$ is flat. Then M_1 and M_2 are flat.

The first step in the proof of Theorem 3.1 is the following

Proposition 4.5. *Consider a pair of skewsymmetric bilinear pairings $(,)_1$, $(,)_2$ on a finite-dimensional complex vector space W. Suppose there is a finite set L and there are families of vectors $w_{l,\lambda} \in W$, $l \in L$, polynomially depending on λ such that $\lambda (w_{l,\lambda}, w)_1 + (w_{l,\lambda}, w)_2 = 0$ for any $w \in W$, $l \in L$, and $\lambda \in \mathbb{C}$. Denote by W_1 the vector subspace spanned by $w_{l,\lambda}$, $l \in L$, $\lambda \in \mathbb{C}$. Suppose that for one particular value of λ_1, λ_2 the corank of the bilinear pairing $\lambda_1 (,)_1 + \lambda_2 (,)_2$ is r. If $\dim W_1 \geq \frac{\dim W + r - 1}{2}$, then the pair $(,)_1$, $(,)_2$ is isomorphic to $\oplus_{t=1}^{r} \mathcal{K}_{2k_t-1}$ with $\sum_t k_t = \dim W_1$. In particular, $\dim W_1 = \frac{\dim W + r}{2}$.*

Proof. We may assume that the pair $(,)_1$, $(,)_2$ is a direct sum of several blocks of the form $\mathcal{J}_{2k,\mu}$ and $\mathcal{K}_{2k-1}$, and that for any $l \in L$ the family $w_{l,\lambda} \not\equiv 0$. We suppose that $(\lambda_1, \lambda_2) \neq (0,0)$, it is easy to consider the remaining case separately.

Start with supposing that there are only blocks of the form $\mathcal{K}_{2k_t-1}$, $t = 1, \dots, T$. Then the only things we need to prove is that $T = r$, and $\dim W_1 \leq \sum_t k_t$. The first statement is obvious.

The following lemma follows immediately from the explicit description of the pair $\mathcal{K}_{2k-1}$:

Lemma 4.6. *For the pair $\mathcal{K}_{2k-1}$ of skewsymmetric pairings there exists a family of vectors $\widetilde{w}_\lambda \in W$ polynomially depending on λ such that $\lambda (\widetilde{w}_\lambda, w)_1 + (\widetilde{w}_\lambda, w)_2 = 0$ for any $w \in W$ and $\lambda \in \mathbb{C}$, and the degree of $\widetilde{w}_\lambda$ in λ is $k-1$. This family is defined uniquely up to multiplication by a constant, and it spans a k-dimensional vector subspace. Any other polynomial family w_λ such that $\lambda (w_\lambda, w)_1 + (w_\lambda, w)_2 = 0$ for any $w \in W$ and $\lambda \in \mathbb{C}$ may be written as $p(\lambda) \widetilde{w}_\lambda$ for an appropriate scalar polynomial p.*

Denote the family $\widetilde{w}_\lambda$ for the Kronecker block $\mathcal{J}_{2k_t-1}$ by $\widetilde{w}_\lambda^{(t)}$. Due to this lemma one can write $w_{l,\lambda} = \sum_{t=1}^{T} p_{lt}(\lambda) \widetilde{w}_\lambda^{(t)}$, thus $\dim W_1 \leq \sum_{t=1}^{r} k_t = \frac{\dim W + r}{2}$. Since $\dim W + r$

is even, this shows that $\dim W_1 = \frac{\dim W + r}{2}$, thus finishes proof of the proposition in the case when there are no Jordan blocks.

Consider now the general case. First of all, $w_{l,\lambda} \neq 0$ for a generic λ, thus $w_{l,\lambda}$ (for a generic λ) is in the null-space of the linear combination $\lambda (,)_1 + (,)_2$. Since for a block of the form $\mathcal{J}_{2k,\mu}$ and generic λ this combination has no null-space, it is obvious that $w_{l,\lambda}$ is in the sum of components of the form $\mathcal{K}_{2k-1}$. Since removing a component of the form $\mathcal{J}_{2k,\mu}$ decreases $\dim W$ by $2k$, does not change $\dim W_1$, and may only decrease r, one can see that conditions of the proposition are applicable to the sum of components of the form $\mathcal{K}_{2k-1}$, but the equality on $\dim W_1$ is sharpened by at least k. However, we have seen that it is not possible to sharpen this inequality more than by $\frac{1}{2}$, which proves that W contains no Jordan components. ☐

Amplification 4.7. *In Lemma 4.6 and Proposition 4.5 one may suppose (without changing the conclusions[10] of these statements) that families $w_{l,\lambda}$ are continuous functions of λ defined on a given open subset $\mathcal{U} \subset \mathbb{C}$ or $\mathcal{U} \subset \mathbb{R}$.*

Corollary 4.8. *In conditions of Theorem 3.1 the dimension of M is odd. There is a point m_1 of M such that the pair of skewsymmetric bilinear pairings in $\mathcal{T}_{m_1}^{*} M$ is isomorphic to $\mathcal{K}_{2k-1}$ with $\dim M = 2k - 1$.*

Proof. In this prove we assume that M is a complex manifold, so that $\mathcal{T}_m^{*} M$ is a complex vector space for any $m \in M$. If M is a C^{∞}-manifold, one should substitute $\mathcal{T}_m^{*} M \otimes \mathbb{C}$ instead of $\mathcal{T}_m^{*} M$ in the arguments below.

In conditions of Theorem 3.1 if m_1 in a neighborhood $\widetilde{U}$ of the point $m_0 \in M$, then vectors $dF_\lambda|_{m_1} \in \mathcal{T}_{m_1}^{*} M$ span a vector subspace $W_1(m_1)$ satisfying $\dim W_1(m_1) > \frac{\dim M}{2}$. There is an open subset $U_r \subset \widetilde{U}$ where $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ has a constant corank r. Obviously, there is $r \in \mathbb{Z}$ such that the point m_0 is in the closure of U_r. Restrict our attention to this value of r. Let m_1 be in U_r, and $W = \mathcal{T}_{m_1}^{*} M$, $L = \{\bullet\}$, and $w_{\bullet,\lambda} = dF_\lambda|_{m_1}$. Then the span W_1 of vectors $w_{\bullet,\lambda}$ considered for all possible $\lambda \in \mathcal{U}$ satisfies $\dim W_1 > \frac{\dim W}{2}$, thus $\dim W_1 \geq \frac{\dim W + 1}{2}$.

The vector space W is equipped with two skewsymmetric bilinear pairings $(,)_1$, $(,)_2$ given by values of η_1, η_2 (see Definition 1.16) at m_1. Obviously, $w_{\bullet,\lambda}$ is in the kernel of $\lambda (,)_1 + (,)_2$.

By the conditions of Theorem 3.1, there is at most one independent Casimir function near m_1, thus $r \leq 1$. Obviously, this is the same r as in Proposition 4.5, thus $\dim M \neq 0$ implies $r \neq 0$. Hence the pair $(,)_1$, $(,)_2$ is isomorphic to $\mathcal{K}_{2k-1}$ for an appropriate k, thus $\dim M$ is odd. ☐

This proves Theorem 3.1. In Section 6 we show that it also allows one to apply the results of [14, 15] to prove Theorem 3.2 as well.

Corollary 4.8 uses a particular case of Proposition 4.5 with $r = 1$. While we will not need it in this paper, it is possible to strengthen Corollary 4.8 so that it uses the

[10] With an obvious exception that p in Lemma 4.6 becomes a continuous function.

full power of Proposition 4.5. This result would move us one step in the direction of Conjecture 16.5.

Amplification 4.9. *Consider a manifold M with two compatible Poisson structures $\{,\}_1$ and $\{,\}_2$. Consider a finite set L, open subsets $\mathcal{U}_l \subset \mathbb{C}$, $l \in L$, and families of smooth functions $F_{l,\lambda}$, $l \in L$, $\lambda \in \mathcal{U}_l$, on M. Suppose that for any $l \in L$ and any $\lambda \in \mathcal{U}_l$ the function $F_{l,\lambda}$ is Casimir w.r.t. the Poisson bracket $\lambda \{,\}_1 + \{,\}_2$, and that $dF_{l,\lambda}|_m \in \mathcal{T}_m^* M$ depends continuously on λ for any $l \in L$ and $m \in M$. For $m \in M$ denote by $W_1(m) \subset \mathcal{T}_m^* M$ the vector subspace spanned by the the differentials $dF_{l,\lambda}|_m$ for all possible l and $\lambda \in \mathcal{U}_l$. If for an appropriate $R \in \mathbb{Z}_{\geq 0}$*

1. *for one particular value $m_0 \in M$ one has $\dim W_1(m_0) \geq \frac{\dim M + R}{2}$;*
2. *for one particular value of $\lambda_1, \lambda_2 \in \mathbb{C}^2$ the Poisson structure $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ has at most R independent Casimir functions on any open subset of M near m_0;*

then $\dim M - R$ is even, $\dim W_1(m_0) = \frac{\dim M + R}{2}$, and the bihamiltonian structure on M is homogeneous of type $(t_1, \dots, t_R)$ on an open subset $U \subset M$ such that m_0 is in the closure of U. Here $t_k \in \mathbb{Z}_{>0}$ are appropriate numbers with $\sum_k t_k = \dim M$.

Proof. First of all, one can proceed as in Corollary 4.8 up to the moment we concluded $r \leq 1$. Under the conditions of the amplification we conclude that $r \leq R$, thus $\dim W_1(m_1) \geq \frac{\dim W + r}{2}$. Proposition 4.5 implies that $\dim W_1(m_1) = \frac{\dim M + r}{2}$, thus $r \geq R$. This shows that in fact $r = R$.

We can conclude that for m in an appropriate open subset $U \subset M$ the pair of bilinear pairings on the vector space $\mathcal{T}_m^* M$ is isomorphic to a direct sum of R Kronecker blocks. What remains to prove is that the dimensions of these blocks do not depend on m in an appropriate open subset of U.

Fix a vector space V. For a sequence $T = (t_1 \leq \dots \leq t_R)$ denote by $\mathfrak{F}_T \subset \Lambda^2 V^* \times \Lambda^2 V^*$ the set of pairs of skewsymmetric bilinear pairings which are isomorphic to $\bigoplus_{a=k}^R \mathcal{K}_{t_k}$. In particular, $\mathfrak{F}_T$ is not empty iff all t_k are odd and $\sum t_k = \dim V$. Moreover, $\mathfrak{F}_T$ is a $\mathrm{GL}(V)$-orbit.

It follows that if $\mathfrak{F}_{T'}$ intersects the closure of $\mathfrak{F}_T$, then $\mathfrak{F}_{T'}$ is contained in this closure. Fix a neighborhood U_1 of m_0, let $T^{(1)}, \dots, T^{(N)}$ be such sequences that there are points m in $U \cap U_1$ where the pair of pairings is in each of $\mathfrak{F}_{T^{(k)}}$, $1 \leq k \leq N$. Suppose that $\mathfrak{F}_{T^{(1)}}, \dots, \mathfrak{F}_{T^{(M)}}$ are of maximal possible dimension among $\mathfrak{F}_{T^{(1)}}, \dots, \mathfrak{F}_{T^{(N)}}$, then the points m in $U \cap U_1$ where the pair of pairings is in any one of $\mathfrak{F}_{T^{(k)}}$, $1 \leq k \leq M$, form an open subset. Obviously, at least one of these subsets has m_0 in its closure. □

Remark 4.10. It is not clear whether one can improve the statement of Amplification 4.9 provided that the rank of $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ is constant near m_0. Recall that in Theorem 3.1 one *could* conclude that the structure is homogeneous in a *neighborhood* of m_0. However, under the condition of constant rank one can weaken the condition on dimension to become $\dim W_1(m_0) \geq \frac{\dim M + R - 1}{2}$.

To recognize a possibility of a jump of the type of decomposition of $\mathcal{T}_m^* M$, consider the vector space with a basis $\boldsymbol{w}_0, \dots, \boldsymbol{w}_4, \boldsymbol{W}$ with the only non-zero pairings being

$$(\boldsymbol{w}_{2l}, \boldsymbol{w}_{2l+1})_1 = 1, \qquad (\boldsymbol{w}_{2l+1}, \boldsymbol{w}_{2l+2})_2 = 1,$$

for $0 \leq l \leq 1$, and $(\boldsymbol{W}, \boldsymbol{w}_1) = (\boldsymbol{W}, \boldsymbol{w}_3) = \varepsilon$. If $\varepsilon \neq 0$, then this pair is of the type $\mathcal{K}_3 \oplus \mathcal{K}_3$, if $\varepsilon = 0$, it is of the type $\mathcal{K}_5 \oplus \mathcal{K}_1$. Thus different orbits $\mathfrak{F}_T$ may be adjacent[11] indeed.

5. Bihamiltonian structures and webs

Consider a manifold M with a Poisson bracket $\{,\}$. To define the notion of a symplectic leaf on M, consider Casimir functions on M. The local classification of Poisson structures of *constant rank* [18, 38] shows that for an arbitrary Poisson bracket there is an open (and in interesting cases dense) subset $U \subset M$ and $k \in \mathbb{Z}_{\geq 0}$ such that on U there are k independent Casimir functions $F_1, \dots, F_k$, and any Casimir function on U may be written as a function of $F_1, \dots, F_k$ (we do not exclude the case $k = 0$). The common level sets $F_1 = C_1, \dots, F_k = C_k$ form an invariantly defined foliation on U, which is called the *symplectic foliation*. Note that one can define this foliation as an equivalence relation given by $m_1 \sim m_2$ iff $F(m_1) = F(m_2)$ for any Casimir function F on U.

Consider now a pair $\{,\}_1$, $\{,\}_2$ of compatible Poisson structures on M (i.e., a bihamiltonian structure). Proceed as with the above construction of leaves, and consider

Definition 5.1. A smooth function F on M is *semi-Casimir* if there is $(\lambda_1, \lambda_2) \neq (0, 0)$ such that F is a Casimir function for $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$.

For any open subset $U \subset M$ define an equivalence relation on U by $m_1 \sim m_2$ iff $F(m_1) = F(m_2)$ for any semi-Casimir function F on U. Denote by $\mathcal{B}_U$ the topological space of equivalence classes. Then any semi-Casimir function F on U induces a continuous function on $\mathcal{B}_U$. Any function on $\mathcal{B}_U$ induces a pull-back function on U.

As a result, to any local bihamiltonian structure $(U, \{,\}_1, \{,\}_2)$ we associated a topological space $\mathcal{B}_U$. Let $\lambda = (\lambda_1 : \lambda_2) \in \mathbb{CP}^1$, let $\mathfrak{C}_\lambda$ be the vector space of functions on $\mathcal{B}_U$ pull-backs of which are Casimir functions for $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$. Note that $\varphi(F_1, F_2, \dots, F_k) \in \mathfrak{C}_\lambda$ if φ is smooth and $F_1, F_2, \dots, F_k \in \mathfrak{C}_\lambda$. This allows one to consider $\mathfrak{C}_\lambda$ as a C^0-analogue of a set of local equations of a foliation.

Later we will see that in the cases we study here $\mathcal{B}_U$ is a manifold, and for any λ the space $\mathfrak{C}_\lambda$ is the set of local equations of a foliation on $\mathcal{B}_U$. The codimension of this foliation is not going to depend on $\lambda \in \mathbb{CP}^1$. Anyway, we come to

[11]The recent preprint [29] contains example of a *bihamiltonian structure* where such an adjacency takes place. Thus the statement of Amplification 4.9 cannot be improved.

Definition 5.2. A *web*[12] is a topological space $\mathcal{B}$ with a given subset $\mathfrak{C}_\lambda$ of the set of continuous functions on $\mathcal{B}$ for any $\lambda \in \mathbb{CP}^1$. We require that $\varphi(F_1, F_2, \dots, F_k) \in \mathfrak{C}_\lambda$ if φ is smooth and $F_1, F_2, \dots, F_k \in \mathfrak{C}_\lambda$.

One can also introduce a notion of $\mathcal{U}$*-web* for any subset $\mathcal{U} \subset \mathbb{CP}^1$, the only change being that $\lambda \in \mathcal{U}$ instead of $\lambda \in \mathbb{CP}^1$.

Proposition 5.3. *To any bihamiltonian structure* $(M, \{\}_1, \{\}_2)$ *one can associate a structure of a web on* $\mathcal{B}_M = M/\sim$.

