NONLINEAR WAVE EQUATION, NONLINEAR RIEMANN PROBLEM, AND THE TWISTOR TRANSFORM OF VERONESE WEBS

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Abstract. Veronese webs are rich geometric structures with deep relationships to various domains of mathematics. The PDEs which determine the Veronese web are overdetermined if dim > 3, but in the case dim = 3 they reduce to a special flavor of a non-linear wave equation. The symmetries embedded in the definition of a Veronese web reveal themselves as Bäcklund–Darboux transformations between these non-linear wave equations.

On the other hand, the twistor transform identifies Veronese webs with moduli spaces of rational curves on certain complex surfaces. These moduli spaces can be described in terms of the non-linear Riemann problem. This reduces solutions of these non-linear wave equations to the non-linear Riemann problem.

We examine these relationships in the particular case of 3-dimensional Veronese webs, simultaneously investigating how these notions relate to general notions of geometry of webs.

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0. Introduction

We denote the d-dimensional coordinate vector space over the base field by $\mathbb{V}^d$, and the corresponding projective space by $\mathbb{P}^{d-1}$. For $(v_1, \dots, v_d) \in \mathbb{V}^d \smallsetminus \{0\}$ we denote by $(v_1 : \dots : v_d)$ the corresponding element of $\mathbb{P}^{d-1}$. By $\mathbb{B}_r^d \subset \mathbb{V}^d$ we denote the open ball of radius r centered at the origin. Then $\left(\mathbb{B}_r^1\right)^d$ is a cube in the real case and a polydisk in the complex case.

As a convention, put $|\infty| = \infty$, so that $\{|z| > 1\}$ includes $z = \infty$.

The word "smooth" can have 3 different meanings: in the case of the base field $\mathbb{R}$ it can mean either C^∞-smooth or real-analytic, in the case of the base field $\mathbb{C}$ it means complex-analitic. When only some of these cases work, we use more specific terms.

In this paper we study a special family of nonlinear wave equations. Elements of this family are parameterized by numbers A, B, C which satisfy

$$A \neq 0,\ B \neq 0,\ C \neq 0,\ A + B + C = 0. \tag{0.1}$$

Given such numbers, the equation is

$$Aw_x w_{yz} + Bw_y w_{xz} + Cw_z w_{xy} = 0;$$

here $w(x, y, z)$ is a function of three variables, If we need to specify A, B, C, we may call this equation the (A, B, C)-*equation*. Whenever we mention an (A, B, C)-equation we assume that A, B, C satisfy (0.1).

In this paper we study only those solutions of (A, B, C)-equations which are in general position, according the the following

Definition 0.1. Say that a function $w(x, y, z)$ is *non-degenerate* if $w_x \neq 0$, $w_y \neq 0$, $w_z \neq 0$ whenever $w(x, y, z)$ is defined.

Definition 0.2. Say that two functions $w(x, y, z)$ and $w'(x, y, z)$ are *gauge transforms* of each other, if $w = \tau \circ w'$ for an appropriate invertible scalar function τ of one variable.

The first target of this paper is the following statement:

Theorem 0.3. *Suppose that triples* (A, B, C) *and* $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$ *satisfy conditions* (0.1). *Consider equations*

$$Aw_x w_{yz} + Bw_y w_{xz} + Cw_z w_{xy} = 0, \tag{0.2}$$

$$\widetilde{A}v_x v_{yz} + \widetilde{B}v_y v_{xz} + \widetilde{C}v_z v_{xy} = 0, \tag{0.3}$$

the system of equations

$$
\begin{aligned}
A\widetilde{B}w_x v_y &= \widetilde{A}B w_y v_x \\
A\widetilde{C}w_x v_z &= \widetilde{A}C w_z v_x,
\end{aligned}
\tag{0.4}
$$

and the equation (with $\alpha = A/\widetilde{A}$, $\beta = B/\widetilde{B}$, $\gamma = C/\widetilde{C}$*)*

$$
(v_x, v_y, v_z) \sim (\alpha w_x, \beta w_y, \gamma w_z),
\tag{0.5}
$$

here for two vector-functions we write $a \sim b$ *if* $a(x,y,z) = \psi(x,y,z)\, b(x,y,z)$ *for an appropriate nowhere-0 scalar function* ψ. *Then locally near* $(x,y,z) = (0,0,0)$

1. *For non-degenerate functions* w, v *System* (0.4) *is equivalent to Equation* (0.5)*;*
2. *Given a solution* (w,v) *of System* (0.4) *with non-degenerate* w *and* v *and any gauge transforms* w_1 *of* w *and* v_1 *of* v *the pair* (w_1, v_1) *is a solution of System* (0.4)*;*
3. *Suppose that* $A\widetilde{B} \neq \widetilde{A}B$. *Given a solution* (w,v) *of System* (0.4) *with non-degenerate* w *and* v, *the function* w *satisfies Equation* (0.2), *the function* v *satisfies Equation* (0.3)*;*
4. *Suppose that* $A\widetilde{B} \neq \widetilde{A}B$. *Given a non-degenerate solution* w *of Equation* (0.2), *there is a non-degenerate function* v *such that the pair* (w,v) *satisfies Equations* (0.4). *As a corollary,* v *satisfies Equation* (0.3)*;*
5. *Such a function* v *is defined uniquely up to a gauge transform.*

Theorem 0.3 is proved in Section 4. While one could prove this theorem purely analytically, we emphasize the geometric meaning of its statements, thus prove it via relationship to 3-dimensional Veronese webs, which are introduced in Sections 1 and 2.

Remark 0.4. One can consider the last two statements of Theorem 0.3 as statements about existence of non-pointwise relationship between Equations (0.2) and (0.3). Given a solution w of Equation (0.2), one obtains a (more or less unique) solution v of Equation (0.3) by solving Equations (0.4). Note that the latter equations are equations of lower order than (0.3) when considered as equations in v.

In other words, System (0.4) provides a Bäcklund–Darboux transform of order 1 between two equations (0.2) and (0.3) of order 2. Moreover, this transform is linear in v.

The second target of this paper is to explicitly solve any (A,B,C)-equation in complex domain in terms of the *nonlinear Riemann problem.* This problem is a straightforward nonlinear analogue of the (linear) Riemann conjugation problem:

Definition 0.5. Consider a complex-analytic function $g(\lambda,t)$ defined for $\varepsilon < |\lambda| < 1/\varepsilon$ and $|t| < \delta$, assume that for any given λ, $\varepsilon < |\lambda| < 1/\varepsilon$, the function $t \mapsto g(\lambda,t)$ is invertible. Suppose that equations

$$
\sigma_-(\lambda) = g(\lambda, \sigma_+(\lambda)) \text{ for } \varepsilon < |\lambda| < 1/\varepsilon, \qquad |\sigma_+(\lambda)| < \delta \text{ for } |\lambda| < 1/\varepsilon,
$$

uniquely determine complex-analytic functions $\sigma_+(\lambda)$ defined for $|\lambda| < 1/\varepsilon$, and $\sigma_-(\lambda)$ defined for $|\lambda| > \varepsilon$. Denote the number $\sigma_+(0)$ by $\mathfrak{R}_{\varepsilon\delta}(g)$.

The function $\mathfrak{R}_{\varepsilon\delta}$ sends a function $g(\lambda,t)$ of two variables to a complex number. Call this function the *non-linear Riemann transform.* Note that changing ε and δ cannot change the value of $\mathfrak{R}_{\varepsilon\delta}(g)$ (though this expression can become undefined), thus we are going to drop ε, δ and denote this function by $\mathfrak{R}$.

The next step is to define a special family $g_{x,y,z}(\lambda,t)$ of functions of two variables, given one function $g(\lambda,t)$. Use notation $F_{\mathrm{M},k}(\lambda)$ for Lagrange interpolation polynomials on points $\mathrm{M} = \{\mu_1,\dots,\mu_m\}$:

$$
F_{\mathrm{M},l}(\lambda) = F_{\mathrm{M}\smallsetminus\{\mu_l\}}(\lambda) / F_{\mathrm{M}\smallsetminus\{\mu_l\}}(\mu_l), \qquad F_{\mathrm{M}}(\lambda) = \prod_{\mu\in\mathrm{M}} (\lambda - \mu_n).
$$

Definition 0.6. Consider sets of k numbers $\Lambda = \{\lambda_1,\dots,\lambda_k\}$ and of m numbers $\mathrm{M} = \{\mu_1,\dots,\mu_m\}$ satisfying $|\lambda_l| > 0$, $|\mu_l| > 0$. Consider a function $g(\lambda,t)$. Let $\lambda_0 = 0$, $\Lambda_0 = \Lambda \cup \{0\}$, $F_+ = F_{\Lambda_0,0}$, $F_{+,l} = F_{\Lambda_0,l}$, $F_-(\lambda) = F_{\mathrm{M}}(\lambda)/\lambda^m$, $F_{-,l} = F_{\mathrm{M},l}(\lambda)\,\mu_l^{m-1}/\lambda^{m-1}$. For collections $\{a_i\}$ and $\{b_i\}$ of k and m numbers correspondingly denote

$$
\mathcal{G}_{\Lambda\mathrm{M},\{a_i\}\{b_i\}}(\lambda,t) = F_-(\lambda)^{-1}\left(g(\lambda,\widetilde{t}) - \sum_{l=1}^{m} b_l F_{-,l}(\lambda)\right), \qquad \widetilde{t} = tF_+(\lambda) + \sum_{l=1}^{k} a_l F_{+,l}(\lambda).
$$

Given 3 numbers $\lambda_1, \lambda_2, \lambda_3$, let $g_{x,y,z}(\lambda,t) \overset{\text{def}}{=} \mathcal{G}_{\{\lambda_1,\lambda_2\}\{\lambda_3\},\{x,y\}\{z\}}(\lambda,t)$.

