# THE SPECTRAL THEORY FOR A PENCIL OF SKEWSYMMETRICAL DIFFERENTIAL OPERATORS OF THE THIRD ORDER

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Abstract. We consider a linear algebra of a pair of skewsymmetrical forms in the space of periodic functions defined by differential operators. By linear transform in the space of functions we reduce this pair to the simplest possible form. In this process we prove the theorem of reduction in rather general context.

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## 0. Introduction

Here we consider a problem to reduce a pair of skewsymmetric bilinear forms

$$
\begin{aligned}
&(\psi_1,\psi_2) \overset{A}{\mapsto} \int \psi_1(x)\left(-\partial^3+2u(x)\partial+2\partial u(x)\right)\psi_2(x)\,dx,\\
&(\psi_1,\psi_2) \overset{B}{\mapsto} 4\int \psi_1(x)\,\partial\psi_2(x)\,dx,\quad \partial=\partial/\partial x,
\end{aligned}
\tag{0.1}
$$

simultaneously to the simplest possible form by a linear transform in the space of 1-periodic functions. Here $u$ is a function with a period 1, the integrals are taken along the interval $[0,1]$. In the following we use two languages identifying a bilinear form $C(v_1,v_2)$ in a space $V$ with an operator $C$ from $V$ to a dual space $V^{\top}$.

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The principal results concerning this problem can be found in the work [1]. The target of this article is to obtain these results by the means of the "pure" theory of operators. In the paper [1] we refered a reader to a work [3]: the main (from the theory of operators point of view) result of the latter paper make it possible to prove "simply" the first of the results in the article [1]. The proof we meant when we wrote [1] as well as the prove in [3] both used the subtle properties of the operators (0.1) and their connection with the operator $Q=-\partial^2+u(x)$. Here we use instead the elaborate theorems of functional analysis of the Keldysh type. Simultaneously we show that the obtained results are very general.

Notwithstanding the above motivation we are not *very* zealous in the task of generalization of the results from [1] and use *some* properties of operators (0.1) of not spectral type. (We believe that this could be avoided by a more elaborate study of the general case.) However, these properties seem much simpler than, say, the used in [3] properties. Moreover, we formulate explicitly the used properties of operators in question.

## 1. The formulation of the theorem

Let $H^s(S^1)$ be a linear space of (generalized) functions on the circle, whose Fourier coefficients $(a_n)$ satisfy the condition

$$((1+|n|)^s a_n)\in l_2.$$

**Theorem 1.1.** *Let $H^\infty(S^1)=\bigcap_s H^s(S^1)$. Let $s\geq 3/2$, $u\in H^{s-1}(S^1)$, $V=H^s(S^1)$. Then*

(1) *The kernel of a non-zero form from the pencil $\mu A+\lambda B$ in the space $V$ is of dimension 1 or 3.*
(2) *The set $\Lambda$ of $\lambda$ such that the form $A+\lambda B$ has 3-dimensional kernel is discrete in $\mathbb{R}$.*
(3) *Let $L_1(V)$ be a closure of the linear envelope for $\operatorname{Ker}(A+\lambda B)$ when $\lambda\notin\Lambda$, $L_2(V)$ be an analogous space for arbitrary $\lambda$:*

$$L_1(V)=\overline{\langle\operatorname{Ker}(A+\lambda B)\mid\lambda\notin\Lambda\rangle},\quad L_2(V)=\overline{\langle\operatorname{Ker}(A+\lambda B)\rangle}.$$

*Then these two spaces are skew-orthogonal respective to any form of the pencil. We mean that for $v_1\in L_1(V)$, $v_2\in L_2(V)$*

$$A(v_1,v_2)=B(v_1,v_2)=0.$$

(4) *These two spaces are skew-orthogonal complements one to another respective to a generic form of the pencil. We mean that if $v_1\perp L_1(V)$ respective to $\mu A+\lambda B$ and $(\lambda,\mu)\neq 0$, then $v_1\in L_2(V)$; if $v_2\perp L_2(V)$ respective to $A+\lambda B$ and $\lambda\notin\Lambda$, then $v_2\in L_1(V)$.*
(5) *There exists an isotropic respective to any form of the pencil complement $K_1$ to the subspace $L_2(V)$ and a skew-orthogonal to $K_1$ complement $K_2$ to the*

*subspace $L_1(V)$ inside the space $L_2(V)$. So*

$$V = L_1(V) \oplus K_2 \oplus K_1, \quad L_2(V) = L_1(V) \oplus K_2.$$

The motivations and comments to this theorem can be found in [1]. Speaking non-formally, the theorem claims that in the space $L_2(V)/L_1(V) \simeq K_2$ the pair of forms can be decomposed in a direct sum of two-dimensional (pairs of) forms. Moreover, this direct sum can be represented as a direct summand in $V$, the remainder (in the space $L_1(V) \oplus K_1$) is undecomposable and can be described as a pair of skewsymmetrical forms of type

$$(-A_1) \oplus A_1^\top, \quad (-B_1) \oplus B_1^\top.$$

Here $A_1$, $B_1$ are operators $L_1(V) \mapsto K_1^\top$ (here $W^\top$ is a dual space to $W$), so $(-A_1) \oplus A_1^\top$ is a mapping $L_1(V) \oplus K_1 \to L_1(V)^\top \oplus K_1^\top$ that can (and will) be considered as a bilinear form in the space $L_1(V) \oplus K_1$.