In [14] and [15] it was shown that in some particularly interesting types of bihamiltonian structures the class of the web $\mathcal{B}_U$ up to an isomorphism determines the class of bihamiltonian structure on U up to an isomorphism (compare Theorem 6.3), at least for small open subsets $U \subset M$. This is going to be the main instrument used in this paper: we show that the bihamiltonian structure from Theorem 3.2 and the structure given by (1.1) are of the type mentioned above, and show that the corresponding webs are locally isomorphic. This will imply a local isomorphism of bihamiltonian structures.

To illustrate advantages of the approach of [14] and [15] introduce

Definition 5.4. A smooth function F on M is an *action* function if locally on each small open subset $U \subset M$ it is a pull-back from a function on $\mathcal{B}_U$.

Obviously, any function of the form $\varphi(F_1, \dots, F_l)$ with semi-Casimir functions $F_1, \dots, F_l$ (not necessarily corresponding to the same λ) is an action function. (The name is related to the fact that in bihamiltonian geometry *action-* and *angle-variables* may be defined by local means. Action function are functions of action variables.)

In these terms the approach of [14] and [15] states that to construct an isomorphism of bihamiltonian structures M' and M'' it is enough to associate to each action function on M' an action function on M'' (with appropriate compatibilities conditions this is equivalent to constructing a diffeomorphism of the webs). One needs not care about "angle" variables. Since explicit constructions of "angle" variables is the most complicated step of integration of a dynamical system, this leads to very significant simplifications.

In particular, we are going to construct an isomorphism of manifolds of (approximately) half the dimension of the initial manifolds. Moreover, these smaller manifolds

[12] The reason for this name is that $\mathcal{B}$ is equipped with a huge family of canonically defined subsets: for any λ one consider intersections of level sets of functions from $\mathfrak{C}_\lambda$. Moreover, one can consider intersections of such subsets for different values of λ. If one assumes that $\mathcal{B}$ and these intersections are manifolds, then one gets a delicate network of submanifolds, with infinitely many of them passing through each given point $b \in \mathcal{B}$.

have a very *rigid* geometric structure[13], so it is quite straightforward to construct an explicit diffeomorphism—the moment one suspects that such a diffeomorphism exists.

6. Webs for odd-dimensional bihamiltonian structures

In this section we suppose that $\dim M = 2k - 1$.

Definition 6.1. A pair of bilinear skewsymmetric pairings on a finite-dimensional vector space V is *indecomposable* if the decomposition of Theorem 4.1 has only one component.

Call a pair of brackets $\{,\}_1$ and $\{,\}_2$ on M *micro-indecomposable* at $m \in M$ if the corresponding pair of bilinear pairings on $\mathcal{T}_m^* M$ is indecomposable.

Definition 6.2. A pair of brackets $\{,\}_1$ and $\{,\}_2$ on M is *generic* at $m \in M$, if two corresponding bilinear pairings on $\mathcal{T}_m^* M$ are in general position[14].

Note that Theorem 4.1 implies that an indecomposable pair of parings on an odd-dimensional vector space W is isomorphic to $\mathcal{K}_{2k-1}$, here $\dim W = 2k - 1$.

Now we can codify the program outlined in Section 5:

Theorem 6.3. ([14, 15]) *Consider a pair of compatible Poisson structures on an odd-dimensional manifold M. This pair is generic at m iff it is micro-indecomposable at m. If it is micro-indecomposable at m, then it is micro-indecomposable at m' for any m' in a neighborhood of m.*

If a pair is micro-indecomposable at $m \in M$, then

1. *The web $\mathcal{B}_U$ is a manifold for any small open neighborhood U of m, in other words, for any open $U \ni m$ there is an open subset U', $m \in U' \subset U$, such that $\mathcal{B}_{U'}$ is a manifold;*
2. *The dimension of the manifold $\mathcal{B}_U$ is $\frac{\dim M + 1}{2}$;*
3. *For any $\lambda \in \mathbb{CP}^1$ there is a foliation $\mathcal{F}_\lambda$ on $\mathcal{B}_U$ of codimension 1 such that the subspace $\mathfrak{C}_\lambda$ consists of smooth functions which are constant on leaves of the foliation $\mathcal{F}_\lambda$.*
4. *Consider a micro-indecomposable pair of compatible Poisson structures on a manifold M', and the corresponding manifold $\mathcal{B}_{U'}$ with foliations $\mathcal{F}'_\lambda$. Suppose that both M and M' are analytic. If there is a diffeomorphism $\xi\colon \mathcal{B}_U \to \mathcal{B}_{U'}$ which sends the foliation $\mathcal{F}_\lambda$ to the foliation $\mathcal{F}'_\lambda$ for any $\lambda \in \mathbb{CP}^1$, then the bihamiltonian structures on M and M' are locally diffeomorphic. This local diffeomorphism is compatible with the diffeomorphism ξ.*

[13] In particular, it has at most 1-dimensional group of automorphisms preserving a given point, as opposed to the group of automorphisms of bihamiltonian structures themselves. Recall that in [14] and [15] it was shown that automorphisms of bihamiltonian structures are enumerated by several functions of two variables.

[14] For the purpose of this discussion, this means that $\mathrm{GL}(\mathcal{T}_m^* M)$-orbit of the given pair of pairings is open.

Remark 6.4. Note that the conjecture of [14] implies that the last statement of this theorem holds in the C^∞-case too. In [36] it is announced that this conjecture holds.

We are not going to repeat the proof of this theorem here, but we sketch some arguments which should convince the reader that the first several statements are true (it is the last one which is complicated).

Note that for the pair (4.1) the pairing $\lambda_1 (,)_1 + \lambda_2 (,)_2$ is degenerate (as any skewsymmetric pairing on an odd-dimensional vector space) for any $(\lambda_1, \lambda_2) \in \mathbb{C}^2$, $(\lambda_1, \lambda_2) \neq (0, 0)$, and has a 1-dimensional null-space. (In other words, the dimension of the kernel does not jump up for any $(\lambda_1, \lambda_2) \neq (0, 0)$.) Moreover, Theorem 4.1 momentarily implies that a pair of pairings is indecomposable iff for any $(\lambda_1, \lambda_2) \neq (0, 0)$ the null-space of $\lambda_1 (,)_1 + \lambda_2 (,)_2$ is 1-dimensional.

This together with compactness of $\mathbb{CP}^1$ immediately implies that a small deformation of $\mathcal{K}_{2k-1}$ is indecomposable, thus isomorphic to $\mathcal{K}_{2k-1}$. In turn, this implies that a Zariski open (thus dense) subset of all possible pairs consists of pairs isomorphic to $\mathcal{K}_{2k-1}$. This shows that the property of being generic coincides with indecomposability.

As a corollary, if a pair of brackets $\{,\}_1$ and $\{,\}_2$ is generic at $m \in M$, then in an appropriate neighborhood U of m the bracket $\{,\}^{(\lambda_1,\lambda_2)} \stackrel{\text{def}}{=} \lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ has a corank 1 in U for any $(\lambda_1, \lambda_2) \in \mathbb{C}^2 \smallsetminus \{0\}$.

Now suppose that the bracket $\{,\}^{(\lambda_1,\lambda_2)}$ is Poisson. Since the rank of the corresponding tensor field η is constant it is easy to see ([18, 38]) that there is a (locally defined) Casimir function F_{λ_1,λ_2}. Since the corank is 1, the level hypersurfaces of F_{λ_1,λ_2} are canonically defined.

On the other hand, the normal direction $\boldsymbol{n}_{\lambda_1,\lambda_2}$ to the level hypersurfaces of F_{λ_1,λ_2} at m is the kernel of the corresponding skewsymmetric pairing on $\mathcal{T}_m^* M$. Let $\lambda = (\lambda_1 : \lambda_2) \in \mathbb{CP}^1$, $\boldsymbol{n}_\lambda \stackrel{\text{def}}{=} \boldsymbol{n}_{\lambda_1,\lambda_2}$. Use the isomorphism of $\mathcal{T}_m^* M$ with the form (4.1) to investigate how $\boldsymbol{n}_\lambda$ depends on λ. It is easy to see that the image of the vectors $\boldsymbol{n}_\lambda$ in the coordinate system of (4.1) is proportional to

$$\boldsymbol{w}_0 + \lambda \boldsymbol{w}_2 + \dots + \lambda^k \boldsymbol{w}_{2k}, \tag{6.1}$$

thus taken for any $k+1$ distinct values $\{\lambda_i\}$ of λ the vectors $\boldsymbol{n}_{\lambda_i}$ span the vector subspace $\langle \boldsymbol{w}_0, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{2k} \rangle$. Translating back to the language of differential geometry, one obtains

Corollary 6.5. *Consider foliations $\mathcal{F}_\lambda$ given by level sets of F_{λ_1,λ_2}, here $\lambda = (\lambda_1 : \lambda_2)$. For any $k+1$ distinct values $\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)} \in \mathbb{CP}^1$ the foliations $\mathcal{F}_{\lambda^{(0)}}, \mathcal{F}_{\lambda^{(1)}}, \dots, \mathcal{F}_{\lambda^{(k)}}$ intersect transversally, and the intersection foliation $\mathcal{F}$ does not depend on the choice of $\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(k)}$. The foliation $\mathcal{F}$ is a subfoliation of the foliation $\mathcal{F}_\lambda$ for any $\lambda \in \mathbb{CP}^1$.*

Now one can momentarily see that the local base of the foliation $\mathcal{F}$ coincides with the web $\mathcal{B}_U$ of the bihamiltonian structure, and the push-forward of $\mathcal{F}_\lambda$ to the base of $\mathcal{F}$ is a foliation on $\mathcal{B}_U$ which corresponds to the subspace $\mathfrak{C}_\lambda$ of functions on $\mathcal{B}_U$.

Remark 6.6. The proof of the last statement of Theorem 6.3 might be broken into two parts. The first one proves this statement under the condition that both bihamiltonian structures admit an involution $i \colon M \to M$ such that $\pi \circ i = i \circ \pi$, and such that $i^* \{f, g\}_a = -\{i^* f, i^* g\}_a$ for any functions f and g on M and any $a = 1, 2$ (here π is the projection from M to its web $\mathcal{B}_M$, and we suppose that M is small enough for the conditions of Theorem 6.3 to be applicable). Given such an involution, one can use the set of fixed points of i as the common level set $\{\varphi_i = 0\}$ of would-be angle variables φ_i. After this choice it is possible to construct the angle variables in purely geometric terms.

The second part of the proof consists of showing local existence of such a section for any bihamiltonian structure of Theorem 6.3. This part of the proof uses a hard cohomological statement related to solvability of some overdetermined partial differential equations with variable coefficients. The constant-coefficient variant of this cohomological statement bears some similarity to the Dolbeault lemma.

In fact for the proof of Theorem 3.2 only this constant coefficients variant is needed, so it is possible that our proof of Theorem 3.2 may be significantly simplified.

Remark 6.7. Note also that in many applications one may avoid using the above cohomological statement, since one may be able to construct an involution i explicitly. Say, for the structure (1.1) the involution is given by $x_j \mapsto (-1)^j x_j$.

7. Criterion of flatness

Here we prove Theorem 3.2. Suppose that conditions of Theorem 3.1 are satisfied. By Corollary 4.8 we know that the manifold in question is odd-dimensional, and the pair of Poisson structures is micro-indecomposable on an open subset U. By Theorem 6.3 the corresponding web $\mathcal{B}_U$ is a manifold with a family of foliations depending on parameter $\lambda \in \mathbb{CP}^1$.

On the other hand, it is easy to describe this web explicitly. By definition, for any given $\lambda \in \mathcal{U}$ the level sets of the function F_λ are unions of fibers of the foliation $\mathcal{F}_\lambda$ on U. Since the foliation $\mathcal{F}$ is a subfoliation of $\mathcal{F}_\lambda$, the function F_λ is constant on leaves of $\mathcal{F}$, thus induces a function $\widetilde{F}_\lambda$ on $\mathcal{B}_U$. We obtain a mapping $\widetilde{F}_\bullet \colon \mathcal{B}_U \times \mathcal{U} \to \mathbb{C}$: $(b, \lambda) \mapsto \widetilde{F}_\lambda(b)$. Considered for variable $\lambda \in \mathcal{U}$, the mapping $\widetilde{F}_\bullet$ induces a mapping from $\mathcal{B}_U$ to the topological vector space $C^0(\mathcal{U})$ of continuous functions on $\mathcal{U}$. This mapping sends a given point $b \in \mathcal{B}_U$ to the function $\widetilde{F}_\lambda(b)$ considered as a function of λ.

The topological space $C^0(\mathcal{U})$ carries a canonical structure of a $\mathcal{U}$-web with $\lambda \in \mathcal{U}$, with the subspace $\mathfrak{C}_\lambda$ which consists of functions on $C^0(\mathcal{U})$ of the form $f \mapsto \varphi(f|_\lambda)$ with an arbitrary smooth function φ. The above description of the mapping $\mathcal{B}_U \mapsto$

$C^0(\mathcal{U})$ shows that the $\mathcal{U}$-web structure on $\mathcal{B}_U$ is induced from the $\mathcal{U}$-web structure on $C^0(\mathcal{U})$. One can also note that by the condition of Theorem 3.1 the mapping $\widetilde{F}_\bullet$ is an immersion. Indeed, the rank of $d\widetilde{F}_\bullet|_b$ is exactly the dimension of the span of $dF_\lambda|_m$ for $\lambda \in \mathcal{U}$ (here b is the projection of m to $\mathcal{B}_U$), which coincides with $\dim \mathcal{B}_U$.

Now suppose that the conditions of Theorem 3.2 are satisfied, so F_λ depends polynomially on λ. Denote the degree of F_λ in λ by d. By conditions of Theorem 3.2 one has $d < \frac{\dim M}{2}$. On the other hand, by Lemma 4.6 the degree in λ of $dF_\lambda|_m$ cannot be less than $\frac{\dim M-1}{2}$. Thus the degree of $F_\lambda(m)$ is exactly $\frac{\dim M-1}{2}$ near m_0.

In particular, for any $b \in \mathcal{B}_U$ the image $\widetilde{F}_\bullet(b)$ of b is in fact inside the $(d+1)$-dimensional vector space of polynomials of degree d in λ. Denote this vector space by $\mathcal{P}_d$. Similarly to $C^0(\mathcal{U})$, it carries a natural structure of $\mathbb{C}$-web, moreover, this structure may be extended to become $\mathbb{CP}^1$-web by noting that $\mathcal{P}_d = \Gamma(\mathbb{CP}^1, \mathcal{O}(d))$. Since $\dim \mathcal{P}_d = d+1 = \dim \mathcal{B}_U$, and $\widetilde{F}$ is an immersion, one can see that

Proposition 7.1. *In conditions of Theorem 3.2*

1. *The mapping $\widetilde{F}$ is a local diffeomorphism compatible with structures of webs on $\mathcal{B}_U$ and $\mathcal{P}_d$;*
2. *The structure of the web on $\mathcal{P}_d$ is invariant w.r.t. parallel translations on $\mathcal{P}_d$;*
3. *For any polynomial $p(\lambda)$ of degree d the family of functions $G_\lambda(m) \stackrel{\text{def}}{=} F_\lambda(m) + p(\lambda)$ on M satisfies the conditions of Theorem 3.2;*
4. *The mapping $\widetilde{G}_\bullet\colon \mathcal{B}_U \to \mathcal{P}_d$ associated to $G_\lambda(m)$ is a parallel translation of the mapping $\widetilde{F}_\bullet$.*

In other words, for any point $p \in \mathcal{P}_d$ and any point $b \in \mathcal{B}_U$ one can find a local diffeomorphism of the webs $\mathcal{B}_U$ and $\mathcal{P}_d$ which sends b to p. This shows that if two bihamiltonian structures satisfy the conditions of Theorem 3.2, then two corresponding webs are isomorphic.

Lemma 7.2. *There is a pair of compatible translation-invariant Poisson brackets on $\mathbb{C}^{2k-1}$ which may be equipped with a family of functions satisfying Theorem 3.2.*

Proof. For translation-invariant brackets the tensor fields η_1 and η_2 are constant, thus to describe the bracket we need to describe the pairing on *one* cotangent space. On the other hand, we know that to satisfy Theorem 3.1 this pairing should be isomorphic to $\mathcal{K}_{2k-1}$, thus any pair which satisfies the lemma is isomorphic to one given by the brackets (1.1).