Given a function $\varphi(\lambda)$, $|\lambda| = 1$, define $\operatorname{ind}\varphi$ as $\frac{1}{2\pi i}\oint_{|\lambda|=1} \frac{d\varphi(\lambda)}{\varphi(\lambda)}$.

Theorem 0.7. *Consider a complex-analytic function* $g(\lambda,t)$ *defined for* $\varepsilon < |\lambda| < 1/\varepsilon$ *and* $|t| < \delta$, *such that* $g(\lambda,0) \equiv 0$ *and* $\operatorname{ind}\frac{\partial g}{\partial t}(\lambda,0) = -2$. *Fix* $0 < r < 1$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{P}^1$, $0 < |\lambda_{1,2}| < r$, $|\lambda_3| > 1/r$. *Then*

1. *the function*

$$
w(x,y,z) = \mathfrak{R}(g_{x,y,z})
\tag{0.6}
$$

is correctly defined for small x, y, z, *is complex-analytic and nondegenerate, and satisfies the equation* (0.2) *with*

$$
A = \lambda_1(\lambda_2 - \lambda_3),\ B = \lambda_2(\lambda_3 - \lambda_1),\ C = \lambda_3(\lambda_1 - \lambda_2);
\tag{0.7}
$$

2. *for any scalar functions* ψ, φ_1, φ_2, φ_3 *of one variable which send 0 to 0 the function* $\widehat{w} = \psi(w(\varphi_1(x), \varphi_2(y), \varphi_3(z)))$ *satisfies the same* (A,B,C)*-equation as* $w(x,y,z)$*;*
3. *for any triple* $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$ *which satisfies conditions* (0.1) *one can find* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{P}^1$ *and* $T \neq 0$ *which satisfy the above inequalities and Equation* (0.7) *with* $A = T\widetilde{A}$, $B = T\widetilde{B}$, $C = T\widetilde{C}$*;*

4. *for any nondegenerate complex-analytic solution* $\widehat{w}(x,y,z)$ *of* (0.2) *defined near (0,0,0) the function* $g(\lambda,t)$ *constructed in Theorem 10.1 (for some particular value of the function* $Y(x)$*) satisfies the conditions above, and* $\widehat{w} = \psi(w(x,\varphi_2(y),z))$*; here* $w(x,y,z)$ *is defined by* (0.6)*,* φ_2 *is the inverse function to* $y = Y(x)$*, and* $\psi(t) = \widehat{w}(t, Y(t), 0)$*.*

This theorem is proved in Section 14.

Remark 0.8. Note that Theorem 10.1 determines the gluing function $g(\lambda,t)$ in terms of the values of w and the normal derivative of w on a hypersurface. Thus Theorem 0.7 can be considered as a procedure to solve Equation (0.2) basing on the Cauchy initial data.

Such an approach would not gain a lot if the nonlinear Riemann problem were complicated to solve. However, in Section 16 we are going to show that it is as complicated as solving an ODE of high dimension.

Plan. In Sections 1 and 2 we define Veronese webs. In Section 3 we show that constructing a 3-dimensional Veronese web is equivalent to solving an (A,B,C)-equation. In Section 4 we prove Theorem 0.3, thus construct Bäcklund–Darboux transformations between different (A,B,C)-equations. In Sections 5 and 6 we show how Veronese webs jump into existence given the statement of Theorem 0.3.

Sections 7 and 8 contain first encounters with the twistor transform of the Veronese web. Although full of technical (and long but simple) statements, these sections enable working with the twistor transform as with a manifold (as opposed to a germ), thus remove many linguistic complications. In Section 9 we introduce convenient coordinate systems on the twistor transform, in Section 10 we describe the gluing functions as solutions of appropriate ODEs.

Section 11 starts dealing with the inverse problem of reconstructing the web by its twistor transform. After recalling what are infinitesimal deformations of submanifolds, we obtain the first solution of the inverse problem, the solution which requires a lot of additional data. Section 12 contains technical results which would allow to drop these additional data in complex-analytic cases: Kodaira–Spencer deformation theory for sections of bundles (Theorem 12.2), and the "inverse" theory (Proposition 12.8) which explicitly constructs a small tubular neighborhood in which the deformation theory works.

Section 13 studies in which cases the "additional data" of the inverse twistor transform can be dropped. We call such webs *airy webs*, and show that Veronese webs are airy. This section also provides an alternative heuristic for utility of so-called *Kronecker webs* introduced in [15]: they are airy webs with the parameter space being $\mathbb{P}^1$.

Section 14 completes the full circle by proving Theorem 0.7, thus providing the explicit construction of the inverse twistor transform. Given a non-degenerate solution of the (A,B,C)-equation, Section 10 had shown how to explicitly calculate gluing functions for the twistor transform via solutions of ODEs. Section 14 shows how to use these gluing data for reconstruction of the initial solution of the (A,B,C)-equation. Theorem 0.7 provides a way to completely integrate the (A,B,C)-equation in the non-degenerate case.

The first Appendix (Section 15) connects the results of Section 10 with Turiel classification of Veronese webs of arbitrary dimension [13, 14]. Additionally, we introduce terms using which one can classify arbitrary airy webs of codimension 1. The second Appendix (Section 16) shows that the nonlinear Riemann problem is not harder to solve than Lipschitz ODEs in Hilbert spaces.

1. Webs

Recall the definition of a foliation.

Definition 1.1. A *prefoliation* $\mathcal{F}$ of codimension r on a manifold M is a representation of M as a disjoint union of subsets called *leaves*, each of which is a connected embedded submanifold of codimension r.

Given an open subset $U \subset M$, one can define a *restriction* $\mathcal{F}|_U$ of $\mathcal{F}$ to U, the leaves of which are connected components of $L \cap U$, L running through leaves of $\mathcal{F}$. Say that $\mathcal{F}$ is *direct* if $M = N \times F$ with a connected F, and leaves are $\{n\} \times F$, $n \in N$. In such a case N is called the *base* of $\mathcal{F}$.

The *tangent space* $\mathcal{T}_m\mathcal{F}$ to $\mathcal{F}$ at $m \in M$ is the tangent space $\mathcal{T}_m L_m$ to the leaf L_m of $\mathcal{F}$ through m, and the *normal space* $\mathcal{N}_m\mathcal{F}$ at $m \in M$ is $\mathcal{T}_m M/\mathcal{T}_m\mathcal{F}$. *Cotangent space* $\mathcal{T}_m^*\mathcal{F}$ and *conormal space* $\mathcal{N}_m^*\mathcal{F}$ at m are defined as dual spaces to the tangent space and the normal space at m. Clearly, $\mathcal{N}_m^*\mathcal{F}$ can be identified with the orthogonal complement $(\mathcal{T}_m\mathcal{F})^\perp$ to $\mathcal{T}_m\mathcal{F} \subset \mathcal{T}_m M$ in $\mathcal{T}_m^* M$.

Definition 1.2. Say that a prefoliation $\mathcal{F}$ is a *foliation* if every point $m \in M$ has a neighborhood U such that $\mathcal{F}|_U$ is diffeomorphic to a direct prefoliation.

Obviously, tangent, cotangent, normal and conormal spaces to a foliation form vector bundles over M, and $\mathcal{TF} \subset \mathcal{T}M$, $\mathcal{N}^*\mathcal{F} \subset \mathcal{T}^*M$ are vector subbundles.

Definition 1.3. A *web* $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ of codimension r on a manifold M is a family of foliations of codimension r on M, one foliation $\mathcal{F}_\lambda$ per each $\lambda \in \Lambda$. Say that a web is *smooth* if Λ is a manifold, and the vector subbundle $\mathcal{N}^*\mathcal{F}_\lambda \subset \mathcal{T}^*M$ depends smoothly on $\lambda \in \Lambda$ (to be more precise, consider $\mathcal{N}^*\mathcal{F}_\lambda$ as a section of the bundle of Grassmannians $\mathrm{Gr}_r(\mathcal{T}^*M)$).

In what follows we use the shortcut $\mathcal{F}_\bullet$ for $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ when we are not interested in the set Λ of parameters of the web.

Definition 1.4. Say that a web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on M is *weakly separating* if for any two points $m_1, m_2 \in M$ there is $\lambda \in \Lambda$ such that m_1 and m_2 are on different leaves of $\mathcal{F}_\lambda$. Say that a web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on M is *weakly separating near* $m \in M$ if $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}|_U$ is weakly separating for an appropriate neighborhood $U \ni m$.

Say that a web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ is *separating* at m if for any tangent vector $v \in \mathcal{T}_m M$, $v \neq 0$, there is $\lambda \in \Lambda$ such that $v \notin \mathcal{T}_m \mathcal{F}_\lambda$.

2. 3-dimensional Veronese webs

Recall the definition of a Veronese web ([4]).

Definition 2.1. Given a web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on M of codimension r, and a point $m \in M$, let $\boldsymbol{n}_m(\lambda) \subset \mathcal{T}_m^* M$, $\lambda \in \Lambda$, be the normal subspace at m to the leaf L of $\mathcal{F}_\lambda$ which passes through m. In the case $r = 1$ one can consider $\boldsymbol{n}_m$ as a mapping from Λ to the projectivization $\mathbb{P}\mathcal{T}_m^* M$ of $\mathcal{T}_m^* M$.