Hence in the direct summand $K_2$ the pair of forms has the "classical" spectral theory, in the rest of the space $V$ the theory is "nonclassical" (it is described in the article [1]).

## 2. Functional analysis theorem

We state here (slightly cheeting) a useful for us theorem 6.12 of [2].

*Theorem A* . Let $G$ be a normal operator in a Hilbert space $\mathcal{H}$ with a compact resolvent such that the spectrum of $G$ is contained in a finite number of rays $\varphi = \alpha_k$ $(k = 1, \dots, n)$. Let for some $p \in [0, 1)$

$$\limsup_{r \to +\infty} \frac{N(r, G)}{r^{1-p}} < \infty,$$

where $N(r, G)$ is a number of $G$ eigenvalues $\lambda$ with $|\lambda| < r$ counted with multiplicities. If an operator $F$ satisfy the $\rho$-inferiority condition:

$$\|Ff\| \leq b\|Gf\|^p \|f\|^{1-p} \quad \text{for } f \in \mathcal{V}(G), \qquad \mathcal{V}(G) \subseteq \mathcal{V}(F),$$

(here $\mathcal{V}(C)$ is a domain of $C$), and $B = G + F$, then $\mathcal{H}$ is decomposed into an unconditional direct sum corresponding to the system of root subspaces for $B$ (i.e., the operator $B$ is a *direct sum* of finite-dimensional operators). We mean that any vector $v \in V$ can be (uniquely) written as $v = \sum v_i$, where sum converges unconditionally and $v_i$ are root vectors for $B$. (In the worst case for this sum to converge maybe we need yet to group the root subspaces with near eigenvalues.)

Here we do not prove the formulated in [1] more strong convergency property for the series $\sum v_i$ (in this article we asserted that the corresponding sum is an $l_2$-sum). However, this property is in the particular case of operators (0.1) almost trivial if we already know the completeness of the system of root subspaces.



## 3. The properties of operators (0.1)

Here we outline the properties of the forms $A$ and $B$ that we use in the proof of the main theorem.

a) There exists a structure of pre-Hilbert space (,) on $V$ such that in the completion $\overline{V}$ of $V$ in this metric to the form $B$ corresponds a Fredholm operator $\boldsymbol{B}$:

$$B(v, w) = (\boldsymbol{B}v, w).$$

Denote as $\mathbf{A}$ the (unbounded) operator in $\overline{V}$ corresponding to the form $A$:

$$A(v, w) = (\boldsymbol{A}v, w).$$

Let $\boldsymbol{B}^{-1}$ be some choice of an "almost inverse" operator to $\boldsymbol{B}$, i.e., the operator such that $\operatorname{rk}(\boldsymbol{B}^{-1}\boldsymbol{B} - \mathbf{1}) < \infty$ (this operator is defined up to a finite-dimensional summand). Then

b) Operator $\boldsymbol{B}^{-1}\boldsymbol{A}$, defined up to a finite-dimensional summand can be represented as a sum from the theorem A. (It is clear that the conditions of this theorem allow addition of any finite-dimensional operator.)

It seems that the properties a) and b) are sufficient to prove some analogues of the sections 2, 3 and 4 of the theorem 1.1. Nevertheless, in what follows we use two more properties, c) and d).

c) The dimension of the form $B$ kernel is no more than the dimension of a kernel for an arbitrary form from the pencil.

To formulate d) let as change the space $V$ to its completion $\overline{V}$ respective to (,). Let us change our notations and denote as $B$ the *extension* of $B$ to a skewsymmetrical form in the space $W = V \oplus \mathbb{C}^k$, $k = \dim \operatorname{Ker} B$:

$$((v, \alpha), (w, \beta)) \mapsto B(v, w) + \sum_j (\beta_j \langle l_j, v\rangle - \alpha_j \langle l_j, w\rangle).$$

Here $(v, \alpha), (w, \beta) \in W = V \oplus \mathbb{C}^k$, $\alpha = (\alpha_i)_{i=1}^k$, $\beta = (\beta_i)_{i=1}^k$; $\{l_i\}_{i=1}^k$ is some fixed set of covectors in $V$, $\langle \cdot, \cdot \rangle$ is a pairing between $V$ and $V^\top$. Suppose that the restriction of the set $\{l_j\}$ to the subspace $\operatorname{Ker} B$ is non-degenerate. (The number of covectors being equal to the dimension of this space, this condition is equivalent to some determinant being non-zero.)