Obviously,

$$F_\lambda = x_0 + \lambda x_2 + \lambda^2 x_4 + \cdots + \lambda^{k-1} x_{2k-2} \tag{7.1}$$

satisfies the conditions of Theorem 3.2, and the bracket $\{,\}_1$ has exactly one independent Casimir function x_{2k-2}. ☐

Corollary 7.3. *In conditions of Theorem 3.2 the web $\mathcal{B}_U$ is locally isomorphic to the web corresponding to the bihamiltonian structure given by Equation* (1.1), *here* $\dim M = 2k-1$.

Indeed, both these webs are locally isomorphic to the web on $\mathcal{P}_{k-1}$.

Now the last part of Theorem 6.3 implies that the bihamiltonian structure on M is isomorphic to the structure given by (1.1), which finishes the proof of Theorem 3.2.

8. Examples of non-flat structures

Here we show that Theorem 3.2 is not a tautology. To do this, we construct a huge pool of bihamiltonian structures which satisfy the conditions of Theorem 3.1, but are not isomorphic to each other (in particular, only one of them is flat). All these structures are integrable by the anchored Lenard scheme (see Section 10, compare with descriptions [23, 12] in symplectic settings).

All the constructions below can be performed in C^∞-geometry and in analytic geometry (unless explicitly specified). We state the C^∞-case only.

Fix an open subset $\mathcal{B} \subset \mathbb{R}^2$ and a smooth function $f(x,y)$ of two variables $(x,y) \in \mathcal{B}$. Consider two brackets on $\mathcal{B} \times \mathbb{R}$ defined by

$$\begin{gathered} \{x,y\}_1 = \{y,z\}_1 = 0, \qquad \{x,y\}_2 = \{x,z\}_2 = 0, \\ \{x,z\}_1 = \frac{\partial f(x,y)}{\partial y}, \qquad \{y,z\}_2 = -\frac{\partial f(x,y)}{\partial x}. \end{gathered} \tag{8.1}$$

Obviously, these two brackets form a bihamiltonian structure on $\mathcal{B} \times \mathbb{R}$.

Definition 8.1. Denote this bihamiltonian structure on $\mathcal{B} \times \mathbb{R}$ by M_f.

Assume that both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ do not vanish in $\mathcal{B}$. One can see that any non-zero linear combination $\lambda_1\{,\}_1 + \lambda_2\{,\}_2$ has rank 2, thus has exactly one independent Casimir function near $b \times z \in \mathcal{B} \times \mathbb{R}$. Moreover, it is easy to construct a family of Casimir functions for different $\lambda \stackrel{\text{def}}{=} \lambda_1 : \lambda_2$ which depend smoothly on $\lambda \in \mathcal{U} \subset \mathbb{R}$ (compare with Section 10). Thus the structure (8.1) satisfies conditions of Theorem 3.1, and is homogeneous of type[15] (3).

One can write explicitly Casimir functions for the values $\lambda_1 : \lambda_2$ being ∞, 0, and 1, i.e., for $\{,\}_1$, for $\{,\}_2$, and for $\{,\}_1 + \{,\}_2$. They have the form $F_\infty(y)$, $F_0(x)$, and $F_1(f(x,y))$ for arbitrary functions F_0, F_1, F_∞. This implies

Lemma 8.2. *Consider a bihamiltonian structure $(M, \{,\}_a, \{,\}_b)$ and a mapping $p\colon M \to M_f$ which is a local isomorphism of bihamiltonian structures. Consider x, y, f as functions on M_f. Then $x \circ p$ is a Casimir function for $\{,\}_b$, $y \circ p$ is a Casimir function for $\{,\}_a$, and $f \circ p$ is a Casimir function for $\{,\}_a + \{,\}_b$.*

[15]Using Theorem 6.3, it is easy to show that any *analytic* homogeneous bihamiltonian structure of type (3) is locally isomorphic to this structure for an appropriate f. For the discussion of global geometry for such structures, see [30].

Since $\{,\}_1$, $\{,\}_2$, and $\{,\}_1+\{,\}_2$ are of corank 1, the structure $(M,\{,\}_a,\{,\}_b)$ of the lemma determines the functions $x'=x\circ p$, $y'=y\circ p$, and $f'=f\circ p$ uniquely up to transformations of the form $\widetilde{A}=\alpha_A(A)$, $A\in\{x',y',p'\}$. Thus $(M,\{,\}_a,\{,\}_b)$ determines the function $f(x,y)$ up to transformations of the form $\widetilde{f}=\gamma(f(\alpha(x),\beta(y)))$. Indeed, the graph of $f(x,y)$ coincides with the image of M w.r.t. the mapping $x'\times y'\times f'$.

Lemma 8.3. *Consider a smooth function $f\colon U_1\times U_2\to U_3$, $U_{1,2,3}\subset\mathbb{R}$, and diffeomorphisms $\alpha\colon U_1'\to U_1$, $\beta\colon U_2'\to U_2$, $\gamma\colon U_3\to U_3'$. Let*

$$g(x,y)\stackrel{\text{def}}{=}\gamma(\alpha(x),\beta(y)).$$

Then the bihamiltonian structures M_f and M_g are isomorphic.

Proof. It is easy to write the diffeomorphism explicitly as $x'=\alpha(x)$, $y'=\beta(y)$, $z'=\xi(x,y)z$ with an appropriate function $\xi(x,y)$. □

The next step is to introduce additional conditions on a function $g(x,y)$ which would define the diffeomorphisms α, β, γ almost uniquely. First, assume $(0,0)\in\mathcal{B}$, $f(0,0)=0$. Consider $O=(0,0,0)$ as a marked point on $\mathcal{B}\times\mathbb{R}$. Then the condition that $x'-$ and y'-coordinates of O remain 0 leads to $\alpha(0)=0$ and $\beta(0)=0$.

Given a function $f(x,y)$ such that $f(0,0)=0$, $\partial f/\partial x\neq 0$ and $\partial f/\partial y\neq 0$ near $x=y=0$, one can find local coordinate changes $x'=\alpha(x)$, $y'=\beta(y)$, and $f'=\gamma(f)$, $\alpha(0)=\beta(0)=\gamma(0)=0$, such that $\varphi(x',y')\stackrel{\text{def}}{=}\gamma(f(\alpha^{-1}(x'),\beta^{-1}(y')))$ satisfies

$$\begin{gathered}\frac{\partial\varphi}{\partial x'}|_{x'=0}=1,\ \frac{\partial\varphi}{\partial y'}|_{x'=0}=1,\\ \frac{\partial\varphi}{\partial x'}|_{y'=0}=\frac{\partial\varphi}{\partial y'}|_{y'=0},\end{gathered}\qquad \varphi(0,0)=0. \tag{8.2}$$

Moreover, such a coordinate change and the function φ are defined uniquely up to simultaneous multiplication of x', y' and φ by the same constant. If $\frac{\partial^2\varphi}{\partial x'\partial y'}|_{(0,0)}\neq 0$, then this last degree of freedom may be eliminated by a requirement that $\frac{\partial^2\varphi}{\partial x'\partial y'}|_{(0,0)}=1$. (In fact if $\varphi(x',y')\not\equiv x'+y'$, then one can fix coordinates x' and y' by normalizing an appropriate derivative of φ of higher order.)

These arguments lead to the following statement from ([14, 15]):

Theorem 8.4. *Consider two functions $\varphi(x,y)$ and $\varphi'(x,y)$ defined in a neighborhoods $\mathcal{B}$, $\mathcal{B}'$ of (0,0). Suppose that both φ and φ' satisfy* (8.2)*. There exists a local diffeomorphism between M_φ and $M_{\varphi'}$ which preserves the point (0,0,0) iff there exists $C\in\mathbb{R}$, $C\neq 0$, such that $\varphi(Cx,Cy)\equiv C\varphi'(x,y)$ near $x=y=0$.*

Corollary 8.5. *If $\varphi(x,y)$ satisfies* (8.2)*, then M_φ is flat iff $\varphi(x,y)=x+y$.*

Proof. Indeed, $\varphi(x,y)=f(x,y)=x+y$ defines a structure with constant coefficients (compare with (1.1)), thus a flat one. Any other function $\varphi(x,y)$ which satisfies (8.2) will define a non-isomorphic bihamiltonian structure, thus a non-flat one. □

As a corollary, one obtains a lot of structures which are not flat, thus cannot satisfy the conditions of Theorem 3.2. The next logical step is to check whether these bihamiltonian structures are "integrable". To do so, we need a formalization of the notion of integrability. One of the simplest such notions is integrability by the anchored Lenard scheme, which is introduced in Section 10. Example 10.15 will demonstrate that any homogeneous bihamiltonian structure M_f is integrable[16] in this sense.

9. One counterexample

The examples of Section 8 show that one cannot expect to prove the conclusion of Theorem 3.2 in conditions of Theorem 3.1, even if one requires $\mathcal{U}$ to become the whole complex plane $\mathbb{C}$, and requires F_λ to depend analytically on λ. Recall that the notation M_f was introduced in Definition 8.1.

To show that in fact even the restriction on the degree of the polynomial in Theorem 3.2 cannot be improved if $k=2$, consider

Definition 9.1. Consider a bihamiltonian structure on M and a smooth function on $\mathbb{R}\times M$, $(\lambda,m)\mapsto C_\lambda(m)$. Call this function a $[d]$*-family* if for any fixed $m\in M$ the function $C_\lambda(m)$ of λ depends on λ as a polynomial of degree d or less, and for any fixed $\lambda\in\mathbb{R}$ the function C_λ of m is a Casimir function for $\lambda\{,\}_1+\{,\}_2$.

Proposition 9.2. *There exists a function $\varphi(x,y)\not\equiv x+y$ which satisfies* (8.2)*, and such that the bihamiltonian structure M_φ admits a* [2]*-family F_λ.*

Proof. Actually it is possible to describe *all* analytic functions $f(x,y)$ such that M_f admits a [2]-family F_λ, at least if we are allowed to restrict our attention to smaller open subsets.

If G_λ is a [1]-family, then $(d\lambda+e)G_\lambda+a\lambda^2+b\lambda+c$, $a,b,c,d,e\in\mathbb{C}$, gives a [2]-family. Call such families *simple* families. By Theorem 3.2 a simple family may exist on a flat bihamiltonian structure only.

Given an open subset $\mathcal{U}\subset\mathbb{C}$ with two analytic functions $\eta,\zeta\colon\mathcal{U}\to\mathbb{C}$, and an open subset $\mathfrak{B}\subset\mathbb{C}\times\mathbb{C}$ with an analytic function $\Lambda\colon\mathfrak{B}\to\mathcal{U}$, let

$$F_\lambda^{(\eta\zeta\Lambda)}(x,y)\stackrel{\text{def}}{=}(\Lambda(x,y)-\lambda)^2y+\zeta(\Lambda(x,y))+\lambda\eta(\Lambda(x,y)).$$

Lemma 9.3. *Consider an analytic bihamiltonian structure M_f with a* [2]*-family F_λ which is not simple. Then there exists an open subset $\mathcal{U}\subset\mathbb{C}$ with two analytic*

[16]In fact Proposition 11.7 will show that *any* homogeneous structure is Lenard-integrable.

functions $\eta, \zeta \colon \mathcal{U} \to \mathbb{C}$, and an open subset $\mathfrak{B} \subset \mathbb{C} \times \mathbb{C}$ with an analytic function $\Lambda \colon \mathfrak{B} \to \mathcal{U}$ such that $F_\lambda(x, y, z) = F_\lambda^{(\eta\zeta\Lambda)}(x, y)$ and

$$\frac{d\zeta}{dt} + t\frac{d\eta}{dt} = 0 \quad \text{if } t \in \mathcal{U}, \qquad x = \Lambda^2(x, y)\, y + \zeta(\Lambda(x, y)) \quad \text{if } (x, y) \in \mathfrak{B},$$

In particular, M_f is locally isomorphic to an open subset of $M_{F_1^{(\eta\zeta\Lambda)}}$.

Proof. Write $F_\lambda = \lambda^2 H_0 + \lambda H_1 + H_2$. Being a Casimir function for $\{,\}_1$, H_2 depends on x only, similarly H_0 depends on y only. If H_0 does not depend on y, then $H_0 = \text{const}$, thus F_λ is simple. Similarly one can exclude the case when H_2 does not depend on x. Restricting to a smaller open subset, one can assume that $x = \alpha(H_2)$, $y = \beta(H_0)$, thus one can consider H_0 and H_2 instead of coordinates x and y on $\mathcal{B}$. In particular, we may assume that $H_2 = x$, $H_0 = y$.

By Lemma 4.6, given a point $(x, y) \in \mathcal{B}$, $dF_\lambda|_{(x,y)}$ may be written as $p(\lambda)\, \widetilde{w}_\lambda$, here $p(\lambda)$ is a scalar polynomial in λ, and $\widetilde{w}_\lambda$ is a vector-valued polynomial of degree 1 in λ. Thus $\deg p = 1$, denote the zero of p by Λ. We conclude that for each (x, y) there is $\Lambda(x, y)$ such that $dF_\lambda|_{(x,y)} = 0$ if $\lambda = \Lambda(x, y)$.

Restricting to an appropriate open subset of $\mathcal{B}$, we may assume that Λ depends analytically on x and y. If Λ is constant, then $dF_\Lambda = 0$ implies that $\frac{F_\lambda(x,y) - F_\lambda(x_0,y_0)}{\lambda - \Lambda}$ is linear, thus F_λ is simple. Thus, decreasing $\mathcal{B}$ again, we may assume that either (Λ, y) or (Λ, x) give a local coordinate system on $\mathcal{B}$. Assume that (Λ, y) is a local coordinate system.

The condition $dF_\Lambda|_{(x,y)} = 0$ implies

$$dx = -\Lambda^2 dy - \Lambda dH_1, \tag{9.1}$$

Thus 2-form $d(-\Lambda^2 dy - \Lambda dH_1)$ vanishes, in other words, $2\Lambda d\Lambda dy + d\Lambda dH_1 = 0$. One can conclude that in the coordinate system (Λ, y) one has $\frac{\partial H_1}{\partial y}|_{\Lambda=\text{const}} = -2\Lambda$, or $H_1 = -2\Lambda y + \eta(\Lambda)$ with an unknown function $\eta(t)$. Equation (9.1) leads to

$$x = \Lambda^2 y + \zeta(\Lambda), \qquad \frac{d\zeta}{dt} = -t\frac{d\eta}{dt}. \tag{9.2}$$

This leads to a formula for F_λ in coordinates y and Λ:

$$F_\lambda = (\lambda - \Lambda)^2 y + \zeta(\Lambda) + \lambda\eta(\Lambda), \qquad \frac{d\zeta}{dt} = -t\frac{d\eta}{dt}. \tag{9.3}$$

By (8.1), $F_1 = \gamma(f)$ for an appropriate function γ. If $F_1(x, y) \equiv \text{const}$, then $\frac{F_\lambda - F_1}{\lambda - 1}$ is linear, thus F_λ is simple. Hence decreasing $\mathcal{B}$ we may assume that one can write $f = \varepsilon(F_1)$ for an appropriate function ε. Thus M_{F_1} locally isomorphic to M_f.

Moreover, (9.2) implies that (Λ, x) is also a coordinate system on an open subset of $\mathcal{B}$. Exchanging x and y, we see that our assumption that (Λ, y) is a local coordinate system is *always* satisfied. ☐

Lemma 9.4. *There is a way to associate to an open subset $\mathcal{U} \subset \mathbb{C}$ and a function $\eta \colon \mathcal{U} \to \mathbb{C}$ a homogeneous bihamiltonian structure $M^{(\eta)}$ of type (3) with a family of function $F_\lambda^{(\eta)}$ which is quadratic in λ. In conditions of Lemma 9.3 there is a diffeomorphism of an open subset of M_f with an open subset of $M^{(\eta)}$ and $C \in \mathbb{C}$ such that the diffeomorphism sends one bihamiltonian structure to another, and the family F_λ to the family $F_\lambda^{(\eta)} + C$. A change of η of the form $\eta'(t) = \eta(t) + at + b$ leads to an isomorphic bihamiltonian structure with the isomorphism sending $F_\lambda^{(\eta)}$ to $F'^{(\eta)}_\lambda + A\lambda^2 + B\lambda + C$, $A, B, C \in \mathbb{C}$.*

Proof. Indeed, given the functions ζ and η, let $\bar{\Sigma} = \{(x, y, \Lambda) \in \mathbb{C}^3 \mid x = \Lambda^2 y + \zeta(\Lambda)\}$. Let $\Sigma = \left\{(x, y, \Lambda) \in \bar{\Sigma} \mid y \neq \frac{1}{2}\frac{d\eta}{d\Lambda}, \Lambda \neq 0\right\}$ be the subset of $\bar{\Sigma}$ where x and y provide a local coordinate system. Plugging into (9.3), one obtains functions Λ, x, y, F_λ defined on Σ.