Definition 2.2. A *Veronese web* on a manifold M is a smooth separating web $\{\mathcal{F}_\lambda\}_{\lambda\in\mathbb{P}^1}$ on M of codimension 1, such that for any point $m \in M$, $\boldsymbol{n}_m$ is a regular mapping $\mathbb{P}^1 \to \mathbb{P}\mathcal{T}_m^* M$ of degree $d = \dim M - 1$.

Remark 2.3. The condition that $\mathcal{F}_\bullet$ is separating is equivalent to $\operatorname{Im} \boldsymbol{n}_m$ being not contained in any proper projective subspace of $\mathbb{P}\mathcal{T}_m^* M$. Recall that all regular mappings $\nu\colon \mathbb{P}^1 \to \mathbb{P}^d$ of degree d which satisfy this property differ only by a projective transformation of $\mathbb{P}^d$. Moreover, the projective transformation $T\colon \mathbb{P}^d \to \mathbb{P}^d$ such that $T \circ \nu_1 = \nu_2$ is uniquely defined if ν_1 and ν_2 are two such mappings. A convenient model of such a mapping is given by $(x : y) \mapsto \left(x^d : x^{d-1} y : \dots : x y^{d-1} : y^d\right)$.

These curves are *Veronese curves* in the terminology of [4], or *rational normal curves* in the terminology of algebraic geometry. The name *Veronese web* suggests relationship with Veronese curves; in turn, the name *Veronese curve* was introduce in recognition of the fact that the *Veronese surface* $\mathbb{P}^2 \to \mathbb{P}^5$ has the same property: any deformation of it differs by a fraction-linear transformation $\mathbb{P}^5 \to \mathbb{P}^5$ only.

Restrict our attention to the particular case of 3-dimensional Veronese webs. In this case the only requirement on the family $\{\mathcal{F}_\lambda\}_{\lambda\in\mathbb{P}^1}$ is that for any $m \in M$ the points $\boldsymbol{n}_m(\lambda)$, $\lambda \in \mathbb{P}^1$ form a smooth (parameterized) quadric in the two-dimensional projective plane $\mathbb{P}\mathcal{T}_m^* M$. Here the parameterization differs from the parameterization given by any stereographic projection by a fraction-linear transformation $\mathbb{P}^1 \to \mathbb{P}^1$ only. In what follows we consider such parameterizations of quadrics only (any smooth parameterization is such in the complex-geometry case).

Lemma 2.4. *A parameterized quadric $\gamma\colon \mathbb{P}^1 \to \mathbb{P}^2$ is uniquely determined by $\gamma(\lambda_i)$, $i = 1, 2, 3, 4$. Here $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is an arbitrary set of 4 points on $\mathbb{P}^1$.*

For any 4 points $P_i \in \mathbb{P}^1$, $i = 1, 2, 3, 4$, on $\mathbb{P}^2$ such that no 3 of these points are on the same line one can find a parameterized quadric $\gamma\colon \mathbb{P}^1 \to \mathbb{P}^2$ such that $P_i = \gamma(\lambda_i)$, $i = 1, 2, 3, 4$.

Proof. Recall that given a point $p \in \mathbb{P}^N$, one can consider a projection π_p with the center at p, which sends $\mathbb{P}^N \smallsetminus \{p\}$ onto a projective space $\mathbb{P}\mathcal{T}_p\mathbb{P}^N$ of tangent directions at p. Here $\pi_p(q)$ is the direction of the line (pq).

Consider compositions $\pi_{P_i} \circ \gamma\colon \mathbb{P}^1 \to \mathbb{P}\mathcal{T}_p\mathbb{P}^2 \simeq \mathbb{P}^1$, $i = 1, 2, 3, 4$. Since $\pi_{P_i}|_{\operatorname{Im}\gamma}$ is a stereographic projection, these compositions are fraction-linear mappings between projective lines. Thus they are determined by images of any 3 distinct points on $\mathbb{P}^1$. Thus the line $(P_i\gamma(\lambda))$ is uniquely determined by $P_{1,2,3,4}$. Since $\gamma(\lambda) = (P_1\gamma(\lambda)) \cap (P_2\gamma(\lambda))$ if $\lambda \neq \lambda_{1,2}$, γ is uniquely determined by $P_{1,2,3,4}$.

To show the existence take any parameterized quadric $\widetilde{\gamma}\colon \mathbb{P}^1 \to \mathbb{P}^2$, let $\widetilde{P}_i = \widetilde{\gamma}(\lambda_i)$, $i = 1, 2, 3, 4$. Then no 3 points out of $\widetilde{P}_1, \dots, \widetilde{P}_4$ are on the same line, thus there is a projective mapping $T\colon \mathbb{P}^2 \to \mathbb{P}^2$ such that $\xi\left(\widetilde{P}_i\right) = P_i$, $i = 1, 2, 3, 4$. Then $T \circ \widetilde{\gamma}$ is the parameterized quadric we need. ☐

Corollary 2.5. *Consider a manifold M, $\dim M = 3$. A Veronese web $\mathcal{F}_\lambda$ on M can be reconstructed given 4 foliations $\mathcal{F}_{\lambda_i}$, $i = 1, 2, 3, 4$, on M. Here $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is an arbitrary set of 4 points on $\mathbb{P}^1$.*

Proof. Since $\boldsymbol{n}_m(\lambda_i)$, $i = 1, 2, 3, 4$, are known for any $m \in M$, by Lemma 2.4 one can find $\boldsymbol{n}_m(\lambda)$ for any $\lambda \in \mathbb{P}^1$ and $m \in M$. This uniquely determines $\mathcal{F}_\lambda$ for any $\lambda \in \mathbb{P}^1$. ☐

Fix 4 points $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subset \mathbb{P}^1$. Given a Veronese web on M and a point $m_0 \in M$, consider a small neighborhood U of m_0 in M. One may assume that in U the foliations $\mathcal{F}_{\lambda_i}$, $i = 1, 2, 3, 4$ can be written by equations $x = \text{const}$, $y = \text{const}$, $z = \text{const}$, $W = \text{const}$; here x, y, z, W are functions on U. Moreover, $dx|_{m_0}$, $dy|_{m_0}$ and $dz|_{m_0}$ are linearly independent. Indeed, the directions of these 3 vectors are 3 distinct points on a quadric in the projective plane, thus are not on the same line.

Consider x, y, z as 3 components of a vector-function $\varphi\colon U \to \mathbb{V}^3$, let $V = \varphi(U)$. We know that the derivative of this function at $m_0 \in M$ is non-degenerate, thus decreasing U we may assume that φ gives a diffeomorphism $U \to V$. Then $w = W \circ \varphi^{-1}$ is a function on V, and $W(m) = w(x(m), y(m), z(m))$ if $m \in U$.

Lemma 2.6. *The scalar function w on $V \subset \mathbb{V}^3$ and $\varphi(m_0) \in V$ uniquely determine the Veronese web $\mathcal{F}_\lambda$ up to a local diffeomorphism near $m_0 \in M$.*

Proof. Instead of determining a web up to a local diffeomorphism near $m_0 \in M$ it is enough to uniquely determine the diffeomorphic image $\varphi_*(\mathcal{F}_\bullet)$ of this web, which is a web on a neighborhood of $\varphi(m_0) \in \mathbb{V}^3$. By Corollary 2.5 it is enough to determine $\varphi_*(\mathcal{F}_{\lambda_i})$, $i = 1, 2, 3, 4$. However, leaves of $\varphi_*(\mathcal{F}_{\lambda_1})$, $\varphi_*(\mathcal{F}_{\lambda_2})$, $\varphi_*(\mathcal{F}_{\lambda_3})$ are given by equations $x = \text{const}$, $y = \text{const}$, $z = \text{const}$; here (x, y, z) is the standard coordinate system on $\mathbb{V}^3$. Similarly, leaves of $\varphi_*(\mathcal{F}_{\lambda_4})$ are given by the equation $w(x, y, z) = \text{const}$. ☐

A change of equations x, y, z of foliations $\mathcal{F}_{\lambda_i}$, $i = 1, 2, 3$, to $x + C_1$, $y + C_2$, $z + C_3$ corresponds to a translation of V and w by (C_1, C_2, C_3), thus one may assume that $\varphi(m_0) = (0, 0, 0)$. Similarly, one may assume that $w(0, 0, 0) = 0$.

3. Nonlinear wave equation as an integrability condition

Fix a set of 4 points $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subset \mathbb{P}^1$.

Definition 3.1. Say that a function w on an open subset $M \subset \mathbb{V}^3$ is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-*admissible* if there is a Veronese web $\mathcal{F}_\lambda$ on M such that foliations $\mathcal{F}_{\lambda_i}$, $i = 1, 2, 3, 4$, are given by equations $x = \text{const}$, $y = \text{const}$, $z = \text{const}$, $w(x, y, z) = \text{const}$; here (x, y, z) is the standard coordinate system on $\mathbb{V}^3$.

First of all, if w is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-admissible, Lemma 2.4 implies that for any point $m \in M$ the directions $dx|_m$, $dy|_m$, $dz|_m$ and $dw|_m$ are in general position. In other words, $w_x \neq 0$, $w_y \neq 0$, $w_z \neq 0$ everywhere in M. Thus w is *non-degenerate* (as defined in Section 0).