In fact we have considered (as in the theory of selfadjoint extensions of the second kind) a minimal non-degenerate extension of the form $B$.

Let as extend the form $A$ by zero as

$$((v, \alpha), (w, \beta)) \mapsto A(v, w).$$

From now on let us denote the "old" forms $A$ and $B$ as $A|_V$ and $B|_V$.

We have resulted in the case of two skewsymmetrical forms in the space $W \supset V$, one if which (we mean $B$) being non-degenerate. This situation lies in the mainstream of the theory of operators and is easy to investigate: an introduction of an operator $X = B^{-1}A$ reduces this problem to investigation of operators, not bilinear forms.

To satisfy the property d) we need the specific choice of the objects from a), b), c) for the pair of forms $A$, $B$ from (0.1). Let $(,)$ be the Sobolev norm $(,)_{H^{1/2}}$, i.e., $(\varphi, \varphi) = \sum (1+|n|)\,|\varphi_n|^2$, here $\varphi_n$ are the Fourier coefficients for $\varphi$. In this case $k=1$, so the set $\{l_i\}$ consists of one covector. Let $l_1$ be $\varphi \mapsto \int \varphi$. For simplicity of notations suppose that the form $A$ has one-dimensional kernel (the general case can be reduced to this by a shift $A_1 = A + \varepsilon B$, that does not essentially change the pencil $\lambda A + \mu B$). Now we can formulate the last property d) as:

d) The operator $X$ has no Jordan blocks[1] in its root decomposition[2] and no eigenspaces of dimension greater than 2. The spectrum of $X$ is real.

## 4. The proof of the theorem in the conditions of the section 3

First of all, the operator $X$ can be decomposed in a direct sum of Jordan cells by the theorem A and the property b). Really, in the property b) of the section 3 we could choose an arbitrary "almost inverse" operator, since the condition of $p$-inferiority from the theorem A is invariant respective to addition of finite-dimensional operator (if the operator $G$ has no kernel). Indeed, it is sufficient to show that an operator of rank 1 is $p$-inferior to any operator without a kernel, and the last assertion is clear by continuity of $G^{-1}$. Therefore the condition of $p$-inferiority is in fact also invariant respective to a finite-dimensional extensions. (It is clear that for a given $B$ we can always choose the operator $G$ that has no kernel.)

Taking all this into account, we see that the operator $X$ satisfies the conditions of the theorem A. (The condition d) shows that this operator is in fact diagonalizable.)

For a while we use only the properties a) and b) of the section 3—it seems to us that this does not overweight the proof too much. However, we hope that it can simplify the acquaintance with a structure of the proof to a reader[3]. We will state it explicitly when the moment comes we need to use more subtle properties c) and d). However, the reader can right now change the words "root subspace" to the words "eigenspace" and skip the parts of the proof that are connected to the study of Jordan blocks.

Extend the covectors $v_i$ to the space $W$ by $v_i|_{\mathbb{C}^k} = 0$. Call the root subspace for $X$ corresponding to an eigenvalue $\lambda$ as $W_\lambda$, the corresponding eigenspace as $\widetilde{W}_\lambda$. Let $V_\lambda = W_\lambda \cap V$, $\widetilde{V}_\lambda = \widetilde{W}_\lambda \cap V$. It is clear that the space $\widetilde{V}_\lambda \subset V$ is a subspace of $\operatorname{Ker}(A-\lambda B)|_V$ as an intersection of $V$ and the $\operatorname{Ker}(A-\lambda B)$. Consider an arbitrary root space (eigenspace) for the operator $X$, corresponding, say, to an eigenvalue $\lambda$.

[1] I.e., no Jordan cells of the size greater than $1 \times 1$.

[2] The existence of this decomposition follows, as it is shown below, from the previous properties.

[3] As we have already noted, it seems very reasonable that almost all the claims of the theorem A can be deduced without the properties c) and d). In fact the property c) guaranties that the pair $A$, $B$ has no Jordan blocks of infinite size. It is clear from the compactness property that such a block can appear only at infinity.