Functions x and y provide a local coordinate system near any point of Σ. On an open subset $\Sigma_1 \subset \Sigma$ one has $\frac{\partial F_1}{\partial x} \neq 0$, $\frac{\partial F_1}{\partial y} \neq 0$, thus putting F_1 into (8.1) instead of f defines a homogeneous bihamiltonian structure on $M^{(\eta,\zeta)} \stackrel{\text{def}}{=} \Sigma_1 \times \mathbb{C}$. As we have seen in the proof of Lemma 9.3, an open piece of M_f is isomorphic to an open piece of $M^{(\eta,\zeta)}$, moreover, the families F_λ are preserved by this isomorphism.

By (9.2), (9.3), a change of the form $\eta'(t) = \eta(t) + 2at + b$ together with the change $\zeta'(t) = \zeta(t)\text{-at}^2 + d$ would lead to a parallel translation of the surface $\bar{\Sigma}$, and to the required change of functions F_λ. In particular, a change in ζ only will not change $M^{(\eta,\zeta)}$, and will change the family F_λ by an additive constant only. ☐

Lemma 9.5. *In the conditions of Lemma 9.4 the family $F_\lambda^{(\eta)}$ on $M^{(\eta)}$ is a [2]-family.*

Proof. By construction of bihamiltonian structure on $M^{(\eta)}$, the functions $F_0^{(\eta)} \equiv x$, $F_1^{(\eta)}$, and the leading coefficient H_0 of the quadratic family $F_\lambda^{(\eta)}$ are Casimir functions for $\{,\}_1$, $\{,\}_1 + \{,\}_2$, and $\{,\}_2$ correspondingly. Fix $m_0 \in M$. Then $dF_\lambda^{(\eta)}|_{m_0}$ is a vector-function which is quadratic in λ. Moreover, at $\lambda = \Lambda(m_0)$ this vector-function vanishes, thus $dF_\lambda^{(\eta)}|_{m_0} = (\lambda - \Lambda(m_0))\, w(\lambda)$, here $w(\lambda)$ is a vector-function of degree 1 in λ. Extend w to become a homogeneous function $w(\lambda_1, \lambda_2)$ of homogeneity degree 1, here $\lambda = \lambda_1 : \lambda_2$.

This function $w(\lambda_1, \lambda_2)$ is in the null-space of $\lambda_1(,)_1 + \lambda_2(,)_2$ for three values 0, 1 and ∞ of $\lambda_1 : \lambda_2$. However, the null-space for a pair of pairing which is isomorphic to $\mathcal{K}_3$ depends linearly on $\lambda_1 : \lambda_2$ (compare with (4.1)). We conclude that $w(\lambda_1, \lambda_2)$ is in the null-space for $\lambda_1(,)_1 + \lambda_2(,)_2$ for any λ_1, λ_2, thus $F_\lambda^{(\eta)}$ is a [2]-family indeed. ☐

Lemma 9.6. *The bihamiltonian structure $M^{(\eta)}$ of Lemma 9.4 is flat on any open subset iff $\frac{d^2\eta}{dt^2} \equiv 0$.*

Proof. By arguments of Section 8, $M_{F_1^{(\eta)}}$ is flat on an open subset iff there is a local dependence between $F_1^{(\eta)}$, x and y of the form $a\left(F_1^{(\eta)}\right) + b(x) + c(y) = 0$. If $\eta = \zeta \equiv 0$, then $F_\lambda^{(\eta)} = \lambda^2 y \pm 2\lambda\sqrt{xy} + x$, thus $F_1^{(\eta)} = \left(\sqrt{x} \pm \sqrt{y}\right)^2$, thus $M_{F_1^{(\eta)}}$ is flat. By Lemma 9.4 this proves the "if" part.

If a dependence $a\left(F_1^{(\eta)}\right)+b(x)+c(y)=0$ exists, then $\frac{\partial}{\partial\Lambda}|_{y=\mathrm{const}}\left(a\left(F_1^{(\eta)}\right)+b(x)\right)=0$, or

$$(\Lambda-1)\,\alpha\left(F_1^{(\eta)}\right)=-\Lambda\beta(x), \tag{9.4}$$

here $\alpha\left(F_1^{(\eta)}\right)=da/dF_1^{(\eta)}$, $\beta(x)=db/dx$.

Taking derivative $\frac{\partial}{\partial y}|_{\Lambda=\mathrm{const}}$ of (9.4), and dividing by the cube of (9.4), one obtains $\left(\alpha^{-3}\frac{d\alpha}{dF_1^{(\eta)}}\right)\left(F_1^{(\eta)}\right)=\beta^{-3}\frac{d\beta}{dx}(x)$. Since $F_1^{(\eta)}$ and x are independent, and $\alpha\not\equiv 0$, $\beta\not\equiv 0$, we conclude that

$$\frac{d\alpha}{dF_1^{(\eta)}}=C\alpha^3,\qquad \frac{d\beta}{dx}=C\beta^3.$$

Thus $\alpha\left(F_1^{(\eta)}\right)=D/\sqrt{F_1^{(\eta)}-\varphi_0}$, $\beta(x)=D/\sqrt{x-x_0}$, $x_0,\varphi_0,D\in\mathbb{C}$, $D\neq 0$. Hence $(\Lambda-1)^2(x-x_0)=\Lambda^2(F_1-\varphi_0)$ by (9.4), and $(-2\Lambda+1)(\zeta(\Lambda)-\zeta_0)=\Lambda^2(\eta(\Lambda)-\eta_0)$ for appropriate $\eta_0,\zeta_0\in\mathbb{C}$. Together with $\zeta'(t)=-t\eta'(t)$ this shows that $\eta(\Lambda)=C\Lambda+E$. □

This finishes the proof of Proposition 9.2. □

10. Anchored Lenard scheme

Recall how Lenard scheme works[17]. Since descriptions of Lenard scheme are in many cases based on an assumption that the Poisson bracket is symplectic, here we supply as many details as possible to unravel the relation of the anchored Lenard scheme with Casimir functions which depend on parameters (such functions do not exists in symplectic situation). In turn, such functions are directly related to webs.

Remark 10.1. Before we proceed with description of the problem which the Lenard scheme solves, we need to resolve a possible ambiguity. The Lenard scheme "as a method of integration" consists of recurrence relations and initial data for these relations. However, the existing *formalizations* of Lenard scheme (e.g., [23, 12, 19]) consider the recurrence relations only, omitting the initial data. The latter approach has an advantage of being more general, in particular, it works in symplectic case too. However, this approach does not address the question when recurrence relations have solutions (these relations are overdetermined), in particular they do not specify how to find the initial data which would make the Lenard scheme succeed.

Since in our settings symplectic Poisson structures have only a tangential rôle, here we consider only the variant of Lenard scheme which is important in applications, when both the initial data and the recurrence relation are specified. To avoid any confusion, we call this variant the *anchored Lenard scheme*. For this scheme we

[17]For history of Lenard scheme and of the term "Lenard scheme" see [19].

will not only be able to describe when recurrence relations are solvable, we will also describe which bihamiltonian systems are completely integrable by Lenard scheme. As Theorem 11.6 will show, such systems are never symplectic, which may explain why the anchored Lenard scheme was not formalized before.

The method of *first integrals* to "integrate" a system of ordinary differential equations $\frac{d}{dt}m(t)=v(m(t))$, $m\in M$, starts with writing the *Hamiltonian representation* for this system, i.e.,

$$\frac{d}{dt}\left(f|_{m(t)}\right)=\{H,f\}|_{m(t)}\qquad\text{for any function } f \text{ on } M.$$

To do this one needs to find the Poisson bracket $\{,\}$ and the function H (called the *Hamiltonian* of the equation). Note that H and $\{,\}$ uniquely determine the initial vector field v.

Additionally, one needs to find a large enough independent collection of functions H_i on M which all commute with each other w.r.t. $\{,\}$ and such that H can be expressed as a function of H_i. Alternatively, one starts with a given bracket $\{,\}$ and a function H, then the problem is to find the family H_i. In fact, given the family H_i, one can take as H any function of H_i.

Thus to construct an integrable system a key problem is to find a large family of independent functions H_i *in involution*, i.e., such that $\{H_i,H_j\}=0$. The anchored Lenard scheme is a particular algorithm to construct such a family on a bihamiltonian manifold.

Start with a way to find many functions in involutions, not necessarily independent. Most statements below are applicable both in C^∞-geometry and in analytic geometry. In such cases we state the smooth variant only, for the corresponding analytic statement one needs to substitute $\mathbb{RP}^1$ by $\mathbb{CP}^1$.

Definition 10.2. Consider a bihamiltonian structure on M, an open subset $\mathcal{U}\subset\mathbb{RP}^1$ and a smooth function on $\mathcal{U}\times M$, $(\lambda,m)\mapsto C_\lambda(m)$. Consider this function as a family C_λ, $\lambda\in\mathcal{U}$, of functions on M. Call C_λ a λ-*Casimir family* on M if $C_{\lambda_1:\lambda_2}$ is a Casimir function for $\lambda_1\{,\}_1+\lambda_2\{,\}_2$ for $(\lambda_1:\lambda_2)\in\mathcal{U}$.

Proposition 10.3. *Consider a bihamiltonian structure M and a point $m_0\in M$. Fix $\lambda^0\in\mathbb{RP}^1$. Suppose that the corank of the bracket $\lambda_1\{,\}_1+\lambda_2\{,\}_2$ at m is $r\in\mathbb{Z}$ for any m near m_0 and any $\lambda_1:\lambda_2$ near λ^0. Then there is a neighborhood $U\times\mathcal{U}$ of $(m_0,\lambda^0)\in M\times\mathbb{RP}^1$ and r families $C_{t,\lambda}$, $1\leq t\leq r$, $\lambda\in\mathcal{U}$, of functions on U such that*

1. *for any given t, $1\leq t\leq r$, $C_{t,\lambda}$ is a λ-Casimir family on U, and*
2. *for any given $\lambda\in\mathcal{U}$ the functions $C_{t,\lambda}$, $1\leq t\leq r$, are independent.*

Proof. Let $\lambda^0=(\lambda_1^0:\lambda_2^0)\in\mathbb{RP}^1$. Consider the symplectic leaf of $\lambda_1^0\{,\}_1+\lambda_2^0\{,\}_2$ passing through m_0. The codimension of this leaf is r, fix a transversal manifold N of dimension r, and coordinate functions c_t, $1\leq t\leq r$, on this manifold. Obviously, if $(\lambda_1 : \lambda_2)$ is close to $(\lambda_1^0 : \lambda_2^0)$ and m is close to m_0, then the symplectic leaf of $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ passing through m intersects N in exactly one point, and this point depends smoothly on $\lambda = \lambda_1 : \lambda_2$ and m.

Thus there is exactly one Casimir function $C_{t,\lambda}$ for $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ which coincides with c_t when restricted to N. Obviously, it satisfies the conditions of the proposition. □

Lemma 10.4. *Consider two λ-Casimir families C_λ, $\lambda \in \mathcal{U}$, and C'_λ, $\lambda \in \mathcal{U}'$, on M. Then*

$$\left\{C_\lambda, C'_\mu\right\}_1 = \left\{C_\lambda, C'_\mu\right\}_2 = 0, \qquad \lambda \in \mathcal{U}, \quad \mu \in \mathcal{U}'.$$

Proof. To simplify notations assume $\infty \notin \mathcal{U}$. Let $\{,\}^\lambda \overset{\text{def}}{=} \lambda \{,\}_1 + \{,\}_2$. Since $\{C_\lambda, f\}^\lambda = \left\{f, C'_\mu\right\}^\mu = 0$ for any function f, we see that $\left\{C_\lambda, C'_\mu\right\}^\nu = 0$ if $\nu = \lambda$ or $\nu = \mu$. Since $\{,\}^\nu$ is linear in ν, $\left\{C_\lambda, C'_\mu\right\}^\nu = 0$ for any ν as far as $\lambda \neq \mu$. On the other hand, the same identity is true for $\lambda = \mu$ by continuity in λ. □

Proposition 10.3 provides a way to obtain a giant collection of functions which commute with each other w.r.t. *both* the brackets. Out of this huge collection of functions on M only a finite number of functions are independent (since this number is bounded by the dimension of the manifold). One needs a way to extract a finite subset out of this continuum. The anchored Lenard scheme provides such a way, moreover, it allows one to find this small collection without actually finding the whole continuum of Casimir functions.

The idea of the anchored Lenard scheme is to put $\lambda^0 = \infty$ and write a formal series in[18] λ^{-1} for a λ-Casimir family C_λ defined near λ_0:

$$C_\lambda = H_0 + \lambda^{-1} H_1 + \lambda^{-2} H_2 + \dots.$$

Obviously, commutativity of Casimir functions implies $\{H_i, H_j\}_1 = \{H_i, H_j\}_2 = 0$ for any i, j. On the other hand, the condition

$$\left\{H_0 + \lambda^{-1} H_1 + \lambda^{-2} H_2 + \dots, f\right\}_1 + \lambda^{-1}\left\{H_0 + \lambda^{-1} H_1 + \lambda^{-2} H_2 + \dots, f\right\}_2 = 0,$$

which describes the formal-variables analogue of the condition on a λ-Casimir family, can be written as

1. function H_0 is a Casimir function for $\{,\}_1$;
2. for any function f on M

$$\{H_i, f\}_2 = -\{H_{i+1}, f\}_1. \tag{10.1}$$

Remark 10.5. It is easy to see that given H_i, the relation (10.1) is equivalent for a system of equations of the form $dH_{i+1}|_{\mathcal{L}} = \omega_{\mathcal{L}}$, here $\mathcal{L}$ runs over symplectic leaves of $\{,\}_1$, and $\omega_{\mathcal{L}}$ is a 1-form on $\mathcal{L}$ which is determined by H_i. In particular, if $\{,\}_1$ has a constant rank, then (10.1) has a local solution iff all the forms $\omega_{\mathcal{L}}$ are closed.

[18] To follow the standard description of Lenard scheme, we use expansion in formal variable λ^{-1}, though by exchanging $\{,\}_1$ and $\{,\}_2$ one might use more natural expansion in λ.

If so, one can find H_{i+1} by integrating $\omega_{\mathcal{L}}$. Thus if a solution to (10.1) exists, it is easy to find. One can also see why in Lenard scheme one takes $\lambda^0 = \infty$: in applications $\{,\}_1$ is much simpler than $\{,\}_2$ or any other combination $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$, thus taking $\lambda^0 = \infty$ simplifies the integration of relations (10.1).

Definition 10.6. Consider a formal series $H_0 + \lambda^{-1} H_1 + \lambda^{-2} H_2 + \dots$ in λ^{-1} with H_i being functions on M. Call it a *formal λ-family* on M if the sequence H_k satisfies the recurrence relation (10.1). Call this formal λ-family *anchored* if H_0 is a Casimir function for $\{,\}_1$.