Given non-degeneracy of w, for any $\lambda \in \mathbb{P}^1$ and $m \in M$ the construction of the proof of Corollary 2.5 gives a direction $\boldsymbol{n}_m(\lambda)$ in the projectivization of $(\mathbb{V}^3)^*$. If w is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-admissible, then $m \mapsto \boldsymbol{n}_m(\lambda)$ coincides with the field of normal directions of the foliation $\mathcal{F}_\lambda$.

Obviously,

Lemma 3.2. *Consider a non-degenerate function w defined on $M \subset \mathbb{V}^3$. Suppose that for any $\lambda \in \mathbb{P}^1$ the direction field $\boldsymbol{n}_m(\lambda)$, $m \in M$, given by the construction of the proof of Corollary 2.5 coincides with the field of normal directions of a foliation on M. Then w is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-admissible.*

Thus to check whether a non-degenerate function w is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-admissible it is enough to check whether a given direction field coincides with a normal field to a foliation. Such direction fields can be described by the following particular case of the *Frobenius integrability condition* [12]:

Lemma 3.3. *Consider a* 1-form *ω on a manifold M which does not vanish at any point of M. Call ω* Frobenius integrable *if there exists a foliation $\mathcal{F}$ of codimension* 1 *on M such that $\omega(m)$ is normal to the tangent space at m to the leaf L_m of $\mathcal{F}$ through m for any $m \in M$.*

Then ω is Frobenius integrable *iff $\omega \wedge d\omega = 0$.*

Proof. The "only if" part is simple: in an appropriate neighborhood U of any given point $m_0 \in M$ the foliation $\mathcal{F}|_U$ can be written as $g = \text{const}$; here g is a function on U, and $dg \neq 0$ for any $m \in U$. Thus $\omega = h\,dg$ for an appropriate function h on U, and $\omega \wedge d\omega = h\,dg \wedge dh \wedge dg = 0$.

For the "if" part it is enough to show the existence locally on M, since the foliation is unique if it exists, thus gluing pieces together is not a problem. We may assume that M is an open subset of $\mathbb{V}^n$, and that $\omega|_{m_0} = dx_n|_{m_0}$. Say that a tangent vector v at $m \in M$ is *k-compatible*, $k = 1, \dots, n-1$, if $\langle \omega|_m, v \rangle = 0$ and v is of the form $\frac{\partial}{\partial x_k} + a \frac{\partial}{\partial x_n}$ with an appropriate number a. Obviously, in an appropriate neighborhood of any point $m_0 \in M$ there is exactly one k-compatible vector $v_k(m)$ for $k = 1, \dots, n-1$. Define functions $a_{(k)}(m)$ by $v_k(m) = \frac{\partial}{\partial x_k} + a_{(k)}(m) \frac{\partial}{\partial x_n}$. Then the fundamental relationship between commutator and de Rham differential[1] [12]

$$\langle \omega, [v_k, v_l] \rangle = v_k \cdot \langle \omega, v_l \rangle - v_l \cdot \langle \omega, v_k \rangle + \langle d\omega, v_k \wedge v_l \rangle$$

implies $\langle \omega, [v_k, v_l] \rangle = \langle d\omega, v_k \wedge v_l \rangle$. Since $\omega \wedge d\omega = 0$, one can write $d\omega = \omega \wedge \alpha$; here α is a 1-form defined near m_0. Hence

$$\langle d\omega, v_k \wedge v_l \rangle = \langle \omega, v_k \rangle \langle \alpha, v_l \rangle - \langle \alpha, v_k \rangle \langle \omega, v_l \rangle = 0.$$

Thus $\langle \omega, [v_k, v_l] \rangle = 0$. On the other hand, $[v_k, v_l] = (v_k \cdot a_l - v_l \cdot a_k) \frac{\partial}{\partial x_n}$. Together with $\langle \omega, [v_k, v_l] \rangle = 0$ this implies $[v_k, v_l] = 0$. By the principal theorem of the theory of ODE, one can find local coordinates $(y_1, \dots, y_n)$ such that $v_k = \frac{\partial}{\partial y_k}$, $k = 1, \dots, n-1$. Since ω is orthogonal to v_k, $k = 1, \dots, n-1$, this implies that $\omega = h(y)\,dy_n$, thus $y_n = \text{const}$ gives a foliation with the required properties. ∎

The next step is to provide an explicit construction of the normal directions $\boldsymbol{n}_m(\lambda)$ in terms of w.

Lemma 3.4. *Given a Veronese curve $\gamma(\lambda)$ in $\mathbb{P}^{n-1}$, one can find polynomials $p_1(\lambda), \dots, p_n(\lambda)$ of degree $\leq n-1$ such that $\gamma(\lambda) = (p_1(\lambda) : \dots : p_n(\lambda))$ for $\lambda \neq \infty$. Polynomials $p_k(\lambda)$ are defined uniquely up to multiplication by the same constant.*

Proof. Any Veronese curve in $\mathbb{P}^{n-1}$ is a projective transformation of the closure of the image of the mapping $\lambda \mapsto (1 : \lambda : \dots : \lambda^{n-1})$. A consideration of the corresponding linear transformation of $\mathbb{V}^n$ provides polynomials $p_1, \dots, p_n$.

It is enough to show uniqueness for the curve $(1 : \lambda : \dots : \lambda^{n-1})$. Obviously, $p_k(\lambda) = \lambda^{k-1} p_1(\lambda)$. Moreover, since $\deg p_n \leq n-1$, $p_1(\lambda)$ is a constant. ∎

Thus any Veronese curve in $\mathbb{P}^2$ is a projectivization of a polynomial vector-function $v(\lambda)$ of degree *exactly* 2. Note that $v(\lambda) \neq 0$ for any λ.

This implies that the dependence on λ of the directions $\boldsymbol{n}_m(\lambda)$, $\lambda \neq \infty$, can be described by the direction of the 1-form $\alpha(m) + \lambda\beta(m) + \lambda^2\gamma(m)$; here α, β, γ are appropriate 1-forms on $M \subset \mathbb{V}^3$ which are defined up to multiplication by the same function on M. If $\lambda \neq 0$, $\boldsymbol{n}_m(\lambda)$ is the direction of $\gamma(m) + \lambda^{-1}\beta(m) + \lambda^{-2}\alpha(\lambda)$, taking the limit $\lambda \to \infty$ implies that $\boldsymbol{n}_m(\infty)$ is the direction of $\gamma(m)$.

Lemma 3.5. *Consider vectors v_1, v_2, v_3, v_4 in $\mathbb{V}^3$ such that v_1, v_2, v_3 are linearly independent. Fix a set of 4 points $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subset \mathbb{P}^1 \smallsetminus \{\infty\}$. There is a unique polynomial vector-function $v(\lambda)$ of degree 2 such that $v(\lambda_4) = v_4$, and $v(\lambda_k)$ is proportional to v_k, $k = 1, 2, 3$.*

Proof. Write v_4 as $av_1 + bv_2 + cv_3$. Since $v(\lambda)$ can be written as $\alpha(\lambda)v_1 + \beta(\lambda)v_2 + \gamma(\lambda)v_3$, we know that $\alpha(\lambda_2) = \alpha(\lambda_3) = 0$, $\alpha(\lambda_4) = a$. This uniquely determines the quadratic polynomial $\alpha(\lambda)$. Proceed similarly for $\beta(\lambda)$ and $\gamma(\lambda)$. ∎

[1] One can easily check this relation in local coordinates.

Corollary 3.6. *Fix a set of 4 points $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subset \mathbb{P}^1 \smallsetminus \{\infty\}$. Given a non-degenerate function w on $M \subset \mathbb{V}^3$, the direction $\boldsymbol{n}_m(\lambda)$ defined by the construction of the proof of Corollary 2.5 coincides with $(p_1(\lambda) w_x : p_2(\lambda) w_y : p_3(\lambda) w_z)$; here*

$$p_i(\lambda) = (\lambda_4 - \lambda_i)(\lambda - \lambda_j)(\lambda - \lambda_k), \tag{3.1}$$

for any permutation (ijk) of (123).

Corollary 3.7. *Consider distinct points $\lambda_i \neq \infty$, $i = 1, 2, 3, 4$. Consider a non-degenerate function w on $M \subset \mathbb{V}^3$. Let*

$$\omega_\lambda \stackrel{\text{def}}{=} p_1(\lambda) w_x dx + p_2(\lambda) w_y dy + p_3(\lambda) w_z dz; \tag{3.2}$$

here $p_{1,2,3}(\lambda)$ are from (3.1). Then the following conditions are equivalent:

1. *w is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-admissible;*
2. *$\omega_\lambda \wedge d\omega_\lambda = 0$ for any λ;*
3. *$\omega_\lambda \wedge d\omega_\lambda = 0$ for any 5 distinct values of λ;*
4. *$\omega_\lambda \wedge d\omega_\lambda = 0$ for any $\lambda_0 \notin \{\lambda_1, \dots, \lambda_4\}$;*

If $0 \notin \{\lambda_1, \dots, \lambda_4\}$, these conditions are equivalent to

$$\nu_{23} w_x w_{yz} + \nu_{31} w_y w_{xz} + \nu_{12} w_z w_{xy} = 0, \tag{3.3}$$

here $\nu_{kl} = \lambda_k / (\lambda_4 - \lambda_k) - \lambda_l / (\lambda_4 - \lambda_l)$.