Since $(A-\lambda B)$ sends vector $w = (v, \alpha) \in W$, $\alpha = (\alpha_i)_{i=1}^k$, into covector

$$\left( (A-\lambda B)\, v + \lambda \sum_j \alpha_j l_j, \left(-\lambda\,(l_j, v)\right)_{i=1}^k \right),$$

it is clear that for $\lambda \neq 0$

$$\begin{aligned}(4.2)\quad & w = (v,\alpha) \in \operatorname{Ker}(A-\lambda B) \\ & \iff v \perp l_j, j = 1, \dots, k, \quad (A-\lambda B)|_V\,(v, \cdot) = \langle -\lambda \alpha_j l_j, \cdot \rangle .\end{aligned}$$

Our choice of the way to extend the operator $A$ assures that the value $\lambda = 0$ is always an eigenvalue with multiplicity at least the $\dim \operatorname{Ker} A|_V$ plus $\dim \operatorname{Ker} B|_V$ (i.e., at least $2 \dim \operatorname{KerB}$).

From now on we use the identification of the space of bilinear form on $V$ with the space of mappings $V \to V^\top$ and denote the corresponding to the forms $A$ and $B$ mappings by the same letter.

Now it is clear that the operator $X$ is symmetrical respective to the form $B$:

$$\begin{aligned} B\,(Xv, w) = \langle BXv, w\rangle = \langle Av, w\rangle = A\,(v,w) = -A\,(w,v) = \\ -\langle Aw, v\rangle = -\langle BXw, v\rangle = -B\,(Xw, v) = B\,(v, Xw)\end{aligned}$$

(here $\langle\cdot,\cdot\rangle$ denotes the pairing between $W^\top$ and $W$). Therefore the operator $X-\lambda$ is also symmetrical, hence the operator $(X-\lambda)^{-1}$ is symmetrical if $\lambda$ is not an eigenvalue of $X$:

$$\begin{aligned} \left((X-\lambda)^{-1} v, w\right) = \left((X-\lambda)^{-1} v, (X-\lambda)\,(X-\lambda)^{-1} w\right) = \\ \left((X-\lambda)\,(X-\lambda)^{-1} v, (X-\lambda)^{-1} w\right) = \left(v, (X-\lambda)^{-1} w\right).\end{aligned}$$

Therefore any linear combination of the operators $(X-\lambda)^{-1}$ for different $\lambda$ (i.e., in fact, any function of $X$) is symmetrical. The same properties of symmetry can be deduced now respective to any form of the pencil, for example,

$$\begin{aligned} (A-\lambda B)\,(Xv, w) = \langle (A-\lambda B)\,Xv, w\rangle = \langle B\,(X-\lambda)\,Xv, w\rangle = \\ B\,((X-\lambda)\,Xv, w) = B\,((X-\lambda)\,v, Xw) = (A-\lambda B)\,(v, Xw).\end{aligned}$$

Second, any two subspaces $W_{\lambda_1}$, $W_{\lambda_2}$ $(\lambda_1 \neq \lambda_2)$ are mutually skeworthogonal respective to any form $A-\lambda B$. Really, by definition the corresponding to $W_{\lambda_1}$ eigenspace is skeworthogonal to the whole $W$ respective to $A-\lambda_1 B$. Since $X-\lambda_1$ is invertible in $W_{\lambda_2}$,

$$\begin{aligned} & (A-\lambda_1 B)\,(W_{\lambda_1}, W_{\lambda_2}) \\ & \quad = (A-\lambda_1 B)\left((X-\lambda_1)^k W_{\lambda_1}, (X-\lambda_1)^{-k} W_{\lambda_2}\right) = (A-\lambda_1 B)\,(0, W_{\lambda_2}) = 0.\end{aligned}$$

Hence $W_{\lambda_1}$ and $W_{\lambda_2}$ are mutually orthogonal respective to $A-\lambda_1 B$ and $A-\lambda_2 B$, therefore respective to any form of a pencil. Now we can note that any space $W_\lambda$

is even-dimensional. Indeed, the kernel of $B|_{W_\lambda}$ should lie in the kernel of $B$ on the whole space $W$ (since any element $W_\lambda$ is skeworthogonal—respective to any form—to the direct sum of the remaining spaces $W_\mu$, i.e., to a complement to $W_\lambda$). However, $B$ on $W$ is non-degenerated.