Proposition 10.7. *Given two anchored formal λ-families $H_0 + \lambda^{-1} H_1 + \lambda^{-2} H_2 + \dots$ and $H'_0 + \lambda^{-1} H'_1 + \lambda^{-2} H'_2 + \dots$ one has $\left\{H_i, H'_j\right\}_1 = \left\{H_i, H'_j\right\}_2 = 0$ for any i and j.*

Proof. Put $H_i = H'_i = 0$ for $i < 0$. This makes (10.1) applicable for $i < 0$ too. For any i and j

$$\left\{H_i, H'_j\right\}_1 = -\left\{H_{i-1}, H'_j\right\}_2 = \left\{H'_j, H_{i-1}\right\}_2 = -\left\{H'_{j+1}, H_{i-1}\right\}_1 = \left\{H_{i-1}, H'_{j+1}\right\}_1.$$

Repeating this process $i+1$ times, one gets $\left\{H_i, H'_j\right\}_1 = \left\{H_{-1}, H'_{i+j+1}\right\}_1 = 0$. Moreover, $\left\{H_i, H'_j\right\}_2 = -\left\{H_{i+1}, H'_j\right\}_1 = 0$. □

If one considers one chain of solutions to (10.1), then the anchoring condition may be dropped:

Amplification 10.8. *Given a formal λ-family $H_0 + \lambda^{-1} H_1 + \lambda^{-2} H_2 + \dots$ one has $\{H_i, H_j\}_1 = \{H_i, H_j\}_2 = 0$ for any i and j.*

Proof. For any $i \geq 1$ and $j \geq 0$ one gets $\{H_i, H_j\}_1 = \{H_{i-1}, H_{j+1}\}_1$ again. Repeating this several times, one can decrease $|i-j|$ until it becomes 0 or 1 (depending on $i-j$ being even or odd). If $i-j$ is even, use $\{H_{k-l}, H_{k+l}\}_1 = \{H_k, H_k\}_1 = 0$, if $i-j$ is odd, use $\{H_{k-l+1}, H_{k+l}\}_1 = \{H_{k+1}, H_k\}_1 = -\{H_k, H_k\}_2 = 0$. Thus $\{H_i, H_j\}_1$ is always 0, moreover, $\{H_i, H_j\}_2 = -\{H_{i+1}, H_j\}_1 = 0$. □

Lemma 10.9. *Suppose that conditions of Proposition 10.3 are satisfied. Fix $n \geq 0$. Given $m_0 \in M$ and any sequence of functions $H_0, \dots, H_n$ on M such that H_0 is a Casimir function for $\{,\}_1$, and H_k satisfy equations (10.1) for $i = 0, \dots, n-1$, there exists a neighborhood U of $(m_0, \infty) \in M \times \mathbb{RP}^1$ and a λ-Casimir family $C_\lambda(m)$ defined for $(m, \lambda) \in U$ such that*

$$C_\lambda = H_0 + \lambda^{-1} H_1 + \lambda^{-2} H_2 + \dots + \lambda^{-n} H_n + o\left(\lambda^{-n}\right).$$

In particular, there is a function H_{n+1} defined near m_0 which solves (10.1) for $i = n$.

Proof. To simplify notations, suppose $r = 1$ (the case of general r is absolutely parallel). Then H_0 is defined uniquely up to a change $H'_0 = \varphi_0(H_0)$. Additionally, given H_i, Equation (10.1) determines H_{i+1} up to a change $H'_{i+1} = H_{i+1} + \varphi_{i+1}(H_0)$.

Moreover, the Taylor series for $C_{1,\lambda}$ provides *one* solution to the recursion relations (10.1). Since the change of the form $H_0' = \varphi_0(H_0)$ corresponds to a change of the form $c_1' = \varphi_0(c_1)$ in the proof of Proposition 10.3, we conclude that there is a locally defined solution to the recursion relations (10.1) for any initial data H_0 which is a Casimir function for $\{,\}_1$.

Next, proceed by induction in n. To do the step of induction, it is enough to prove the following statement: given a λ-Casimir family C_λ near $\lambda = \infty$ such that $C_\infty = H_0$, and given any function $\varphi_n(h)$ of one variable defined in a neighborhood of $h = H_0(m_0)$ one can find another λ-Casimir family C_λ' such that $C_\lambda' - C_\lambda = \varphi_n(H_0)\lambda^{-n} + o(\lambda^{-n})$. Putting $C_\lambda' = C_\lambda + \varphi(C_\lambda)\lambda^{-n}$ finishes the proof in the case $r = 1$. □

The following statement is obvious:

Lemma 10.10. *Suppose that $r = 1$ and a sequence (H_i) satisfies conditions of Lemma 10.9. If H_k depends functionally on $H_0, \dots, H_{k-1}$, then H_l depends functionally on $H_0, \dots, H_{k-1}$ for any l such that $k \leq l \leq n$.*

This shows that a maximal independent subset of the sequence (H_l) can be chosen to be the starting subsequence. The situation in the case $r > 1$ is slightly more complicated, however, it is easy to show that

Proposition 10.11. *Consider a maximal collection $H_0^{(1)}, \dots, H_0^{(r)}$ of independent Casimir functions for $\{,\}_1$ near $m_0 \in M$. Let $H_i^{(t)}$, $t = 1, \dots, r$, $i \geq 0$, be solutions to recursion relations (10.1) with $H_0^{(t)}$ as the initial data. Then there are numbers $k_1, \dots, k_r$ such that the collection $\left\{H_i^{(t)}\right\}$, $1 \leq t \leq r$, $0 \leq i \leq k_t$, is functionally independent, and all functions $H_i^{(t)}$, $1 \leq t \leq r$, $i \geq 0$, depend functionally on this collection.*

Definition 10.12. *Anchored Lenard scheme* of finding a large family of functions on a bihamiltonian structure which mutually commute w.r.t. both brackets consists of two steps: first one finds a maximal independent collection of Casimir functions for the bracket $\{,\}_1$, then one solves recurrence relations (10.1) with these functions as initial data until new functions start depend on the old ones.

In fact it is not necessary to consider many chains of solutions of recurrence relations:

Amplification 10.13. *In conditions of Proposition 10.3 suppose that the bihamiltonian structure on M is analytic. Then there is a sequence of functions $H_0, \dots, H_n$ defined near a given point $m_0 \in M$ such that*

1. *function H_0 is a Casimir function for $\{,\}_1$;*
2. *functions (H_i) satisfy the recurrence relation (10.1);*
3. *functions (H_i) are independent, and for any $1 \leq t \leq r$ and λ near ∞ the function $C_{t,\lambda}$ of Proposition 10.11 depends on $(H_0, \dots, H_n)$.*

Proof. Since the Taylor series for $C_{t,\lambda}$ in λ^{-1} converge, it is enough to show that the Taylor coefficients for $C_{t,\lambda}$ depend on $(H_0, \dots, H_n)$. Fix numbers α_t, $2 \leq t \leq r$. Let $n = \sum_{t=1}^r k_t + r - 1$, and

$$C_\lambda = C_{1,\lambda} + \alpha_2 \lambda^{-k_1+1} C_{2,\lambda} + \dots + \alpha_r \lambda^{-k_1 - \dots - k_{r-1}+1} C_{r,\lambda}.$$

Obviously, this is a λ-Casimir family.

It is easy to show that for generic values of $\alpha_2, \dots, \alpha_r$ the first $n+1$ Taylor coefficients $H_0, \dots, H_n$ of C_λ are independent, which finishes the proof. □

Remark 10.14. Since the functions $H_i^{(t)}$ of the anchored Lenard scheme are obtained by doing manipulations (taking Taylor coefficients) with Casimir functions, they can be pushed down to the web $\mathcal{B}_M$ of M. Thus they should be considered as action functions on M (see Section 5).

In interesting cases (see Section 11 and [29]) the functions $H_i^{(t)}$ provide a local coordinate system on $\mathcal{B}_M$. (This shows that in fact $\mathcal{B}_U$ is a smooth manifold if U is a small subset of M.) In these cases the submanifolds $\left\{H_i^{(t)} = \mathrm{const}_i^{(t)} \mid i \geq 0\right\}$ carry a natural local affine structure, thus one can find a complementary set of *angle variables* φ_j such that functions $\{H_i, \varphi_j\}_k \stackrel{\text{def}}{=} c_{\mathrm{ijk}}$ depends on H_l only[19].

Example 10.15. Consider the bihamiltonian structure defined by (8.1). In this case $r = 1$, and Casimir functions are functions of x and y only. Thus H_i are functions of x and y too. Moreover, one can write an explicit formula for H_i.

Indeed, let $\Phi(x, y) = \frac{\partial f / \partial x}{\partial f / \partial y}$. Obviously, the symplectic leaves for $\{,\}_1 + \lambda^{-1}\{,\}_2$ can be described as surfaces $\{(x, y, z) \mid y = \Psi(x)\}$, here Ψ is a solution of the ODE

$$\frac{d\Psi}{dx} = -\lambda^{-1}\Phi(x, \Psi) \tag{10.2}$$

Given (x_0, y_0) which is close to (0,0), let $\Psi_{\lambda, x_0, y_0}(x)$ be the solution of (10.2) which passes through the point (x_0, y_0). Let $F_\lambda(x_0, y_0) \stackrel{\text{def}}{=} \Psi_{\lambda, x_0, y_0}(0)$. Obviously, $F_\lambda(x, y)$ is well-defined for large $|\lambda|$ and small (x, y). Moreover, F_λ is a Casimir function for $\{,\}_1 + \lambda^{-1}\{,\}_2$.

Taking Laurent coefficients of F_λ near $\lambda = \infty$, one obtains functions H_i from the anchored Lenard scheme. Obviously,

$$H_0(x, y) = y, \qquad H_1(x, y) = \int_0^x \frac{\partial f(t, y)/\partial t}{\partial f(t, y)/\partial y} dt = x + o(x).$$

This implies that all other functions H_i depend on H_0 and H_1. One can see that z provides an example of an angle variable, and any other angle variable can be written as $a(x, y) z + b(x, y)$ with arbitrary $a(x, y)$ and $b(x, y)$.

[19] Another problem is to find such change-of-variables in action variables $\mathring{H}_i = \mathring{H}_i(H_0, \dots)$ that the corresponding functions $\mathring{c}_{\mathrm{ijk}}$ become as simple as possible. As Theorem 8.4 shows, in general it is *not possible* to make all $\mathring{c}_{\mathrm{ijk}}$ into constants. However, it is obviously possible for bihamiltonian structures with constant coefficients, thus for flat structures.

Summing up, we obtain

Proposition 10.16. *The bihamiltonian structure M_f given by* (8.1) *is completely integrable by the anchored Lenard scheme. If φ satisfies* (8.2) *and $\varphi(x,y) \not\equiv x+y$, then M_φ is not flat.*

Remark 10.17. It is possible to provide similar examples of homogeneous but not flat bihamiltonian structures of any given type. In Section 11 we will see that all these structures are completely integrable by the anchored Lenard scheme. In the case of type $(2k-1)$, $k \in \mathbb{N}$, one can write such a bihamiltonian structure[20] based on $k-1$ functions $\varphi_1(x,y), \dots, \varphi_{k-1}(x,y)$ of two complex variables (though one cannot do it as explicitly as in (8.1)). Any two of these bihamiltonian structures are not locally isomorphic, thus only one of them (for any given $k \in \mathbb{N}$) is flat. What is very surprising is that (apparently) they did not appear in examples of integrable systems which appear in problems of mathematical physics.

11. Lenard-integrable structures

Here we show that the class of bihamiltonian structures for which the anchored Lenard scheme gives "many" functions in involution coincides with the class of homogeneous structures. In fact, since our approach to Lenard scheme is based on a formal analogue of λ-Casimir families, the result of this section are closely related to ones in [2] (compare with discussion of "completeness" in [29]).

Definition 11.1. The *action dimension* of a Poisson structure $(M, \{,\}_1)$ of constant corank r is $\frac{\dim M + r}{2}$. The action dimension of an arbitrary Poisson structure on M at $m_0 \in M$ is the minimum action dimension of open subsets $U \subset M$ which contain m_0 in its closure, and such that the Poisson structure is of constant corank on U.

This definition gives a *lower* bound on the number of functions in involution which are enough to completely integrate the dynamical system on M given by some Hamiltonian H. Indeed, in the case of constant corank r one needs r functions to disambiguate symplectic leaves, and $\frac{\dim M - r}{2}$ functions to provide action variables inside the leaves.

To do the same in the case of a bihamiltonian structure, introduce

Definition 11.2. The *action dimension* of a complex vector space V with two skewsymmetric bilinear pairings is $\frac{\dim V + r}{2}$, here r is the number of Kronecker blocks of V.

Definition 11.3. The *action dimension* at $m_0 \in M$ of a bihamiltonian structure on M is the lower limit of action dimensions of[21] $\mathcal{T}^*_m M \otimes \mathbb{C}$ for $m \to m_0$.

[20] In a slightly different language such bihamiltonian structures were described in [14] and [15].

[21] If M is analytic, one should consider $\mathcal{T}^*_m M$ instead of $\mathcal{T}^*_m M \otimes \mathbb{C}$.

Note that the number of Kronecker blocks of a pair of skewsymmetric pairings $(,)_1$, $(,)_2$ is equal to $\min_{\lambda_1,\lambda_2} \dim \operatorname{Ker}(\lambda_1(,)_1 + \lambda_2(,)_2)$, here Ker denotes null-space of the pairing. Thus the action dimension of a bihamiltonian structure provides a *lower* bound on the number of functions in involution necessary to completely integrate the structure w.r.t. *at least one particular* Poisson structure of the form $\lambda_1\{,\}_1 + \lambda_2\{,\}_2$ on an open subset of M near m_0.

Definition 11.4. Call a bihamiltonian structure on M *Lenard-integrable* at $m_0 \in M$ if the number of independent functions provided by the anchored Lenard scheme in an appropriate neighborhood of m_0 coincides with the action dimension of M at m_0.

Call a bihamiltonian structure on M *strictly Lenard-integrable* at m_0 if it is Lenard-integrable at m_0 and the sequences $H_i^{(t)}$ of the anchored Lenard scheme can be continued for $i > k_t$ as well.

Remark 11.5. Recall that Section 10 describes the anchored Lenard scheme as a formal-series counterpart of λ-Casimir families. For this description to work one needs to assume some constant rank conditions, as in Proposition 10.3. The condition of Proposition 10.3 was not very restrictive, since one could achieve it by a small deformation of (m_0, λ^0). However, in the anchored Lenard scheme λ^0 is fixed to be ∞, thus the restriction of Proposition 10.3 is in fact non void. Thus Lemma 10.9 *does not* imply that any Lenard-integrable structure is strictly Lenard-integrable.

Theorem 11.6. *If a bihamiltonian system on M is strictly Lenard-integrable at $m_0 \in M$, then it is homogeneous on an open subset U of M with m_0 being in the closure of U.*

Proof. Indeed, if a structure is Lenard-integrable at m_0, then it is also Lenard-integrable at m for m in an appropriate open subset of M. It is easy to show that by decreasing this subset U one may assume that at any point $m \in U$ the sizes of Kronecker blocks of the pair of pairings on $\mathcal{T}^*_m M \otimes \mathbb{C}$ are the same.

Functions $H_i^{(t)}$, $1 \leq t \leq r$, $0 \leq i \leq k_t$, given by the anchored Lenard scheme provide a mapping $\boldsymbol{H}\colon U \to \mathbb{R}^K$, $K = \sum_t (k_t + 1)$. Decreasing U yet more, we may assume that the differential of this mapping is of constant rank K (recall that components of $\boldsymbol{H}$ are independent). Fix a point $m \in U$ and t, $1 \leq t \leq r$. Let $\beta_i = dH_i^{(t)}|_m \in \mathcal{T}^*_m M$. Let $W_{\mathbb{R}} = \mathcal{T}^*_m M$, $W = W_{\mathbb{R}} \otimes \mathbb{C}$. By Equation (10.1), β_0 is in the null space of pairing $(,)_1$ on W, and $(\beta_i, w)_2 = (\beta_{i+1}, w)_1$ for any $w \in W$.

An immediate check shows that if $W \simeq \mathcal{J}_{2k,\mu}$, $\mu \in \mathbb{CP}^1$, then $\beta_i = 0$, $i = 0, \dots$. Similarly, if $W \simeq \mathcal{K}_{2k-1}$, then all vectors β_i are in the subspace $W_1 = \langle \boldsymbol{w}_0, \boldsymbol{w}_2, \dots, \boldsymbol{w}_{2k-2} \rangle$ of $\mathcal{K}_{2k-1}$. The dimension of this subspace is k (it is the same subspace which appears in a similar context in Lemma 4.6). In general case, taking a decomposition of W into a sum of indecomposable components, one can see that all vectors β_i are in the sum of Kronecker blocks of W, moreover, they are in a direct sum of subspaces W_1 for these blocks.

This shows that $N \leq \sum_{t=1}^{r} k_r$, here $\mathcal{K}_{2k_t-1}$, $1 \leq t \leq r$, are Kronecker blocks of W. The restriction on the action dimension shows that $\dim W \leq \sum_{t=1}^{r} (2k_r - 1)$, thus W has no Jordan blocks, which finishes the proof of the theorem. □

Proposition 11.7. *Any homogeneous bihamiltonian structure is strictly Lenard-integrable on small open subsets.*

Proof. A tiny modification of the above proof together with Proposition 10.3 imply this statement immediately. □

This shows that the "strict" anchored Lenard scheme integrates homogeneous structures and only them. Note that a linear combination of brackets of homogeneous structure is never symplectic.