Proof. Obviously, $\omega_\lambda|_m \neq 0$ for any $\lambda \neq \infty$ and any $m \in M$. By Lemma 3.3, $\omega_\lambda \wedge d\omega_\lambda = 0$ is equivalent to existence of a foliation to which ω_λ is normal. Thus by Lemma 3.2 the first statement implies the second one.

If $\omega_\lambda \wedge d\omega_\lambda = 0$ for any λ, then by Corollary 3.6, the required in Lemma 3.2 foliation exists for $\lambda \neq \infty$. However, $\widetilde{\omega}_\lambda = \lambda^{-2}\omega_\lambda$ is defined for $\lambda \in \mathbb{P}^1 \smallsetminus \{0\}$, and $\widetilde{\omega}_\lambda \wedge d\widetilde{\omega}_\lambda$ is a polynomial of degree 4 in λ^{-1}. Thus $\widetilde{\omega}_\lambda \wedge d\widetilde{\omega}_\lambda = 0$, *including* $\lambda = \infty$. Moreover, $\widetilde{\omega}_\infty|_m \neq 0$ for any m, which implies the existence of $\mathcal{F}_\lambda$ for $\lambda = \infty$ as well. Thus the second statement implies the first one.

Since ω_λ is quadratic in λ, $\omega_\lambda \wedge d\omega_\lambda$ is a polynomial of degree 4 in λ. Thus the second statement is equivalent to the third one. By construction $\omega_{\lambda_{1,2,3,4}}$ are proportional to dx, dy, dz, and dw correspondingly. This implies that $\omega_\lambda \wedge d\omega_\lambda = 0$ for $\lambda \in \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Consequently, the fourth statement is equivalent to the third one.

Assume that $\lambda_0 = 0 \notin \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Let $\mu_k = \lambda_4/\lambda_k - 1$, $k = 1, 2, 3$. Then

$$\omega_0 = \lambda_1\lambda_2\lambda_3\widetilde{\omega}, \qquad \widetilde{\omega} \stackrel{\text{def}}{=} \mu_1 w_x dx + \mu_2 w_y dy + \mu_3 w_z dz,$$

and $\widetilde{\omega} \wedge d\widetilde{\omega}$ can be written as

$$\mu_1\mu_2\mu_3 \left(\left(\mu_2^{-1} - \mu_3^{-1}\right) w_x w_{yz} + \left(\mu_3^{-1} - \mu_1^{-1}\right) w_y w_{xz} + \left(\mu_1^{-1} - \mu_2^{-1}\right) w_z w_{xy} \right) dx \wedge dy \wedge dz.$$

(It is clear that $\mu_k \neq 0$ for $k = 1, 2, 3$.) Since $\nu_{kl} = \mu_k^{-1} - \mu_l^{-1}$, the equation $\omega_0 \wedge d\omega_0 = 0$ is proportional to (3.3), which implies the last statement of the corollary. □

Obviously, $\omega_\lambda \wedge d\omega_\lambda = \alpha \prod_{k=1}^4 (\lambda - \lambda_k)$; here α is a 3-form on M which does not depend on λ. Thus the equations $\omega_{\lambda_0} \wedge d\omega_{\lambda_0} = 0$ for different values λ_0 are proportional, and it does not matter much which value of λ_0 one would use. Consequently, any other choice of λ_0 would lead to an equation which is proportional to (3.3), and one can drop the conditions that $0 \notin \{\lambda_1, \dots, \lambda_4\}$. Moreover, it is possible to drop the condition $\infty \notin \{\lambda_1, \dots, \lambda_4\}$ as well:

Theorem 3.8. *A non-degenerate function w on $M \subset \mathbb{V}^3$ is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-admissible iff it satisfies an (A, B, C)-equation (0.2) with $-A/C = (\lambda_1 : \lambda_2 : \lambda_3 : \lambda_4)$; here $(a : b : c : d) = \frac{d-a}{d-c}\frac{b-c}{b-a}$ is the cross-ratio of a, b, c, d.*

Proof. Indeed, a direct calculation shows that $\nu_{12} + \nu_{23} + \nu_{31} = 0$, $\nu_{12} \neq 0$, $\nu_{23} \neq 0$, $\nu_{31} \neq 0$, and $-\nu_{23}/\nu_{12} = (\lambda_1 : \lambda_2 : \lambda_3 : \lambda_4)$. Thus the statement holds for $\infty \notin \{\lambda_1, \dots, \lambda_4\}$. However, if T is a projective transformation, then w is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-admissible iff it is $(T\lambda_1, T\lambda_2, T\lambda_3, T\lambda_4)$-admissible. Since cross-ratio is invariant w.r.t. projective transformations, it is enough to prove the statement for $(T\lambda_1, T\lambda_2, T\lambda_3, T\lambda_4)$ with an arbitrary T. By an appropriate choice of T we can ensure that $\infty \notin \{\lambda_1, \dots, \lambda_4\}$ (and additionally $0 \notin \{\lambda_1, \dots, \lambda_4\}$ if we wish). □

Remark 3.9. Since the cross-ratio of 4 distinct points can take any value distinct from $0, 1, \infty$, one can momentarily see that for any triple (A, B, C) which satisfies (0.1) and for any 3 distinct points $\lambda_1, \lambda_2, \lambda_3$ one can find λ_4 such that $-A/C = (\lambda_1 : \lambda_2 : \lambda_3 : \lambda_4)$. Thus any (A, B, C)-equation can be interpreted as an integrability condition of a Veronese web: any Veronese web gives rise to a non-degenerate solution of such an equation, and any non-degenerate solution can be represented in this form.

Remark 3.10. One can generalize Corollary 3.7 to the case of Veronese webs of arbitrary dimension. In dimension d one still needs one function w of d variables to completely determine a web up to a local diffeomorphism. The foliation $\mathcal{F}_\lambda$ can be described by a 1-form ω_λ which is normal to leaves of $\mathcal{F}_\lambda$, and is given by a formula similar to (3.2).

The 1-form ω_λ depends on λ as a polynomial of degree $d-1$, and the integrability condition $\omega_\lambda \wedge d\omega_\lambda = 0$ is a polynomial of degree $2d-2$. Thus a non-degenerate function w of d variables corresponds to a Veronese web iff $\omega_\lambda \wedge d\omega_\lambda = 0$ for $2d-1$ different values of λ. By its construction, $\omega_\lambda \wedge d\omega_\lambda = 0$ automatically holds for $d+1$ value of λ. Thus a naive generalization (as done in [4]) of Corollary 3.7 would be that it is enough to require $\omega_\lambda \wedge d\omega_\lambda = 0$ at $d-2$ "additional" values of λ.

However, [9, 10] contain a much stronger result: if $\omega_\lambda \wedge d\omega_\lambda = 0$ for any "additional" value of λ, then $\omega_\lambda \wedge d\omega_\lambda = 0$ for any λ, thus w determines a Veronese web. Unfortunately, this condition is on a 3-form in d-dimensional space, thus it is still an overdetermined system of partial differential equations on w, if $d > 3$. It is very interesting to investigate whether arguments of [13, 14] allow extraction of one equation on w which implies $\omega_\lambda \wedge d\omega_\lambda = 0$.

4. Bäcklund–Darboux transformations

By Remark 3.9, any solution of an (A, B, C)-equation gives rise to a Veronese web, which in turn leads to a solution of (A', B', C')-equation, possibly with different (A', B', C').

Corollary 4.1. *Let w be a non-degenerate solution of (A, B, C)-equation in a neighborhood of $(0, 0, 0)$, $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ be numbers such that $-A/C = (\lambda_1 : \lambda_2 : \lambda_3 : \lambda_4)$. Then for any number λ*

1. *there is a function $v(x, y, z)$ defined in a neighborhood of $(0, 0, 0)$ such that the following identity of vector-functions holds:*

$$(v_x, v_y, v_z) = \psi(x, y, z)(\alpha w_x, \beta w_y, \gamma w_z); \tag{4.1}$$

   here $\alpha, \beta, \gamma \alpha = p_1(\lambda)$, $\beta = p_2(\lambda)$, $\gamma = p_3(\lambda)$, p_k are polynomials given by (3.1), and ψ is an appropriate scalar-valued function;
2. *if $\lambda \notin \{\lambda_1, \lambda_2, \lambda_3\}$, then $v(x, y, z)$ can be chosen to be non-degenerate;*
3. *if v is non-degenerate it is $(\lambda_1, \lambda_2, \lambda_3, \lambda)$-admissible;*
4. *if $\lambda \notin \{\lambda_1, \lambda_2, \lambda_3\}$, and $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$ satisfy conditions (0.1), and $-\widetilde{A}/\widetilde{C} = (\lambda_1 : \lambda_2 : \lambda_3 : \lambda)$, then the function $v(x, y, z)$ satisfies $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$-equation.*

Proof. By Corollary 3.7, w is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$-admissible, thus it corresponds to a web $\mathcal{F}_\bullet$. Write the leaves of $\mathcal{F}_\lambda$ as $v(x, y, z) = \text{const}$, and apply Corollary 3.7 again. □

Obviously, the function v of the previous corollary is defined uniquely up to a gauge transformation (see Definition 0.2).