Let us show that

$$(*)\qquad 2\dim \widetilde{V}_\lambda - \dim \widetilde{W}_\lambda = \dim \operatorname{Ker}\left(A-\lambda B\right)|_V - \dim \operatorname{Ker} B|_V.$$

Really,

$$\operatorname{Ker}\left(A-\lambda B\right)|_V / V \cap \operatorname{Ker}\left(A-\lambda B\right) \xrightarrow{A-\lambda B} \left(W/V\right)^\top = V^\perp$$

$$\operatorname{Ker}\left(A-\lambda B\right) / V \cap \operatorname{Ker}\left(A-\lambda B\right) \xrightarrow{\pi} W/V.$$

It is clear that the images of these maps are mutually orthogonal, since if

$$v \in \operatorname{Ker}\left(A-\lambda B\right)|_V, \quad w \in \operatorname{Ker}\left(A-\lambda B\right) / V \cap \operatorname{Ker}\left(A-\lambda B\right),$$

the expression $\left(A-\lambda B\right)\left(v,w\right)$ is defined correctly by definition of $v$ and is 0 by definition of $w$. On the other hand, if $\alpha \in \left(W/V\right)^\top = V^\perp$ is orthogonal to $\operatorname{Ker}\left(A-\lambda B\right)/V \cap \operatorname{Ker}\left(A-\lambda B\right)$, then it is orthogonal also to $\operatorname{Ker}\left(A-\lambda B\right)$, hence lies in $\operatorname{Im}\left(A-\lambda B\right)^\top = \operatorname{Im}\left(A-\lambda B\right)$, therefore also in $\operatorname{Im}\left(A-\lambda B\right) \cap V^\perp$. However, the latter space coincides evidently with the image of

$$\operatorname{Ker}\left(A-\lambda B\right)|_V / V \cap \operatorname{Ker}\left(A-\lambda B\right) \xrightarrow{(A-\lambda B)} \left(W/V\right)^\top = V^\perp,$$

hence the images of two maps in question are orthogonal complements one to another. Comparison of dimensions of these images gives as (*).

In particular, if $\dim W_\lambda = 0$, then also $\dim \widetilde{W}_\lambda = \dim \widetilde{V}_\lambda = 0$, hence

$$\dim \operatorname{Ker}\left(A-\lambda B\right)|_V = \dim \operatorname{Ker} B|_V.$$

Therefore the jumps of dimension of the pencil $\left(A-\lambda B\right)|_V$ kernel can appear only in the eigenvalues of the operator $X$.

Consider $T = \operatorname{KerB}|_V \subset V \subset W$. By definition $BT \perp V$, $BT \subset W^\top$. The computation of dimension shows that $BT = V^\perp$ (the orthogonal complement to $V$ in $W^\top$). Since $T \subset V$, the space $T$ is isotropic respective to $B$. Hence $BT$ is isotropic in $W^\top$ respective to the form $B^{-1}$ in this space:

$$B^{-1}\left(\alpha,\beta\right) = \left\langle \alpha, B^{-1}\beta \right\rangle, \qquad B^{-1}\colon W^\top \to W.$$

Consider $\lambda$ such that $X-\lambda$ is invertible. Let $T_\lambda = \left(X-\lambda\right)^{-1} T \subset W$. Since

$$T_\lambda = \left(X-\lambda\right)^{-1} T = \left(X-\lambda\right)^{-1} \operatorname{Ker} B|_V = \\ \left(AB^{-1}-\lambda\right) \operatorname{Ker} B|_V = \left(A-\lambda B\right)^{-1} B \operatorname{Ker} B|_V = \left(A-\lambda B\right)^{-1} V^\perp,$$

the space

$$T_\lambda = \left\{ w \in W \mid \left(A-\lambda B\right)\left(w,v\right) = 0 \quad \forall v \in V \right\}$$



contains the space $\operatorname{Ker}\left(A-\lambda B\right)|_V$. However, under the condition c)

$$\dim \operatorname{Ker}\left(A-\lambda B\right)|_V \geq \dim T = \dim T_\lambda.$$

Hence under the condition c)

$$T_\lambda = \operatorname{Ker}\left(A-\lambda B\right)|_V \subset V.$$

Now we can consider an eigenvalue $\lambda$ of the operator $X$ and the corresponding projector $P_\lambda$ on the root subspace:

$$P_\lambda = \frac{1}{2\pi i} \oint_C \left(X-\lambda'\right)^{-1} d\lambda',$$

here contour $C$ goes around the point $\lambda$. It is clear that $P_\lambda^\top$ is the projector on a root subspace for $X^\top$ with an eigenvalue $\lambda$. Consider $S_\lambda = P_\lambda^\top\left(BT\right) \subset W^\top$.

First, $S_\lambda$ is orthogonal to $W_\mu$ if $\lambda \neq \mu$. Second, if $w \in W_\lambda$, $w \perp S_\lambda \iff w \perp BT \iff w \in V$. Hence $S_\lambda$ is the space of equations for $V_\lambda$ in $W_\lambda$.