Remark 11.8. The Lenard schemes of [23, 12, 19] differ from what we describe here, the difference being that they consider non-anchored formal λ-families. Though our condition is more restrictive, note that in applications the Lenard scheme usually provides an anchored λ-family. Moreover, in non-symplectic cases there is no simple way to find a non-anchored family, thus it is not obvious whether non-anchored Lenard scheme may be used to integrate a system (unless applied to the traces of powers of recursion operator).

Remark 11.9. The amount of our knowledge about classification of bihamiltonian structures is not enough to describe finite-dimensional Lenard-integrable system which are not strict. The situation is slightly more promising if one consider non-strict structures for which *one* anchored Lenard chain provides enough functions in involution.

In this case slightly more elaborate arguments than those in the proof of Theorem 11.6 show that there is an open subset $U \subset M$ such that at a point m of U the pair of brackets in $\mathcal{T}_m^*M$ has one Jordan block only, and this block is of the form $\mathcal{J}_{2k,\infty}$. (The remaining blocks are Kronecker.)

In particular, if at least one linear combination $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ is symplectic near m_0, then there are no Kronecker blocks. Thus the pairings on $\mathcal{T}_m^*M$ are isomorphic to $\mathcal{J}_{2k,\infty}$ for any $m \in U$.

Such symplectic structures were classified in [35], they turn out to be flat (thus isomorphic to the natural bihamiltonian structure on the dual space to the vector space $\mathcal{J}_{2k,\infty}$). These are exactly the structures for which the arguments of [23] and [12] are actually applicable to *anchored* formal λ-families. It is again an interesting question to find physically interesting bihamiltonian structures of this form.

Remark 11.10. Note asymmetry between Theorem 11.6 and Proposition 11.7: one of them is applicable on small open neighborhoods of any point, another on small open neighborhood of a *dense* collection of points. Note that [29] introduces a more general notion than homogeneity: bihamiltonian structure is *complete* if the pairs of pairings at $\mathcal{T}_m^*M$ for any $m \in M$ do not contain a Jordan block (thus the condition of Kronecker blocks having the same sizes for all the points of M is dropped). For

complete structures Proposition 10.3 is applicable for any point of M, and it is easy to see that the following statement holds:

Amplification 11.11. *The class of bihamiltonian structures which are strictly Lenard-integrable at any point of M coincides with the class of complete bihamiltonian structures.*

12. Bihamiltonian Toda lattices

Definition 12.1. The *open Toda lattice* ([6]) is the $(2k+1)$-dimensional vector space V_{2k+1} over $\mathbb{C}$ with coordinates $v_0, \dots, v_{2k}$ and the two compatible Poisson brackets defined as follows. The bracket $\{,\}_1$ is defined by the condition $\{v_i, v_j\} = 0$ for $|i-j| > 1$, and

$$\{v_{2l}, v_{2l\pm1}\}_1 = \mp v_{2l\pm1}. \tag{12.1}$$

The bracket $\{,\}_2$ is defined by the condition $\{v_i, v_j\} = 0$ for $|i-j| > 2$, and

$$\begin{aligned} \{v_{2l}, v_{2l\pm1}\}_2 &= \mp v_{2l} v_{2l\pm1}, \\ \{v_{2l}, v_{2l+2}\}_2 &= -2v_{2l+1}^2, \\ \{v_{2l-1}, v_{2l+1}\}_2 &= -\frac{1}{2} v_{2l-1} v_{2l+1}, \end{aligned} \tag{12.2}$$

for all l such that the the left-hand sides make sense.

We denote a point of V_{2k+1} by $\boldsymbol{v}$. Define transformation $\mathfrak{T}_\lambda$, $\lambda \in \mathbb{C}$, by

$$\mathfrak{T}_\lambda \colon V_{2k+1} \to V_{2k+1} \colon \boldsymbol{v} \mapsto \boldsymbol{v} + \lambda \boldsymbol{v}^0, \qquad \boldsymbol{v}^0 = (1, 0, 1, 0, 1, \dots, 0, 1). \tag{12.3}$$

Translating bracket $\{,\}_2$ by the transformation $\mathfrak{T}_{-\lambda}$, one obtains a Poisson bracket $\{,\}^{(\lambda)}$ which depends on a parameter λ.

Remark 12.2. Note that for any i, j the bracket $\{v_i, v_j\}_2$ depends linearly on v_{2l}, $l = 0, \dots, k$, thus $\{,\}^{(\lambda)}$ depends linearly on λ. In fact $\{,\}^{(\lambda)}$ may be written as

$$\{,\}^{(\lambda)} = \lambda \{,\}_1 + \{,\}_2 .$$

One can use this remark to simplify the proof of compatibility and Poisson property of brackets $\{,\}_1$ and $\{,\}_2$. Indeed, if we know that $\{,\}_2$ is Poisson, then $\{,\}^\lambda$ is Poisson, thus is $\{,\}_1$ as a limit of $\{,\}^{(\lambda)}/\lambda$. To check that $\{,\}_2$ is Poisson, one can use the symmetry of (12.2) of the form $l \mapsto 2m \pm l$, so it is enough to check Jacobi identity for v_0, v_1, v_2, for v_1, v_2, v_3, for v_0, v_1, v_3, for v_0, v_2, v_3, for v_0, v_2, v_4, and for v_1, v_3, v_5.

Definition 12.3. The *infinite Toda lattice* is the manifold with coordinates v_l, $l \in \mathbb{Z}$, and the Poisson brackets[22] (12.1), (12.2). Considering sequences v_l with period $2k$,

[22] These brackets are well-defined on functions which depend on finite number of coordinates v_l only.

one obtains a pair of well-defined Poisson brackets on a $2k$-dimensional subvariety. Denote this bihamiltonian structure by V_{2k}, call it the *periodic Toda lattice*.

In Sections 13 and 14 we prove that open dense subsets of the bihamiltonian structures V_{2k+1} and V_{2k} are Kronecker bihamiltonian structure. In other words, in these sections we prove the following theorems:

Theorem 12.4. *The open Toda lattice (of dimension $2k-1$) is a bihamiltonian structure which is generically Kronecker of type $(2k-1)$.*

Theorem 12.5. *The periodic Toda lattice (of dimension $2k$) is a bihamiltonian structure which is generically Kronecker of type $(2k-1,1)$.*

Remark 12.6. Note that one can also consider a manifold $\widetilde{V}_{2k}$ with coordinates $v_0,\dots,v_{2k-1}$ and brackets (12.1), (12.2). It is also bihamiltonian, but it is not a Kronecker structure, so it cannot be described by the methods of this paper. Say, at a generic point both the Poisson structures are in fact symplectic, while all linear combinations of Poisson structures of a Kronecker structure are degenerated. While this structure may be described by the means of [35, 24, 25, 26, 15], note that the in applications $\widetilde{V}_{2k}$ appears not by itself, but as a reduction of the structure V_{2k+1} w.r.t. forgetting the variable v_{2k}.

This supports the point of view from Section 16 that Kronecker structures are more important in applications than structures which may be described in symplectic terms.

13. Casimir families on the open Toda lattice

Apply the description of Section 6 to the bihamiltonian Toda structure. First, construct a family of would-be semi-Casimir functions F_λ, $\lambda\in\mathbb{C}$.

Consider the inclusion ι of V_{2k+1} into $\operatorname{Mat}(k+1,k+1)$ which sends $(v_0,\dots,v_{2k})$ to a symmetric 3-diagonal matrix with diagonal elements $(v_0,v_2,\dots,v_{2k})$ and over-diagonal elements $(v_1,v_3,\dots,v_{2k-1})$. Taking determinant of the resulting matrix, one obtains a polynomial function F_0 on V_{2k+1}.

Any proof of integrability of Toda lattice is based on the following statement:

Lemma 13.1. *The function F_0 is Casimir, in other words, for any function f on V_{2k+1} the Poisson bracket $\{F_0,f\}_2$ is identically 0.*

Proof. Let d_{2m} be the determinant of the upper-left minor of $\iota(\boldsymbol{v})$ of size $(m+1)\times(m+1)$. We need to show that $\{v_l,d_{2k}\}_2=0$, $0\leq l\leq 2k$. Let us show that $\{v_l,d_{2m}\}_2=0$, $0\leq l\leq 2m$, $m\leq k$.

Use induction in m. Plugging the identity

$$d_{2m}=v_{2m}d_{2m-2}-v_{2m-1}^2d_{2m-4}$$

into $\{v_l,d_{2m}\}_2$ shows that the step of induction will work as far as $l\leq 2m-4$. On the other hand, due to obvious symmetry $v_t\iff v_{2m-t}$ of brackets (12.2) and the determinant d_{2m}, it is enough to check $\{v_l,d_{2m}\}_2=0$ for $0\leq l\leq m$. Moreover, if we know $\{v_l,d_{2m}\}_2=0$ for $0\leq l\leq m-1$, then we know it for $m+1\leq l\leq 2m$, thus $\{d_{2m},d_{2m}\}=\frac{\partial d_{2m}}{\partial v_m}\{v_m,d_{2m}\}$. Since all these expressions are polynomials in v_i, and $\frac{\partial d_{2m}}{\partial v_m}\not\equiv 0$, one would be able to conclude that $\{v_m,d_{2m}\}=0$.

Thus the only relations to check are $\{v_l,d_{2m}\}_2=0$ for $0\leq l\leq m-1$ such that $2m-l\leq 3$. This leaves only $\{v_0,d_2\}$ and $\{v_1,d_4\}$, which are easy to check (using one step of induction for the latter one). ☐

Remark 13.2. In Remark 12.2 we used the fact that the right-hand sides of (12.2) are linear in variables v_{2l}. The last sentence of the above proof is the only other place were we use the particular form of right-hand sides of (12.2).

Consider the translation $\mathfrak{T}$ defined in (12.3). Motivated by the above lemma, define $F_\lambda\overset{\text{def}}{=}\mathfrak{T}_\lambda^*F$, $\lambda\in\mathbb{C}$. By definition of $\{,\}^{(\lambda)}$, the bracket $\{F_\lambda,f\}^{(\lambda)}$ is identically 0 for any function f. On the other hand, for any given $\boldsymbol{v}\in V_{2k+1}$ the function $F_\lambda(\boldsymbol{v})$ of λ is the characteristic polynomial of $\iota(\boldsymbol{v})$. Thus the degree of $F_\lambda(\boldsymbol{v})+(-1)^k\lambda^{k+1}$ in λ is k. We obtain

Proposition 13.3. *The family $\mathring{F}_\lambda(\boldsymbol{v})\overset{\text{def}}{=}F_\lambda(\boldsymbol{v})+(-1)^k\lambda^{k+1}$ of functions on V_{2k+1} depends polynomially on λ with the degree being k. For each λ the function $\mathring{F}_\lambda$ is a Casimir function for the bracket $\lambda\{,\}_1+\{,\}_2$.*

However, this proposition is not yet enough to put us in the context of Theorem 3.2, since we do not know the dimension of the span of $d\mathring{F}_\lambda|_{\boldsymbol{v}}$ for any given $\boldsymbol{v}$ and variable λ. To find this dimension, we need to investigate the functions F_λ in more details.

Denote the set of polynomials of degree d in λ with the leading coefficient $(-1)^d$ by $\mathfrak{P}_d$. Functions F_λ (considered as polynomials in λ) define a mapping $F_\bullet\colon V_{2k+1}\to\mathfrak{P}_{k+1}$, $\boldsymbol{v}\mapsto F_\bullet(\boldsymbol{v})$.

To describe the geometry of this mapping, associate with each $\boldsymbol{v}=(v_i)\in V_{2k+1}$ a finite sequence of polynomials C_{I_p} in λ. First, construct a partition of the set of even numbers $\{0,2,\dots,2k\}$: consider numbers $2l+1$ such that $v_{2l+1}=0$ as walls, they separate $\{0,2,\dots,2k\}$ into continuous intervals $I_1,\dots,I_q$, which we call *runs*. To each run $I_p=\{2l_p,2l_p+2,\dots,2l_{p+1}-2\}$ associate the characteristic polynomial C_{I_p} of the corresponding principal minor (with columns and rows $l_p+1,\dots,l_{p+1}$) of the matrix $\iota(\boldsymbol{v})$. Obviously, $\det(\iota(\boldsymbol{v})-\lambda)$ coincides with the product of polynomials C_{I_p}.

Call $\boldsymbol{v}\in V_{2k+1}$ *S-generic* if any two of polynomials C_{I_p} are mutually prime. Non-S-generic points form a submanifold of codimension 2: one of v_{2l+1} should vanish, and two polynomials should have a common zero.

Proposition 13.4. *At an S-generic point $\boldsymbol{v}\in V_{2k+1}$ the mapping $F_\bullet\colon V_{2k+1}\to\mathfrak{P}_{k+1}$ is a submersion[23]. At non-S-generic points it is not a submersion.*

[23] I.e., its derivative is an epimorphism.

Proof. It is enough to consider the case when no v_{2l+1} vanishes. Indeed, if we leave all the variables v_m except v_{2l+1} fixed, then $\det \iota(\boldsymbol{v})$ is quadratic in v_{2l+1} without the linear term. Thus $v_{2l+1} = 0$ implies $\frac{\partial \det}{\partial v_{2l+1}} = 0$. On the other hand, if $v_{2l+1} = 0$, the matrix breaks into two blocks, and the derivatives w.r.t. other variables can be calculated when we consider two blocks separately. Now the case when some v_{2l+1} vanish can be proved by induction using the following obvious

Lemma 13.5. *The multiplication mapping $\mathfrak{P}_a \times \mathfrak{P}_b \to \mathfrak{P}_{a+b}$ is a submersion at (P_1, P_2) iff P_1 and P_2 are mutually prime.*

In the case when all $v_{2l+1} \neq 0$ the matrix $\iota(\boldsymbol{v})$ is similar to a 3-diagonal matrix with diagonal entries v_{2l}, above-diagonal entries 1, and below-diagonal entries v_{2l+1}^2. Denote by Q_{k+1} the set of 3-diagonal $(k+1) \times (k+1)$ matrices with the above-diagonal entries being 1. Denote by $\widetilde{F}_{\bullet}$ the mapping $Q_{k+1} \to \mathfrak{P}_{k+1}$ of taking the characteristic polynomial. Denote the diagonal entries of $q \in Q$ by a_l, $l = 0, \dots, k$, the below-diagonal entries by b_l, $l = 1, \dots, k$. Now the proposition is an immediate corollary of the following

Lemma 13.6. *The mapping $\widetilde{F}_{\bullet}$ restricted on the subset $b_l \neq 0$, $l = 1, \dots, k$, is a submersion.*

To prove this lemma, denote the characteristic polynomial of the upper-left principal $l \times l$ minor by d_l. The lemma is an immediate corollary of

Lemma 13.7. *The mapping $(d_k, d_{k+1}) : Q_{k+1} \to \mathfrak{P}_k \times \mathfrak{P}_{k+1}$ restricted on the subset $b_l \neq 0$, $l = 1, \dots, k$, is a bijection onto the subset of mutually prime polynomials $(P_1, P_2) \in \mathfrak{P}_k \times \mathfrak{P}_{k+1}$.*

This lemma is a direct discrete analogue of the inverse problem for Sturm–Liouville equation by the spectrum with fixed ends and normalizing numbers (compare [22]). In fact zeros of d_{k+1} determine the spectrum, and values of d_k at these points determine the normalizing numbers.

Proof. Indeed, extending the sequence d_l by $d_0 = 1$, $d_{-1} = 0$, one can see that this sequence is uniquely determined by the recurrence relation

$$d_l = (a_{l-1} - \lambda) d_{l-1} - b_{l-1} d_{l-2}.$$

From this relation one can immediately see that if b_m, $m < l$, do not vanish, then d_l and d_{l-1} are mutually prime. On the other hand, given mutually prime $d_l \in \mathfrak{P}_l$ and $d_{l-1} \in \mathfrak{P}_{l-1}$, one can uniquely determine $d_{l-2} \in \mathfrak{P}_{l-2}$ and two numbers a_{l-1} and b_{l-1} from the above relation, and $b_{l-1} \neq 0$. □

This finishes the proof of the proposition. □

We conclude that at an S-generic point $\boldsymbol{v}$ the derivatives $d\mathring{F}_\lambda|_{\boldsymbol{v}}$ span $k+1$-dimensional space (since $\dim \mathfrak{P}_{k+1} = k+1$). Now the only condition of Theorem 3.2 (in fact, of Amplification 3.3) which is missing is the calculation of the rank of $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ for an appropriate λ_1 and λ_2. One can easily see that

Lemma 13.8. *The rank of the bracket $\{,\}_1$ at the point $\boldsymbol{v}$ is $2k - 2d$, here d is the number of indices $l = 0, \dots, k-1$, such that $v_{2l+1} = 0$.*

This shows that on the subset $v_{2l+1} \neq 0$, $l = 0, \dots, k-1$, the bihamiltonian structure satisfies conditions of Theorem 3.2 and Amplification 3.3, thus is flat indecomposable. This finishes the proof of Theorem 12.4.