Let us find relationships between 9 constants (A, B, C), $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$ and (α, β, γ) which appear in the statements of this section. Construct $\widetilde{\nu}_{kl}$ basing on the 4-tuple $(\lambda_1, \lambda_2, \lambda_3, \lambda)$ using the same formula as used to construct ν_{kl} basing on $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Let

$$\tau = (\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)\frac{\lambda}{\lambda_4}.$$

Then it is easy to check that $\alpha = \tau\nu_{23}/\widetilde{\nu}_{23}$, $\beta = \tau\nu_{13}/\widetilde{\nu}_{13}$, $\gamma = \tau\nu_{12}/\widetilde{\nu}_{12}$. Since simultaneous multiplication of α, β, γ by the same non-zero number does not change the meaning of Equation (4.1), we conclude that one can take $\alpha = A/\widetilde{A}$, $\beta = B/\widetilde{B}$, $\gamma = C/\widetilde{C}$.

Proof of Theorem 0.3 . The first statement is obvious, and the second one is the corollary of the first since $dv_1 \sim dv$ if v_1 is a gauge transform of v. The third and the fourth statements are reformulations of parts of Corollary 3.7. The last statement is a direct corollary of the first one and of the following obvious statement:

Lemma 4.2. *Given $(v_x, v_y, v_z) \sim \left(v'_x, v'_y, v'_z\right)$ for two non-degenerate functions v and v' defined in a neighborhood of (0,0,0) in $\mathbb{V}^3$, one can decrease the neighborhood so that the functions become gauge transforms of each other.*

This finishes the proof of Theorem 0.3. □

To enhance the statements about Equation (0.5), note the following two lemmas.

Lemma 4.3. *Given numbers $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$ such that $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$, there exist two triples (A, B, C) and $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$ which both satisfy conditions (0.1), and $\alpha = A/\widetilde{A}$, $\beta = B/\widetilde{B}$, $\gamma = C/\widetilde{C}$. The numbers $A, B, C, \widetilde{A}, \widetilde{B}, \widetilde{C}$ are defined uniquely up to multiplication by the same constant.*

Proof. Given $\widetilde{A}, \widetilde{B}, \widetilde{C}$ put $A = \alpha\widetilde{A}$, $B = \beta\widetilde{B}$, $C = \gamma\widetilde{C}$. The conditions (0.1) on (A, B, C) can be translated to an additional linear equation $\alpha\widetilde{A} + \beta\widetilde{B} + \gamma\widetilde{C} = 0$ on $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$. This equation is independent of $\widetilde{A} + \widetilde{B} + \widetilde{C} = 0$, thus there is a unique (up to proportionality) solution $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$ of these two equations. What remains to check is that this solution does not contradict the conditions $\widetilde{A} \neq 0$, $\widetilde{B} \neq 0$, $\widetilde{C} \neq 0$. However, $\widetilde{A} = 0$ contradicts $\beta \neq \gamma$, etc. □

Lemma 4.4. *Given two triples (A, B, C) and $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$ which both satisfy conditions (0.1), put $\alpha = A/\widetilde{A}$, $\beta = B/\widetilde{B}$, $\gamma = C/\widetilde{C}$. Then either $\alpha = \beta = \gamma$, or $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$.*

This statement is elementary.

5. Inverse construction

Of course, Theorem 0.3 can be proven by elementary methods without any reference to Veronese webs. However, Veronese webs are not useful because this theorem can be proven "naturally" by using Veronese webs. In fact Veronese webs appears naturally as reformulations of the *statement* of this theorem.

Indeed, given a solution of Equation (0.2), consider Systems (0.4) for all possible triples $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$. Since proportional triples $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$ give essentially the same systems, we can enumerate all the triples by the ratio $\lambda = -\widetilde{A}/\widetilde{C}$, which can be considered as an element of $\mathbb{P}^1$ with the only restrictions being $\lambda \neq \infty$, $\lambda \neq 0$, $\lambda \neq 1$.

For any such value of λ one obtains a solution $v^{[\lambda]}$ of Equation (0.3). This solution is defined in a neighborhood U_λ of (0,0,0), and it is easy to show that this neighborhood may be chosen independently of λ, denote it by U. The solution $v^{[\lambda]}$ is not unique, but the foliation $\mathcal{F}_\lambda$ of U defined by $v^{[\lambda]} = \text{const}$ is uniquely defined. Moreover, $\mathcal{F}_\lambda$ depends smoothly on $\lambda \in \mathbb{P}^1 \smallsetminus \{0, 1, \infty\}$. What remains it to consider what happens near $\lambda = 0$, near $\lambda = 1$, and near $\lambda = \infty$.

If $\lambda \approx 0$, then $\widetilde{A}$ is very small, thus Equation (0.5)

$$(v_x, v_y, v_z) \sim \left(Aw_x, \widetilde{A}B\widetilde{B}^{-1}w_y, \widetilde{A}C\widetilde{C}^{-1}w_z\right)$$

becomes close to $(v_x, v_y, v_z) \sim (Aw_x, 0, 0)$, or, in other words, to $(v_x, v_y, v_z) \sim (1,0,0)$. The solution to this equation is $v = v(x)$, thus the foliation $\mathcal{F}_\lambda$ has a limit $x = \text{const}$ when $\lambda \to 0$. Similarly, the limit when $\lambda \to 1$ is $y = \text{const}$, when $\lambda \to \infty$ is $z = \text{const}$.

Thus an investigation of the statement of Theorem 0.3 directly leads to a family a foliations which depend smoothly on a parameter $\lambda \in \mathbb{P}^1$. In the following section we show that the conditions that the normal directions to the foliations span a quadratic cone is also related to the elementary theory of Equation (0.2).

Additionally, the following statement is easy to obtain elementary, but it is an immediate corollary of Theorem 0.3:

Corollary 5.1. *If w is a non-degenerate solution of Equation* (0.2), *then any gauge transform of w is also a solution of Equation* (0.2).

6. Linearization

Given a solution $\bar{\kappa}$ of a non-linear (system of) equation(s) $F(\kappa) = 0$, the *linearized* equation at $\bar{\kappa}$ is the equation $F(\bar{\kappa} + \varepsilon\kappa) = O(\varepsilon^2)$. It is a (system of) linear equation(s) on κ with the coefficients being partial derivatives of F at $\bar{\kappa}$.

Obviously, given a solution $\bar{w}$ of Equation (0.2), the linearization is

$$A\bar{w}_x w_{yz} + B\bar{w}_y w_{xz} + C\bar{w}_z w_{xy} + A\bar{w}_{yz} w_x + B\bar{w}_{xz} w_y + C\bar{w}_{xy} w_z = 0, \tag{6.1}$$

The left-hand side is a linear differential operator of second order in w, denote this operator $l_{\bar{w}}$ or just l. The principal symbol of $l_{\bar{w}}$ is

$$\Lambda(x, y, z, \xi, \eta, \zeta) = A\bar{w}_x\eta\zeta + B\bar{w}_y\xi\zeta + C\bar{w}_z\xi\eta. \tag{6.2}$$

This is a non-degenerate quadratic form in (ξ, η, ζ) iff $\bar{w}$ is non-degenerate. Moreover, it vanishes if $(\xi, \eta, \zeta) = (1, 0, 0)$, or $(\xi, \eta, \zeta) = (0, 0, 1)$, or $(\xi, \eta, \zeta) = (0, 0, 1)$. This shows that the linearization is hyperbolic iff $\bar{w}$ is non-degenerate. *This* is why it makes sense to call the equation (0.2) a *nonlinear wave equation.*

Fix a point (x, y, z). Recall that a covector (ξ, η, ζ) at (x, y, z) is *characteristic* if $\Lambda(x, y, z, \xi, \eta, \zeta) = 0$. Characteristic covectors of a hyperbolic linear differential equation form a cone in the cotangent space, this cone is called a *wave cone.* Since our equation is of second order, it is a quadratic cone. Recall that a surface in $\mathbb{V}^3$ is called *characteristic* if the normal direction to this surface at any point is characteristic. One can define similar notions for square systems of equations by taking $\det \Lambda$ instead of Λ.

Recall how to construct characteristic surfaces. Consider an expression $l\left(e^{ik\varphi(x,y,z)}\right)$ when $k \to \infty$. It can be written as $\Phi_\varphi(k, x, y, z)\, e^{ik\varphi(x,y,z)}$; here Φ_φ depends polynomially on k, the degree being 2 or less. Say that φ is an *eikonal* solution if Φ_φ is a

polynomial in k of degree ≤ 1. If $\Phi_{\varphi,2}(x, y, z)$ is the coefficient at k^2 in Φ_φ, then the equation

$$\Phi_{\varphi,2}(x, y, z) = 0$$

is a non-linear differential equation of the first order on φ. Call this equation the *eikonal equation.*

Obviously, eikonal solutions coincide with solutions to the eikonal equation. Moreover, it is easy to see that the eikonal equation is equivalent to the surfaces $\varphi = c$ being characteristic surfaces for any constant c.

A similar statements holds for square systems of differential equations if one considers $l\left(v \cdot e^{ik\varphi(x,y,z)}\right)$ as a linear function of a vector v. Then Φ_φ becomes a square matrix, and we can consider the degree of $\det \Phi$ in k instead of the degree of Φ in k.