We want to prove that the subspace $V_\lambda$ is coisotropic in $W_\lambda$ respective to the form $B|_{W_\lambda}$. It is sufficient to show that $V_\lambda^\perp$ is isotropic in $W_\lambda^\top$ respective to $\left(B|_{W_\lambda}\right)^{-1}$, or, what is the same, that $S_\lambda$ is isotropic in $W^\top$ respective to $B^{-1}$. However,

$$B^{-1}\left(S_\lambda, S_\lambda\right) = \left\langle B^{-1}P_\lambda^\top BT, P_\lambda^\top BT \right\rangle.$$

Now we can note that

$$B^{-1}P_\lambda^\top B = B^{-1}\frac{1}{2\pi i}\oint_C \left(\left(B^{-1}A-\lambda'\right)^\top\right)^{-1} d\lambda' B = \\ B^{-1}\frac{1}{2\pi i}\oint_C \left(AB^{-1}-\lambda'\right)^{-1} d\lambda' B = \frac{1}{2\pi i}\oint_C \left(B^{-1}A-\lambda'\right)^{-1} d\lambda' = P_\lambda.$$

Hence

$$B^{-1}\left(S_\lambda, S_\lambda\right) = \left\langle P_\lambda T, P_\lambda^\top BT \right\rangle = \left\langle P_\lambda^2 T, BT \right\rangle = \left\langle P_\lambda T, BT \right\rangle = 0,$$

since $P_\lambda T \subset V$ (as a linear combination of $\left(X-\lambda'\right)^{-1} T = T_{\lambda'} \subset V$). So $V_\lambda$ is indeed a coisotropic subspace in $W_\lambda$. Let $V_\lambda' \subset V_\lambda$ be a skeworthogonal complement to $V_\lambda$ in $W_\lambda$ respective to $B$ (hence respective to any form of a pencil). Note that if $\dim \operatorname{Ker}\left(A-\lambda B\right)|_V = \dim \operatorname{KerB}|_V$, then $2\dim \widetilde{V}_\lambda = \dim \widetilde{W}_\lambda$, hence $V_\lambda$ is a lagrangian subspace of $W_\lambda$ and $V_\lambda' = V_\lambda$, if $\widetilde{W}_\lambda = W_\lambda$.

Now it is the time to leave the general point of view and to return to our particular choice of the pair of forms. Since in this case any root vector of $X$ is an eigenvector, the space $V_\lambda$ lies in the kernel of $A-\lambda B$. Hence $\sum_\lambda V_\lambda \subset L_2\left(V\right)$, and the skeworthogonal complement to $L_2\left(V\right)$ respective to a generic form from the pencil should be contained in $\sum_\lambda V_\lambda'$. Let us show that $\sum_\lambda V_\lambda' \subset L_1\left(V\right)$. It is sufficient to prove that if $V_\lambda' \neq 0$, then $\dim \operatorname{Ker}\left(A-\lambda B\right)|_V = \dim \operatorname{KerB}|_V$.

However, if $\dim \operatorname{Ker}\left(A-\lambda B\right)|_V > \dim \operatorname{KerB}|_V = 1$ and $V_\lambda' \neq 0$, then (*) shows that $\operatorname{Ker}\left(A-\lambda B\right)$ should be at least 4-dimensional, what contradicts d).

Hence in our case either $\dim \operatorname{Ker}(A-\lambda B)|_V = 1$, or $\dim \operatorname{Ker}(A-\lambda B)|_V = 3$, and in the latter case $\lambda$ is an eigenvalue of $X$ and $\dim W_\lambda = \dim V_\lambda = 2$. This proves the parts 1 and 2 of the theorem. The part 3 can be proved in the same way as that the subspaces $W_\lambda$ are skeworthogonal. To prove 4 let us note that a skeworthogonal to $L_2(V)$ vector in $V$ should be $B$-skeworthogonal to $V_\lambda$ (since a generic form from the pencil $A-\mu B$ is proportional to $B$ on $W_\lambda$). Hence the $W_\lambda$-component of this vector in the decomposition of $W$ in root subspaces for $X$ should lie in $V'_\lambda$. Therefore this component is 0 if $\dim \operatorname{Ker}(A-\lambda B)|_V \neq 1$. If $\dim \operatorname{Ker}(A-\lambda B)|_V = 1$, then $V'_\lambda = V_\lambda = \operatorname{Ker}(A-\lambda B)|_V$ (use the formula (*)), hence $V'_\lambda \subset L_1(V)$, and one part of 4 is proved. In particular, a sum $\sum_\lambda V'_\lambda$ is dense in $L_1(V)$, since any vector of the latter space is skeworthogonal to $L_2(V)$.

If a vector $v \in V$ is skeworthogonal to $L_1(V)$ respective to $A-\mu B$, then it is skeworthogonal to $V'_\lambda$, hence its $W_\lambda$-component should lie in $V_\lambda$ for $\lambda \neq \mu$. Let $v_1$ be the $W_\mu$-component for $v$, $v_2 = v - v_1$. The previous discussion shows that $v_2 \in V$, hence $v_1 \in V$, hence $v_1 \in V_\mu$, i.e., *all* the $W_\lambda$-components for $v$ lie in $V_\lambda$, hence $v \in L_2(V)$.