Moreover, since for a flat indecomposable structure both brackets have corank 1 everywhere, Lemma 13.8 implies that in a neighborhood of a point $\boldsymbol{v}$ with $v_{2l+1} = 0$ for some $l = 0, \dots, k-1$ the bihamiltonian open Toda structure is *not* flat indecomposable.

14. Periodic Toda lattice

Recall that V_{2k} denotes the periodic Toda lattice.

Lemma 14.1. *The function $N = v_1 v_3 \dots v_{2k-1}$ on V_{2k} is Casimir w.r.t. both Poisson brackets $\{,\}_1$ and $\{,\}_2$.*

Proof. Since this function is invariant w.r.t. translation $\mathfrak{T}_\lambda$, it is enough to show this for the bracket $\{,\}_2$. When one calculates $\{N, v_{2l}\}$, only the factor $v_{2l-1}v_{2l+1}$ of N matters, and by (12.2) $\{v_{2l-1}v_{2l+1}, v_{2l}\}$ vanishes. Similarly, for $\{N, v_{2l-1}\}$ only $\{v_{2l-3}v_{2l+1}, v_{2l-1}\}$ matters, and it also vanishes. □

Since dimension of V_{2k} is even, this shows that symplectic leaves of $\lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ have codimension at least 2. Any hypersurface $N = \text{const}$ is decomposed into a union of such leaves for any $(\lambda_1, \lambda_2) \neq (0, 0)$. In particular, each hypersurface $N = \text{const}$ carries an odd-dimensional bihamiltonian structure.

Theorem 14.2. *For any $c \neq 0$ the bihamiltonian structure on the hypersurface $N = c$ is generically flat indecomposable.*

Note that this theorem implies Theorem 12.5, since one can easily modify Theorem 6.3 to cover families of bihamiltonian structures as well:

Amplification 14.3. *Consider a family of bihamiltonian structures $\left(\{,\}_1^{(\mu)}, \{\}_2^{(\mu)}\right)$ on a manifold M which depends smoothly on a parameter $\mu \in \mathcal{M}$. Suppose that for any μ the bihamiltonian structure is flat indecomposable. Then for any $m_0 \in M$ and $\mu_0 \in \mathcal{M}$ there is a neighborhood U of m, a neighborhood U' of μ_0 and a family of coordinate system $\left(x_i^{(\mu)}\right)$ on U depending smoothly on a parameter $\mu \in U'$ such that the bihamiltonian structure $\left(\{,\}_1^{(\mu)}, \{\}_2^{(\mu)}\right)$ in the coordinate system $\left(x_i^{(\mu)}\right)$ is given by (1.1) for any $\mu \in U'$.*

Since the bihamiltonian structure corresponding to $\mathcal{K}_1$ has both bracket being 0, this amplification implies Theorem 12.5.

Proof of Theorem 14.2 . Associate to a point $\boldsymbol{v}$ of the infinite Toda lattice an infinite 3-diagonal matrix $\iota(\boldsymbol{v})$ in the same way we did it in Section 13. Consider a matrix equation $\iota(\boldsymbol{v})\boldsymbol{x} = 0$, here $\boldsymbol{x} \in \mathbb{C}^{\infty}$ is a two-side-infinite vector. Since this equation may be written as the recursion relation

$$v_{2l-1}x_{l-1} + v_{2l}x_l + v_{2l+1}x_{l+1} = 0, \qquad l \in \mathbb{Z}, \tag{14.1}$$

this matrix equation has a two-dimensional space of solutions if $v_{2l-1} \neq 0$ for any $l \in \mathbb{Z}$.

If $\boldsymbol{v}$ is in the periodic Toda lattice, then the equation $\iota(\boldsymbol{v})\boldsymbol{x} = 0$ is invariant with respect to the shift $x_l \mapsto x_{l+k}$ of coordinates of $\boldsymbol{x}$. This shift induces a linear transformation $\mathcal{M} = \mathcal{M}(\boldsymbol{v})$ of monodromy in the 2-dimensional vector space of solutions. As in Section 12, denote by $\boldsymbol{v}^0$ an element of $\mathbb{C}^{\infty}$ with 1 on even positions, 0 on odd positions.

Lemma 14.4. *If $v_{2l-1} \neq 0$ for any $l \in \mathbb{Z}$, then $\det \mathcal{M} = 1$, and $\operatorname{Tr} \mathcal{M}(\boldsymbol{v} - \lambda\boldsymbol{v}^0)$ is a polynomial of degree k in λ with the leading coefficient N^{-1}.*

Proof. Indeed, the recursion (14.1) induces a linear transformation $(x_l, x_{l+1}) = m_l(x_{l-1}, x_l)/v_{2l+1}$, $m_l = \begin{pmatrix} 0 & v_{2l+1} \\ -v_{2l-1} & -v_{2l} \end{pmatrix}$. In an appropriate basis $N\cdot\mathcal{M}$ can be written as $m_k m_{k-1} \dots m_1$, and each matrix $m_l = m_l(\boldsymbol{v})$ has determinant $v_{2l-1}v_{2l+1}$. Moreover, $m_l(\boldsymbol{v} - \lambda\boldsymbol{v}^0)$ is of degree 1 in λ with the leading term $\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$.

Thus $N\cdot\mathcal{M}(\boldsymbol{v} - \lambda\boldsymbol{v}^0)$ is a polynomial in λ of degree k with the leading term being $\begin{pmatrix} 0 & 0 \\ 0 & \lambda^k \end{pmatrix}$, which finishes the proof. ◻

Lemma 14.5. *The function $\operatorname{Tr} \mathcal{M}(\boldsymbol{v})$ defined on the open subset $v_{2l-1} \neq 0$, $l = 1, \dots, k$, of V_{2k} is a Casimir function for the Poisson bracket $\{,\}_2$.*

We do not prove this standard statement about the periodic Toda lattice. As in the case of Lemma 13.1, the proof is reduced to a check of a finite number of identities.

The following lemma is obvious:

Lemma 14.6. *On the open subset $v_{2l-1} \neq 0$, $l = 1, \dots, k$, of V_{2k} the Poisson bracket* (12.1) *has symplectic leaves of codimension 2 given by the equations $v_0 + v_2 + \dots + v_{2k-2} = C_0$, $v_1v_3 \dots v_{2k-1} = C_1$.*

This shows that $r = 2$ in Proposition 4.5.

To demonstrate Theorem 14.2 the only thing which remains to be proved is that at a generic point $\boldsymbol{v} \in V_{2k}$ the differentials $d \operatorname{Tr} \mathcal{M}(\boldsymbol{v} - \lambda\boldsymbol{v}^0)|_{\boldsymbol{v}}$ for different $\lambda \in \mathbb{C}$ and the differential of $N \equiv v_1v_3 \dots v_{2k-1}$ span a $k+1$-dimensional vector subspace of $\mathcal{T}^*_{\boldsymbol{v}}V_{2k}$. It is enough to show that for a generic $\boldsymbol{v}$ the differentials of $N\cdot\mathcal{M}(\boldsymbol{v} - \lambda\boldsymbol{v}^0)$ for different $\lambda \in \mathbb{C}$ span a k-dimensional vector subspace of the hyperplane $d(v_1v_3 \dots v_{2k-1}) = 0$ in $\mathcal{T}^*_{\boldsymbol{v}}V_{2k}$.

The leading coefficient in λ of $N \cdot \operatorname{Tr} \mathcal{M}(\boldsymbol{v} - \lambda\boldsymbol{v}^0)$ is 1, thus the function $N \cdot \operatorname{Tr} \mathcal{M}(\boldsymbol{v} - \lambda\boldsymbol{v}^0) - \lambda^k$ defines a mapping $\mathfrak{M}\colon V_{2k} \to \mathcal{P}_{k-1}$. Again, it is enough to show that the restriction of this polynomial mapping to $H_c = \{v_1v_3 \dots v_{2k-1} = c\}$ is a submersion for a generic $\boldsymbol{v}$ and $c \neq 0$. On the other hand, multiplication of v_i by the same non-zero constant does not change $\mathcal{M}(\boldsymbol{v})$, thus if we prove this statement for one $c \neq 0$, is it true for any $c \neq 0$. Thus it is enough to demonstrate this statement for $c \approx 0$, $c \neq 0$. Again, it is enough to show that the restriction of $\mathfrak{M}$ to an open subset of $c = 0$ is a submersion.

However, if $v_1 = v_2 = \dots = v_{2k-1} = 0$, then

$$\lambda^k + \mathfrak{M}(\boldsymbol{v}) = (\lambda - v_2)(\lambda - v_4)\dots(\lambda - v_{2k}),$$

thus the restriction of $\mathfrak{M}$ to $\{v_1 = v_2 = \dots = v_{2k-1} = 0\}$ is a surjection, thus is a submersion in a generic point. This shows that Theorem 3.2 is applicable, thus the bihamiltonian structure is indeed flat indecomposable at a generic point. ◻

15. Lax structures

The following definition is inspired by [19]. In this paper a notion of a Lax operator is introduced, this is a matrix-valued function on a bihamiltonian structure which satisfies some compatibility relations. However, since these relations are expressed in terms of the characteristic polynomial of the matrix, it is more convenient to work directly with the mapping into polynomials.

Recall that $\mathcal{P}_n$ was defined in Section 7. Denote the value at λ of a polynomial $p \in \mathcal{P}_n$ by $p|_{\lambda}$.

Definition 15.1. Consider a bihamiltonian structure $(M, \{,\}_1, \{,\}_2)$. Consider a mapping $\boldsymbol{L}$ from M to the set $\mathcal{P}_{n-1}$ of polynomials of degree $n-1$. This mapping is a *weak Lax structure* on M of rank n if for any $\lambda \in \mathbb{R}$ the function C_{λ} on M defined by $m \mapsto \boldsymbol{L}(m)|_{\lambda}$ is a Casimir function for $\lambda\{,\}_1 + \{,\}_2$.

Consider a point $m_0 \in M$. Suppose that the action dimension of M at $m_0 \in M$ is n. A *Lax structure* on M near m_0 is a weak Lax structure $\boldsymbol{L}$ of rank n such that the mapping $\boldsymbol{L}$ is a submersion.

Note that if the bihamiltonian structure is in fact analytic, then C_{λ} is Casimir for complex λ too (since the conditions of being a λ-Casimir family are polynomial in λ).

Theorem 15.2. *If an analytic bihamiltonian structure on M admits a Lax structure near $m_0 \in M$, and for one particular $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ the Poisson structure $\lambda_1\{,\}_1 + \lambda_2\{,\}_2$ has a constant corank 1, then the bihamiltonian structure is a Kronecker structure of type* $(\dim M)$ *near m_0.*

In other words, the manifold M is odd-dimensional and one can find a local coordinate system where both brackets have constant coefficients and are given by (1.1). In particular, all such bihamiltonian structures of the same dimension are locally isomorphic.

Proof. Reduce this statement to one of Amplification 3.3.

In our case $d = n - 1$, and, by submersion condition, $\dim W_1 = n$. Thus the only thing one needs to show is that $\dim M = 2n - 1$. This momentarily follows from the definition of the action dimension. ☐

Remark 15.3. In applications the Poisson bracket $\{,\}_1$ usually has a much simpler form than $\{,\}_2$, thus most of the time one would check the rank condition for the bracket $\{,\}_1$. (Recall that for Kronecker structures *all* the nonzero linear combinations of brackets have the same rank.)

Let us spell out the relation of our definition with one of [19]. Consider the Newton symmetric functions $s_k = \sum_i \lambda_i^k$ of roots $\{\lambda_i\}$ of polynomial $\lambda^n + p(\lambda)$, $p \in \mathcal{P}_{n-1}$ as functions on $\mathcal{P}_{n-1}$, let $H_{k-1} \stackrel{\text{def}}{=} s_k \circ \boldsymbol{L}/k$. Then the conditions of [19] are that H_k, $k \geq 0$, satisfy Lenard recursion relations (10.1). As in Section 10, consider a formal power series $c(t) = \sum_{k\geq 1} s_k t^{1-k}/k$ in t^{-1} with coefficients in functions on $\mathcal{P}_{n-1}$. Then $e^{-c(t)/t} = \Pi_i (1 - \lambda_i/t) = t^{-n}(t^n + p(t))$, $p \in \mathcal{P}_{n-1}$. Let $C(t) = \sum_{k\geq 0} H_k t^{-k}$, then $e^{-C(t)/t}|_m = t^{-n}\boldsymbol{L}(m)|_t$ for any $m \in M$.

Since the latter expression is a formal series in t^{-1} with a finite number of non-zero coefficients, it is an anchored formal λ-family iff it is a λ-Casimir family. Since for any function α of one variable $\alpha(C)$ is a Casimir function if C is such, we conclude that $\boldsymbol{L}|_t$ is a λ-Casimir family iff $C(t)$ is an anchored formal λ-family. Thus the condition that $\boldsymbol{L}$ is a weak Lax structure is equivalent to the pair of conditions: of H_k satisfying Lenard recursion relations (10.1), *and additionally* of H_0 being a Casimir function for $\{,\}_1$. This shows

Proposition 15.4. *Suppose that $L\colon M \to \operatorname{Mat}(n)$ is a Lax operator in the sense of* [19]. *Let $\boldsymbol{L}$ be the mapping*

$$M \to \mathcal{P}_{n-1} \colon m \mapsto \det(t\mathbf{1} - L(m)) - t^n.$$

Then $\boldsymbol{L}$ is a weak Lax structure iff $\operatorname{Tr} L$ is a Casimir function for $\{,\}_1$.

As in Section 11, note that in applications the Lenard scheme is most frequently used when $\operatorname{Tr} L$ is a Casimir function for $\{,\}_1$. Note also that one can consider a weak Lax structure as an "anchored" variant of a Lax operator of [19] (compare with Remark 10.1 and Definition 10.6).

Remark 15.5. By Theorem 12.4, in conditions of Theorem 15.2 an open subset of the bihamiltonian structure is locally isomorphic to the structure of Toda lattice. This isomorphism provides the subset U with a Lax operator in the most usual sense of

this word, i.e., with a mapping $L\colon U \to \operatorname{Mat}(n)$ such that for any action function[24] H on U there is a mapping $A_H\colon U \to \operatorname{Mat}(n)$ such that H-Hamiltonian flow on U corresponds to $\frac{dL}{dt} = [A_H, L]$.

In other words, Theorem 15.2 provides a partial explanation for the relation between Lax operator and Lax–Nijenhuis operators discovered in [19].

Remark 15.6. Note that the conditions of Theorem 15.2 break into four separate parts: the condition of being a weak Lax structure, the condition that coefficients of $\boldsymbol{L}$ provide enough functions to completely integrate M, the submersion condition, and the condition of having small corank. Note that the corank of the structure cannot be less than 1, since we *require* existence of Casimir function for any λ. Thus two last conditions taken together may be interpreted as conditions of non-degeneracy of the Lax structure.

Question . *Which conditions on a weak Lax family imply that the bihamiltonian structure is Kronecker at generic points?*

Conjecture 16.2 claims that many bihamiltonian structures which admit a Lax structure are in fact Kronecker at generic points. An answer on the above question might have provided a better understanding for the statement of Conjecture 16.2.

16. Geometric conjectures

Note that the Theorems 12.4, 12.5, and 15.2 run against the common intuition, which says that integrable systems should be expressed as direct products of two-dimensional blocks. However, this point of view comes from the symplectic approach to integrable systems, where everything is *forced* to be even-dimensional.

The above theorems show that this common intuition has historical roots only, and some new type of intuition for geometric approach to integrable systems may be needed.