Proposition 6.1. *Consider a non-degenerate solution w of Equation* (0.2). *Then w is also a solution of the linearized equation* (6.1) *at w. Moreover, w is also an eikonal solution for this linearized equation.*

Proof. To prove the first statement, apply Corollary 5.1. Since w is a solution, so is $w + \varepsilon\omega$ for any ε. Similarly, since $w + \varepsilon e^{ikw}$ is a solution for any ε and k, w is an eikonal solution as well. ☐

Proposition 6.2. *Consider a solution (w, v) of System* (0.4) *with non-degenerate w and v. Let $l^{\{1\}}$, $l^{\{2\}}$ be the linearizations of Equations* (0.2), (0.3) *at w, and $l^{\{3\}}$ be the linearization of Equation* (0.4) *at (w, v). Then*

1. *Characteristic cones of $l^{\{1\}}$, $l^{\{2\}}$, $l^{\{3\}}$ coincide.*
2. *The function v is a solution of the eikonal equation for $l^{\{1\}}$.*

Proof. It is easy to check the first claim by a direct calculation. In the second claim we already know that v is a solution of the eikonal equation for $l^{\{2\}}$. Since characteristic cones coincide, v is also a solution of the eikonal equation for $l^{\{1\}}$. ☐

Remark 6.3. Let us provide a more conceptual heuristic proof of the first claim of the proposition. It is enough to consider characteristic cones for $l^{\{1\}}$ and $l^{\{3\}}$. If φ is a solution of the eikonal equation for $l^{\{3\}}$, then $l^{\{3\}}(\widetilde{w}, \widetilde{v}) = O(1)$ when $k \to \infty$; here

$$\widetilde{w}(x, y, z) = W e^{ik\varphi(x,y,z)}, \qquad \widetilde{v}(x, y, z) = V e^{ik\varphi(x,y,z)}$$

and W and V are appropriate constants. The usual arguments of calculus of asymptotics (see, for example, [7]) show that by allowing W and V depend smoothly on x, y, z, k^{-1} one can ensure that $l^{\{3\}}(\widetilde{w}, \widetilde{v})$ is asymptotically 0 when $k \to \infty$.

In other words, starting with a solution of the eikonal equation for $l^{\{3\}}$, one can construct an asymptotic solution for $l^{\{3\}}$. Since the relationship between $l^{\{3\}}$ and $l^{\{1\}}$ is a linearization of relation between System (0.4) and Equation (0.2), we conclude that $W(x, y, z, k^{-1})\, e^{ik\varphi(x,y,z)}$ is an asymptotic solution for $l^{\{1\}}$ (as given this argument is heuristic only, one needs to check that the order of taking limits in k and in ε is correct). Thus φ is also a solution of the eikonal equation for $l^{\{1\}}$. Since the characteristic cone is spanned by differentials of eikonal solutions, the characteristic cone for $l^{\{3\}}$ is a subset of a characteristic cone for $l^{\{1\}}$.

On the other hand, characteristic cones of $l^{\{1\}}$ and $l^{\{3\}}$ are quadratic cones, thus they should coincide.

Remark 6.4. Let us repeat the arguments of Section 5 in the linearized situation. For any $\lambda \in \mathbb{P}^1$ we can construct a corresponding triple $\left(\widetilde{A}, \widetilde{B}, \widetilde{C}\right)$ with $-\widetilde{A}/\widetilde{C} = \lambda$, and a solution $v^{[\lambda]}$ of the corresponding Equation (0.3), thus of $l^{\{2\}}$. Then $v^{[\lambda]}$ is a solution of the eikonal equation for $l^{\{1\}}$. Its level surfaces are characteristic surfaces of $l^{\{1\}}$. For each value of λ we obtain one characteristic surface passing through a given point.

Moreover, when we vary λ the coefficients $A/\widetilde{A}$, $B/\widetilde{B}$, $C/\widetilde{C}$ in Equation (0.5) vary as well. They cannot be proportional for different values of λ, thus all the above characteristic surfaces passing through a given point have different directions.

In other words, at a given point we obtain a family of characteristic directions parameterized by $\mathbb{P}^1$. But characteristic directions span a quadratic cone, and the base of this cone is $\mathbb{P}^1$. It easily follows that given a characteristic direction at a given point one can find a value of $\lambda \in \mathbb{P}^1$ such that $dv^{[\lambda]}$ at the given point goes in the prescribed direction.

This concludes arguments of Section 5, since using elementary arguments we concluded that results of Theorem 0.3 imply that normal directions to $\mathcal{F}_\lambda$ at a given point should span a quadratic cone.

7. $\mathcal{F}_\bullet$-convex sets and the twistor transform

Definition 7.1. Given a foliation $\mathcal{F}$ on M and an open subset $U \subset M$, we say that U is *$\mathcal{F}$-convex* if there is an open subset $V \supset U$ such that $\mathcal{F}|_V$ is direct (as defined in Section 1), and for any leaf L of $\mathcal{F}|_V$ the set $L \cap U$ is connected. Call U *strictly $\mathcal{F}$-convex* if additionally the image of U under the natural projection $U \to \mathfrak{B}_{\mathcal{F}|_V}$ is homeomorphic to a ball.

It is obvious that any point $m \in M$ has a strictly $\mathcal{F}$-convex neighborhood. For example, any direct neighborhood (see Section 1) goes.

Definition 7.2. Given a foliation $\mathcal{F}$ on M, denote by $\mathfrak{B}_\mathcal{F}$ the set of leaves of $\mathcal{F}|_U$, and by $b: M \to \mathfrak{B}_\mathcal{F}$ the natural projection. Given an $\mathcal{F}$-convex subset U, the set $\mathfrak{B}_{\mathcal{F}|_U}$ has a natural structure of a manifold. Obviously, when one decreases an $\mathcal{F}$-convex subset U, the base $\mathfrak{B}_{\mathcal{F}|_U}$ decreases as well. In particular, if $m \in M$, then the germ[2] of $\mathfrak{B}_{\mathcal{F}|_U}$ near $b(m)$ does not depend on the $\mathcal{F}$-convex neighborhood U of m. Call this germ the *local base* of the foliation $\mathcal{F}$ near m.

[2]Given a manifold M with a closed submanifold N, an open submanifold $U \subset M$ is *compatible* with M if $U \supset N$. Extend compatibility relation to an equivalence relation $\sim$ between manifolds $M_1 \supset N$. Call equivalence classes $\widetilde{M}$ *germs near* N. A *mapping of germs* (or a *germ of a mapping*)

Definition 7.3. Given a web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on M, call an open subset $U \subset M$ (strictly) *$\mathcal{F}_\bullet$-convex* if U is (strictly) $\mathcal{F}_\lambda$-convex for all the foliations $\mathcal{F}_\lambda$.

Recall that a *section* of a mapping $\pi: M \to N$ is a right inverse to π mappings $N \to M$.

Definition 7.4. Consider a smooth web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on M, and an $\mathcal{F}_\bullet$-convex subset $U \subset M$. For any fixed $\lambda \in \Lambda$ consider the manifold $\mathfrak{B}_{\mathcal{F}_\lambda|_U}$. Taken together, they form a manifold $\mathfrak{T} = \mathfrak{T}_{\mathcal{F}_\bullet} = \coprod_{\lambda\in\Lambda} \mathfrak{B}_{\mathcal{F}_\lambda}$ equipped with a projection $\mathfrak{T} \xrightarrow{\pi} \Lambda$ (which sends $\mathfrak{B}_{\mathcal{F}_\lambda} \to \{\lambda\}$). Call the pair $(\mathfrak{T}, \pi)$ the *twistor transform* of $\mathcal{F}_\bullet|_U$.

Given a point $m \in M$, let $\Sigma_m(\lambda)$ be a leaf of $\mathcal{F}_\lambda$ which passes through m. Consider $\Sigma_m(\lambda)$ as a point of $\mathfrak{T}$. Then $\Sigma_m: \Lambda \to \mathfrak{T}$ is a section of the projection π. If it cannot lead to a confusion, denote the image of this map by the same symbol Σ_m.

Describe in more details how the bases of $\mathcal{F}_\lambda|_U$ for different λ fit together inside $\mathfrak{T}$. Call a submanifold $S \subset M$ a *cross-sections* of a foliation $\mathcal{F}$ on M if S is transversal to the leaves of $\mathcal{F}$, and each leaf of $\mathcal{F}$ intersects S at most once. Obviously, cross-sections exist after restriction of $\mathcal{F}$ to an appropriate open subset U, and are identified with open subsets of the base $\mathfrak{B}_{\mathcal{F}|_U}$.

Moreover, if $\mathcal{F}_\bullet$ is a smooth web, and S is a cross-section to $\mathcal{F}_{\lambda_0}$, then for any point $m \in S$ there is a neighborhood $U \subset S$, $U \ni m$, and a neighborhood $V \subset \Lambda$, $V \ni \lambda_0$, such that U is a cross-section for $\mathcal{F}_\lambda$, $\lambda \in U$. This gives a local identification of bases of $\mathcal{F}_\lambda$, $\lambda \in V$, thus a structure of a manifold on $\mathfrak{T}$.

Remark 7.5. One can show that for a smooth web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on M with a compact manifold Λ, any point $m \in M$ has an $\mathcal{F}_\bullet$-convex neighborhood U. Different choices of U lead to different twistor transforms, but all of them contain Σ_m. Thus in such a case the germ of $\mathfrak{T}_{\mathcal{F}_\bullet|_U}$ near Σ_m does not depend on U.

In fact, this germ is well-defined for any smooth web $\mathcal{F}_\bullet$. Indeed, the construction with cross-sections allows gluing local bases for $\mathcal{F}_\lambda$ near m into a germ of a manifold near Σ_m.