To prove 5, take as $K'_1$ a sum of orthogonal complements to $V'_\lambda$ in $W_\lambda$ (respective to the hermitian form $(\cdot,\cdot)$) for $\lambda$ such that $\dim \operatorname{Ker}(A-\lambda B)|_V = 1$. This subspace of $W$ contains a complement to $V$ in $W$ (consider $\lambda = 0$). Let $K_1 = V \cap K'_1$, $K_2 = \sum V_\lambda$ for $\dim \operatorname{Ker}(A-\lambda B)|_V = 3$. It is clear that $K_1$ is isotropic and orthogonal to $K_2$ respective to any form of the pencil and that $K_2$ is a complement to $L_2(V)$ in $L_1(V)$, $K_1$ is a complement to $L_2(V)$.

## 5. The proof of properties from the section 3

The condition a) is evident respective to the bracket $(,)_{H^{1/2}}$. In the condition b) take as $G$ the operator $\frac{1}{4}\partial^2 - 1$, and as $F$ the operator $-u - \frac{1}{2}\partial^{-1} \circ u'$ (here $\partial^{-1}$ is 0 on constant functions and $\int \partial^{-1}\varphi = 0$). Take $p = \frac{1}{2}$; now the condition of the theorem A is satisfied since the operator $F : H^{1/2}(S^1) \to H^{-1/2}(S^1)$ is bounded and the function $s \mapsto \log \|\cdot\|_s$ is convex.

To prove c) consider an operator $Q = -\partial^2 + u(x)$. We can easily prove a standard in studying the operators $A$ and $B$ property

$$(Q-\lambda)\varphi = 0, \quad (Q-\lambda)\psi = 0 \Rightarrow (A-\lambda B)\varphi\psi = 0. \tag{5.1}$$

Let $\varphi$ and $\psi$ be Bloch functions for the operator $Q-\lambda$ (there are two of them, if the monodromy operator has no Jordan blocks). Then $\varphi\psi \in \operatorname{Ker}(A-\lambda B)$. If the monodromy operator is Jordan, then instead of $\varphi\psi$ we can take a square of the (only) Bloch function.

To prove d) we need a subtler discussion. If the kernel of $A-\lambda B$ is of dimension more than 2, then its dimension is at least 4. (Here we bravely use the claims from the section 4 and leave the proof of the absence of logical cycles to a reader.) Hence all the elements of the space $\operatorname{Ker}(A-\lambda B)|_V$ (whose dimension cannot exceed 3 by the obvious reason) should be orthogonal to $l_1$ (by (4.2)) if $\lambda \neq 0$. However, if $\operatorname{Ker}(A-\lambda B)|_V$ is a 3-dimensional space, then the operator $Q-\lambda$ should have a double eigenvalue with periodic or antiperiodic boundary conditions. (Really, by (5.1) the monodromy operator for $(A-\lambda B)$ is isomorphic to a symmetrical square of the monodromy operator $\mathfrak{T}_\lambda$ for $Q-\lambda$, and the equation $S^2Z = \mathbf{1}$ has only 2 solutions in the matrices $2\times 2$, the operators $\pm\mathbf{1}$—it is sufficient to consider a Jordan form.) However, in this case the function $\varphi^2+\psi^2$ (here $\varphi$, $\psi$ are eigenfunctions for $Q$) is an element of a kernel of $A-\lambda B$ that isn't orthogonal to $l_1$. A contradiction.



To prove that the spectrum of $X$ is real, note that a function

$$\Phi_\lambda : x \mapsto \frac{\delta}{\delta u(x)} \operatorname{Tr} \mathfrak{T}_\lambda \tag{5.2}$$

is proportional to a vector from $\operatorname{Ker}(A-\lambda B)|_V$ (see [3]), hence any eigenvalue of $X$ that isn't a double eigenvalue of $Q$ corresponds to $l_1 \perp \operatorname{Ker}(A-\lambda B)|_V$, i.e., to $\frac{\partial}{\partial\lambda}\operatorname{Tr}\mathfrak{T}_\lambda = 0$, hence $\lambda$ is real.