Our meta-conjecture is that the mindset of "everything is a product of odd-dimensional components (given by (1.1))" is much more appropriate for the geometric study of bihamiltonian structures, compare with Remark 16.1 and Conjecture 16.2.

Again, if one believes in the above meta-conjecture, one can see that the Procrustean approach of symplectic geometry forces a reduction of dimension (as in Remark 12.6, which gives an analogue of restriction to a hypersurface), which reduces a feature-rich bihamiltonian structure to a non-rigid symplectic structure.

Remark 16.1. Definition 1.20 provides an example of micro-local approach to bihamiltonian systems. By Theorem 4.1, in each tangent space any bihamiltonian structure decomposes into a direct sum of Jordan blocks and Kronecker blocks. Thus a natural question arises: given a bihamiltonian structure M, which indecomposable pairs $\mathcal{J}_{2k,\lambda}$ and $\mathcal{K}_{2k-1}$ appear at which points of M?

[24] See Section 5.

Theorems 12.4 and 12.5 answer this question for generic points of the open and the periodic *Toda lattice*. We think we can answer this question[25] for generic points of the odd-dimensional open or even-dimensional periodic *Kac–van Moerbeke–Volterra system* [17, 8], of the *full Toda lattice* [20], and of the multidimensional *Euler top* [27]. In tangent spaces at generic points the open Toda lattice is an indecomposable Kroneker block, the periodic Toda lattice is a direct product of indecomposable 1-dimensional and $2k-1$-dimensional Kroneker blocks. The complete Toda lattice and the multidimensional Euler top are products of Kroneker blocks with the dimensions of components being $(2k-1, 2k-3, 2k-5, \dots)$ and $(2k-1, 2k-5, 2k-9, \dots)$ correspondingly.

Additionally, results of [29] show that a similar decomposition exists for the regular case of Example 1.12. In this case the dimensions of components have the form $2e_1-1, \dots, 2e_r-1$, e_i being the exponents of the Weyl group of $\mathfrak{g}$, r being the rank of $\mathfrak{g}$.

The above descriptions of tangent spaces together with Theorems 12.4 and 12.5 suggest the following

Conjecture 16.2. The odd-dimensional open Volterra system, the even-dimensional periodic Volterra system, the full Toda lattice, the multidimensional Euler top, and the regular case of Example 1.12 are[26] generically Kronecker bihamiltonian structures.

As shown in this paper, the powerful methods of [14, 15] are enough to translate some simple properties[27] of the open and the periodic Toda lattices into description of the *local* geometry of these structures. One may hope that it is possible to generalize the results of [14, 15] so that they cover structures with geometry of tangent spaces as in Remark 16.1. This would allow one to prove Conjecture 16.2 using some simple results about these integrable systems[28].

Using language of Section 5, one can state such conjectures in the following form.

Conjecture 16.3. Suppose that two bihamiltonian structures $\left(M, \{\}_1, \{\}_2\right)$ and $\left(M', \{\}_1', \{\}_2'\right)$ are both homogeneous. Consider webs[29] $\mathcal{B}_U$ and $\mathcal{B}_{U'}$ which correspond to small open subsets $U \subset M$, $U' \subset M'$. If webs $\mathcal{B}_U$ and $\mathcal{B}_{U'}$ are locally isomorphic, then the bihamiltonian structures on M and M' are locally isomorphic. In particular, the types of M and M' coincide.

This conjecture may be augmented by the following description of webs for homogeneous structures [29]:

Proposition 16.4. *The web $\mathcal{B}_U$ corresponding to a small open subset U of homogeneous bihamiltonian structure of type $(2k_1-1, 2k_2-1, \dots, 2k_l-1)$ is a manifold of dimension $k_1+k_2+\dots+k_l$, and the subspace $\mathfrak{C}_\lambda$ of the space of functions on $\mathcal{B}_U$ consists of local equations of a foliation $\mathcal{F}_\lambda$ on $\mathcal{B}_U$ of codimension l.*

Conjecture 16.3, together with Amplification 4.9, lead to the following

Conjecture 16.5. Consider a manifold M with two compatible Poisson structures $\{,\}_1$ and $\{,\}_2$. Consider a finite set L with r elements. Consider families of smooth functions $F_{l,\lambda}$, $l \in L$, $\lambda \in \mathbb{C}$, on M such that for any $l \in L$ and any $\lambda \in \mathbb{C}$ the function $F_{l,\lambda}$ is Casimir w.r.t. the Poisson bracket $\lambda\{,\}_1+\{,\}_2$. Suppose that $F_{l,\lambda}$ depends polynomially on λ

$$F_{l,\lambda}(m) = \sum_{k=0}^{d_l} f_{l,k}(m)\,\lambda^k,$$

with smooth coefficients $f_{l,k}(m)$. For $m \in M$ denote by $W_1(m) \subset \mathcal{T}_m^*M$ the vector subspace spanned by the the differentials $df_{l,k}|_m$ for all possible l and $0 \leq k \leq d_l$. If

1. for one particular value $m_0 \in M$ one has $\dim W_1(m_0) \geq \frac{\dim M + r}{2}$;
2. for one particular value of $\lambda_1, \lambda_2 \in \mathbb{C}^2$ the Poisson structure $\lambda_1\{,\}_1+\lambda_2\{,\}_2$ has at most r independent Casimir functions on any open subset of M near m_0;
3. the degrees d_l satisfy $\sum_L (2d_l+1) \leq \dim M$;

then $\dim M - r$ is even, $\dim W_1(m_0) = \frac{\dim M + r}{2}$, the degrees d_l satisfy $2\sum_L d_l + r = \dim M$, and the bihamiltonian structure on M is Kronecker of type $(2d_1+1, \dots, 2d_r+1)$ on an open subset $U \subset M$ such that m_0 is in the closure of U.

Conjecture 16.5 immediately implies Conjecture 16.2, since the explicit formulae for Hamiltonians for the dynamic systems of Conjecture 16.2 are well-known and may be included into families as in Conjecture 16.5.

To understand the significance of Conjecture 16.5, note that by Remark 4.3 all the Kronecker structures of the given type are locally isomorphic, and obviously satisfy the conditions of the conjecture. Thus this conjecture provides a criterion of being a Kronecker structure in terms of the mutual position of Casimir functions for the combinations of brackets of bihamiltonian structure.

Conjecture 16.6. In the settings of Conjecture 16.5 if one supposes that the Poisson structure $\lambda_1\{,\}_1+\lambda_2\{,\}_2$ has constant corank r, then one may weaken the condition on $\dim W_1$ to become $\dim W_1(m_0) \geq \frac{\dim M + r - 1}{2}$, and amplify the conclusion to so that the open subset U contains m_0.

[25]After the initial release of this paper M. Gekhtman explained us that the result on the open Toda lattice implies the statements about the open odd-dimensional Kac–van Moerbeke–Volterra lattice, as well as a similar statement about the open relativistic Toda lattice [32].

This is an immediate corollary of the existence of local isomorphisms of these bihamiltonian systems similar to those constructed in [5, 3], see [11] and [7].

[26]Paper [39] contains a proof of the part of the conjecture related to Example 1.12, see the previous footnote for some other cases.

[27]The existence of Casimir functions given by Lemmas 13.1 and 14.5.

[28]Again, since the geometry of these system is very well investigated, it may be possible to prove this conjecture directly using appropriate systems of action-angle variables for these manifolds.

However, an approach based on Conjecture 16.3 would allow one to prove Conjecture 16.2 using only simple-to-obtain action variables, i.e., families of Hamiltonians for the above manifolds.

[29]See Section 5.

The above theorems and conjectures lead one to the following

Question . *Why each "classical" finite-dimensional bihamiltonian structure has an open subset which is Kronecker, or may be "naturally" considered as a reduction of dimension starting from a larger bihamiltonian structure which is Kronecker?*

This question is amplified by the fact that in [14, 15] we constructed a huge family of non-Kronecker integrable bihamiltonian structures (see also examples in Section 8 for the dimension being 3). Such integrable systems are *actually nonlinear*, as opposed to *manifestly nonlinear* systems, which may become linear after an appropriate coordinate change (compare with Definition 1.6). One would see that an answer to the above question would unravel some mechanism by which the actually nonlinear integrable systems avoid attention of mathematical physicists.

Note that Theorem 12.4 allows one to restate the above question using direct products of open Toda lattices instead of Kronecker structures:

Why many "classical" bihamiltonian structures are (in generic points) locally isomorphic to direct products of open Toda lattices?

While Section 15 singles out flat indecomposable structures as those which admit non-degenerate Lax structures, we do not consider this as a legitimate explanation to the above selection principle. Lax representation is only one of multiple approaches to integration of dynamical systems, so explaining the above selection principle by using Theorem 15.2 just substitutes one question (why all the classical systems are flat) by another one (why all the classical systems admit Lax representation).

Remark 16.7. Note that a flat bihamiltonian structure of dimension d may be extended locally to a $d(d-1)/2$-parametric linear family of Poisson structures: those which have constant coefficients in the above coordinate system. Our meta-conjecture about the rôle of Kronecker structures may explain an abundance of multi-hamiltonian structures in mathematical physics (for example, see [4, 28, 34, 1]).[30]

References

1. H. S. Blas Achic, L. A. Ferreira, J. F. Gomes, and A. H. Zimerman, *Some comments on the bi(tri)-Hamiltonian structure of generalized AKNS and DNLS hierarchies*, Phys. Lett. A **237** (1998), no. 4-5, 225–233.
2. A. V. Bolsinov, *Compatible Poisson brackets on Lie algebras and the completeness of families of functions in involution*, Izv. Akad. Nauk SSSR Ser. Mat. **55** (1991), no. 1, 68–92.
3. Pantelis A. Damianou, *Multiple Hamiltonian structures for Toda-type systems*, J. Math. Phys. **35** (1994), no. 10, 5511–5541, Topology and physics.
4. Pantelis A. Damianou, Paschalis Paschalis, and Christodoulos Sophocleous, *A tri-Hamiltonian formulation of the full Kostant-Toda lattice*, Lett. Math. Phys. **34** (1995), no. 1, 17–24.

[30]However, note that what is commonly called a "multi-hamiltonian" structure is frequently just a figure of speech: the additional "brackets" which augment the bihamiltonian structure are not only not Poisson (thus do not satisfy Jacobi condition), but not even brackets (thus $\{f,g\}_3$ would be defined for *some* f and g only).

5. Percy Deift and Luen Chau Li, *Poisson geometry of the analog of the Miura maps and Bäcklund-Darboux transformations for equations of Toda type and periodic Toda flows*, Comm. Math. Phys. **143** (1991), no. 1, 201–214.
6. L. D. Faddeev and L. A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer-Verlag, Berlin, 1987, Translated from the Russian by A. G. Reyman [A. G. Reĭman].
7. Leonid Faybusovich and Michael Gekhtman, *Elementary Toda orbits and integrable lattices*, preprint, September 1999.
8. Rui L. Fernandes and João P. Santos, *Integrability of the periodic KM system*, Proceedings of the XXIX Symposium on Mathematical Physics (Toruń, 1996), vol. 40, 1997, pp. 475–484.
9. A. S. Fokas and B. Fuchssteiner, *On the structure of symplectic operators and hereditary symmetries*, Lett. Nuovo Cimento (2) **28** (1980), no. 8, 299–303.
10. F. R. Gantmacher, *The theory of matrices. Vols. 1, 2*, Chelsea Publishing Co., New York, 1959, Translated by K. A. Hirsch.
11. M. I. Gekhtman and M. Z. Shapiro, *Noncommutative and commutative integrability of generic Toda flows in simple Lie algebras*, Comm. Pure Appl. Math. **52** (1999), no. 1, 53–84.
12. I. M. Gel′fand and I. Ja. Dorfman, *Hamiltonian operators and algebraic structures associated with them*, Funktsional. Anal. i Prilozhen. **13** (1979), no. 4, 13–30, 96.
13. Israel M. Gelfand and Ilya Zakharevich, *Spectral theory for a pair of skew-symmetrical operators on S^1*, Func. Anal. Appl. **23** (1989), no. 1, 85–93.
14. ______, *Webs, Veronese curves, and bihamiltonian systems*, J. of Func. Anal. **99** (1991), 150–178.
15. ______, *On the local geometry of bihamiltonian structures*, The Gelfand mathematical seminar, 1990–1992 (Boston), Birkhäuser, 1993, pp. 51–112.
16. ______, *The spectral theory for a pencil of skewsymmetrical differential operators of the third order*, Comm. Pure Appl. Math. **47** (1994), no. 8, 1031–1041.
17. M. Kac and Pierre van Moerbeke, *On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices*, Advances in Math. **16** (1975), 160–169.
18. A. A. Kirillov, *Local Lie algebras*, Uspehi Mat. Nauk **31** (1976), no. 4(190), 57–76.
19. Y. Kosmann-Schwarzbach and F. Magri, *Lax-Nijenhuis operators for integrable systems*, J. Math. Phys. **37** (1996), no. 12, 6173–6197.
20. Bertram Kostant, *The solution to a generalized Toda lattice and representation theory*, Adv. in Math. **34** (1979), no. 3, 195–338.
21. Peter D. Lax, *Almost periodic solutions of the KdV equation*, SIAM Rev. **18** (1976), no. 3, 351–375.
22. B. M. Levitan, *Inverse Sturm-Liouville problems*, VSP, Zeist, 1987, Translated from the Russian by O. Efimov.
23. Franco Magri, *A simple model of the integrable Hamiltonian equation*, Journal of Mathematical Physics **19** (1978), no. 5, 1156–1162.
24. ______, *On the geometry of soliton equations*, Preprint, 1988.
25. ______, *On the geometry of soliton equations*, Acta Appl. Math. **41** (1995), no. 1-3, 247–270, Geometric and algebraic structures in differential equations.
26. Henri McKean, private communication, 1990.
27. Carlo Morosi and Livio Pizzocchero, *On the Euler equation: bi-Hamiltonian structure and integrals in involution*, Lett. Math. Phys. **37** (1996), no. 2, 117–135.
28. Peter J. Olver and Philip Rosenau, *Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support*, Phys. Rev. E (3) **53** (1996), no. 2, 1900–1906.
29. Andriy Panasyuk, *Veronese webs for bihamiltonian structures of higher corank*, Banach Center Publications, Poisson geometry, Proceedings of the conference dedicated to the memory of

Stanislaw Zakrzewski, Warsaw 1998 (Warszawa) (Pawel Urbański and Janusz Grabowski, eds.), Instytut Matematyczny PAN, 1999, to appear.
30. Marie-Hélène Rigal, *Tissus de Véronèse en dimension 3*, Séminaire Gaston Darboux de Géométrie et Topologie Différentielle, 1994–1995 (Montpellier), Univ. Montpellier II, Montpellier, 1995, pp. iv, 63–68.
31. ______, *Systèmes bihamiltoniens en dimension impaire*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 3, 345–359.
32. S. N. M. Ruijsenaars, *Relativistic Toda systems*, Comm. Math. Phys. **133** (1990), no. 2, 217–247.
33. Robert C. Thompson, *Pencils of complex and real symmetric and skew matrices*, Linear Algebra Appl. **147** (1991), 323–371.
34. Takayuki Tsuchida, Yasumasa Kajinaga, and Miki Wadati, *Tri-Hamiltonian structure and complete integrability of Volterra model*, J. Phys. Soc. Japan **66** (1997), no. 9, 2608–2617.
35. Francisco-Javier Turiel, *Classification locale d'un couple de formes symplectiques Poisson-compatibles*, Comptes Rendus des Seances de l'Academie des Sciences. Serie I. Mathematique **308** (1989), no. 20, 575–578.
36. ______, *C^∞-équivalence entre tissus de Veronese et structures bihamiltoniennes*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), no. 10, 891–894.
37. H. W. Turnbull and A. C. Aitken, *An introduction to the theory of canonical matrices*, Dover Publications Inc., New York, 1961.
38. Alan Weinstein, *The local structure of Poisson manifolds*, J. Differential Geom. **18** (1983), no. 3, 523–557.
39. Ilya Zakharevich, *Kronecker webs, bihamiltonian structures, and the method of argument translation*, Archived as math.SG/9908034.

Dept. of Mathematics, Rutgers University, Hill Center, New Brunswick, NJ, 08903
E-mail address: `igelfand@math.rutgers.edu`

Department of Mathematics, Ohio State University, 231 W. 18 Ave, Columbus, OH, 43210
E-mail address: `ilya@math.ohio-state.edu`