Indeed, $\operatorname{Tr}\mathfrak{T}_\lambda$ is an entire function of order $\frac{1}{2}$, hence it should be uniquely determined by zeros of $\operatorname{Tr}\mathfrak{T}_\lambda - 2$ (or $\operatorname{Tr}\mathfrak{T}_\lambda + 2$). These zeros correspond to $Q$ eigenvalues with periodic or antiperiodic (respectively) boundary conditions. Hence they are real. A change of $u(x)$ results in a continuous change of these numbers. However, a zero of $\operatorname{Tr}\mathfrak{T}_\lambda - 2$ can never coinside with a zero of $\operatorname{Tr}\mathfrak{T}_\lambda + 2$. Hence the order of this points on a line should remain the same as the function $u$ changes, hence it is the same as for $u = 0$. In this case pairs of roots for $\operatorname{Tr}\mathfrak{T}_\lambda + 2$ are divided by pairs of roots for $\operatorname{Tr}\mathfrak{T}_\lambda - 2$. Hence the extrema of $\operatorname{Tr}\mathfrak{T}_\lambda$ should be simple and real.

In what follows we use the fact that the family of subspaces $\operatorname{Ker}(A-\lambda B)|_V$, $\lambda \notin \Lambda$, can be completed to a continuous family on the whole complex plane. Really, the function $\Phi_\lambda \in \operatorname{Ker}(A-\lambda B)|_V$ depends analytically in $\lambda$. However, $\Phi_{\lambda_0} \equiv 0$ if $\lambda_0 \in \Lambda$. Nevertheless, the consideration of $\frac{\Phi_\lambda}{\lambda-\lambda_0}$ proves the claim. It is easy to see that $\frac{\Phi_\lambda}{\lambda-\lambda_0}|_{\lambda=\lambda_0}$ is proportional to $\varphi^2+\psi^2$, $\varphi$ and $\psi$ being the orthonormal basis in $\operatorname{Ker}(Q-\lambda)$.

Now suppose that the operator $X$ has a Jordan block:

$$(X-\lambda)w = 0, \quad (X-\lambda)v = w.$$

Then

$$w \in \operatorname{Im}(X-\lambda) \iff w \perp \operatorname{Ker}(X-\lambda)^\top \iff B(w, \operatorname{Ker}(X-\lambda)) = 0, \tag{5.4}$$

since $(X-\lambda)^\top = B(X-\lambda)B^{-1}$ and $\operatorname{Ker}(X-\lambda)^\top = B\cdot\operatorname{Ker}(X-\lambda)$. Hence $w \in \operatorname{Ker B}|_{\operatorname{Ker}(A-\lambda B)}$. Since $\operatorname{Ker}(A-\lambda B)$ is two-dimensional, $B|_{\operatorname{Ker}(A-\lambda B)} = 0$.

Now we can look on the space $\operatorname{Ker}(A-\lambda B)|_V$, so we need to consider two cases: when dimension of this space is 3 and when it is 1.

In the first case $\lambda$ is a double eigenvalue of $Q$. We claim that $B|_{\operatorname{Ker}(A-\lambda B)|_V}$ is also 0. Really, the vector $w_0 = \varphi^2+\psi^2$ that complements the former space $\operatorname{Ker}(A-\lambda B)$

to the latter $\operatorname{Ker}(A-\lambda B)|_V$ should lie in $\operatorname{KerB}|_{\operatorname{Ker}(A-\lambda B)|_V}$ (because it allows a deformation to a vector $w_\varepsilon \in \operatorname{Ker}(A-(\lambda+\varepsilon)B)$). As in (5.4) we can deduce that for any $w$ such that $(A-\lambda B)|_V w = 0$ there exists $v$ such that $(A-\lambda B)|_V v = B|_V w$. Hence $(A-(\lambda+\varepsilon)B)(w+\varepsilon v) = \mathcal{O}(\varepsilon^2)$. Since the operator $(A-\lambda B)$ has three periodical eigenfunctions, the change of the equation $(A-\lambda B)\varphi = 0$ to the corresponding Volterra equation shows that the monodromy operator for $(A-(\lambda+\varepsilon)B)$ differs of $\mathbf{1}$ on $\mathcal{O}(\varepsilon^2)$, hence $\mathfrak{T}_\lambda = \pm\mathbf{1} + \mathcal{O}(\varepsilon^2)$. Since the function Tr on $\mathrm{SL}_2$ has a critical point at $\mathbf{1}$, $\operatorname{Tr}\mathfrak{T}_\lambda = \pm 2 + \mathcal{O}(\varepsilon^4)$. However, the last formula means that the extremum of $\operatorname{Tr}\mathfrak{T}_\lambda$ is triple, what is impossible.

In the second case $\lambda$ isn't a double eigenvalue of $Q$. Now $\lambda$ corresponds to $\frac{\partial}{\partial\lambda}\operatorname{Tr}\mathfrak{T}_\lambda \propto \langle\Phi_\lambda, 1\rangle = 0$ (see (4.2), (5.2)), and the same discussion with $v \perp l_1$ shows that $\frac{\partial}{\partial\lambda}\operatorname{Tr}\mathfrak{T}_\lambda$ has zero of the second order, what is impossible.

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