To simplify the following exposition, we pretend that the twistor transform is well-defined after a restriction of the web to an appropriate small open subset of M. This is always so if Λ is compact. The general case can be always treated honestly by switching to the language of germs.

8. Explicit construction of the twistor transform

In the case of codimension 1 the construction of $\mathcal{F}$-convex subsets can be easily made explicit. Moreover, such an explicit construction would make statements in the rest of the paper simpler to formulate.

$\left(\widetilde{M}, N\right) \to \left(\widetilde{M}', N'\right)$ is a smooth mapping $f: M \to M'$ such that $f(N) \subset N'$; here M, M' are some representatives of classes $\widetilde{M}$, $\widetilde{M}'$. Such mappings are considered up to the natural equivalence relation induced by restriction to compatible open subsets.

Put $\Delta(a,b) = |a|/|b| + |b|/|a|$, $\Delta(a_1,\dots,a_d) = \sum_{1\leq k<l\leq d}\Delta(a_k,a_l)$. Consider the following condition on a 1-form α on $U\subset\mathbb{V}^d$, $0\in U$:

$$\sum_{\substack{k=1\\l=1}}^{d}\left|\frac{\partial\alpha_k}{\partial x_l}\right|^2 \leq \frac{1}{E\Delta(a_1,\dots,a_d)^P r^2}\sum_{k=1}^{d}|\alpha_k|^2 \tag{8.1}$$

here E, r and P are numbers, $\alpha_k(x_1,\dots,x_d)$, $k=1,\dots,d$, are components if α, and $a_k=\alpha_k(0,\dots,0)$, $1\leq k\leq d$. This condition makes sense if $a_1,\dots,a_d\neq 0$, but if $P=0$, then it makes sense for any α.

The following lemma is not surprising:

Lemma 8.1. *Fix an integer $d>0$. Consider a 1-form α defined on $\mathbb{B}^d_r\subset\mathbb{V}^d$ and a foliation $\mathcal{F}$ on $\mathbb{B}^d_r$ of codimension 1. Suppose that $\alpha|_0\neq 0$, and $\alpha(x)$ is normal to L_x for any $x\in\mathbb{B}^d_r$; here L_x is the leaf of $\mathcal{F}$ which passes through x. There are numbers $D,E>0$ (which depend on d only) such that for any $0<\rho<r/D$*

1. *if α satisfies (8.1) with $P=0$ in $\mathbb{B}^d_r$, then $\mathbb{B}^d_\rho$ is strictly $\mathcal{F}$-convex;*
2. *if α satisfies (8.1) with $P=2$ in $\mathbb{B}^d_r$, and $a_k\neq 0$, $1\leq k\leq d$, then $\left(\mathbb{B}^1_\rho\right)^d$ is strictly $\mathcal{F}$-convex;*

Proof. Transposing coordinates x_k, one can ensure that $|a_d|\geq|a_k|$, $k=1,\dots,d-1$. Changing α to α/a_d allows us to assume that $a_d=1$.

Obviously, one can find D and E such that the condition above implies that in $\mathbb{B}^d_r$ one has $|\alpha_d-1|\leq 1/2$ and $\sum_{k=1}^{d-1}|\alpha_k|^2\leq 2d$. Consequently, in $\left(\mathbb{B}^{d-1}_{r/D}\times\mathbb{V}^1\right)\cap\mathbb{B}^d_r$ one can write any leaf of $\mathcal{F}$ which passes through $(0,\dots,0,c)$, $|c|<4\sqrt{d}r/D$, as $x_d=\varphi_c(x_1,\dots,x_{d-1})$, and $\sum_{k=1}^{d-1}|\partial\varphi_c/\partial x_k|^2<3\sqrt{d}$. Thus one can include $\mathbb{B}^d_\rho$ and $\left(\mathbb{B}^1_\rho\right)^d$ into a chart-like subset of $\mathbb{B}^{d-1}_r\times\mathbb{V}^1$.

The next step is to show that the leaves intersected with $\mathbb{B}^d_\rho$ or $\left(\mathbb{B}^1_\rho\right)^d$ are connected. In the case of the ball it is enough to show that

$$N_c(x_1,\dots,x_{d-1}) = |\varphi_c(x_1,\dots,x_{d-1})|^2+\sum_{k=1}^{d-1}|x_k|^2$$

is concave on $\mathbb{B}^{d-1}_{r/D}$ for $|c|<4\sqrt{d}r/D$. It is enough to show that the Hessian $\partial^2|\varphi_c|^2/\partial x_k\partial x_l$ of $|\varphi_c|^2$ on $\mathbb{B}^{d-1}_{r/D}$ cannot have a large negative eigenvalue under an appropriate choice of constants E and D. This Hessian is a sum of a non-negative part $2(\partial\varphi_c/\partial x_k)(\partial\varphi_c/\partial x_l)$ and of $2\varphi_c\partial^2\varphi_c/\partial x_k\partial x_l$.

In turn, it is enough to show that[3] $|\varphi_c|^2\sum_{\substack{k=1\\l=1}}^{d-1}|\partial^2\varphi_c/\partial x_k\partial x_l|^2$ can be made bounded by $1/16$. Since $|\varphi_c|$ can be bounded by $7\sqrt{d}r/D$, it is enough if we can bound second derivatives of φ_c as $O(1/r)$.

[3]In the complex-analytic case one needs to consider $\partial\bar{\partial}/\partial x_k\bar{\partial}x_l$ as well as $\partial^2/\partial x_k\partial x_l$.

However, the estimates on α_k, $k=1,\dots,n$, given above allow one to estimate second derivatives of φ_c in terms of derivatives of α_k. This finishes the proof of $\mathcal{F}$-convexity in the case of the ball.

Investigate strict $\mathcal{F}$-convexity in the case of the ball. It is clear that one can invert φ_c and write $c=\psi(x_1,\dots,x_d)$. It is enough to prove that the ψ-image of a small ball is convex, which follows from the following simple

Lemma 8.2. *There is a number E (which depends on d only) such that given a function ψ on $\mathbb{B}^d_r$ such that $d\psi$ satisfies (8.1) with $P=0$ in $\mathbb{B}^d_r$, then the image $\psi\left(\mathbb{B}^d_\rho\right)$ is convex for $0<\rho<r$.*

Investigate the case of the polydisk. The stronger assumptions we have in the polydisk case allow ensuring $|\alpha_k-a_k|<|a_k|/F$ for any given $F>0$. Now the statement follows from the following

Lemma 8.3. *Given d, there are numbers F and D which satisfy the following condition. Given a smooth function $\psi(x_1,\dots,x_d)$ defined on $\left(\mathbb{B}^1_r\right)^d$ and any numbers $a_1,\dots,a_d$, and c, if ψ satisfies*

$$|\partial\psi/\partial x_k-a_k|<|a_k|/F,\ k=1,\dots,d, \tag{8.2}$$

on $\left(\mathbb{B}^1_r\right)^d$, then $\psi\left(\left(\mathbb{B}^1_\rho\right)^d\right)$ is convex, and $\psi^{-1}(c)\cap\left(\mathbb{B}^1_\rho\right)^d$ is connected if non-empty for any $0<\rho<r/D$.

Proof. The statement is obvious in the real case, so assume complex-analytic situation. Start with the case $d=1$. Put $D=2$, $F=4$. We may assume $r=1$, $a_1=1$, then $|\psi''|<1/2$ on $\mathbb{B}^1_{1/2}$. Thus the direction of the tangent line l_τ to the curve $\psi\left(e^{i\tau}/2\right)$ rotates counterclockwise when τ grows, with the angular velocity being close to 1. This implies convexity of $\psi\left(\mathbb{B}^1_{r/D}\right)$. The connectivity of $\psi^{-1}(c)$ is obvious.

In the case $d>1$ the convexity follows from similar arguments: the boundary of the image of $\left(\mathbb{B}_\rho\right)^d$ is the curve $\Psi(\tau_1)=\psi\left(e^{i\tau_1}\rho,e^{i\tau_2(\tau_1)}\rho,\dots,e^{i\tau_d(\tau_1)}\rho\right)$; here τ_k are appropriate functions, $d\tau_k/d\tau\approx 1$, and the direction of the tangent line the curve $\Psi(\tau)$ behaves as in the case $d=1$.

For connectivity proceed by induction in d. We may assume that $|a_d|\geq|a_k|$, $k=1,\dots,d-1$. Increasing F and D, one can ensure that $\psi^{-1}(c)\cap\left(\left(\mathbb{B}^1_\rho\right)^{d-1}\times\mathbb{B}^1_r\right)$ is given by $x_d=\varphi_c(x_1,\dots,x_{d-1})$ if $c\in\psi\left(\left(\mathbb{B}^1_\rho\right)^d\right)$, and φ_c satisfies (8.2) with $d-1$ taken instead of d. Thus $\varphi_c^{-1}(c_1)\cap\left(\mathbb{B}^1_\rho\right)^{d-1}$ is connected if non-empty. On the other hand, $\psi^{-1}(c)\cap\left(\mathbb{B}^1_\rho\right)^d$ is diffeomorphic to $\varphi_c^{-1}\left(\mathbb{B}^1_\rho\right)\cap\left(\mathbb{B}^1_\rho\right)^{d-1}$. Since $\varphi_c\left(\left(\mathbb{B}^1_\rho\right)^{d-1}\right)\cap\mathbb{B}^1_\rho$ is convex, it is connected, thus $\psi^{-1}(c)\cap\left(\mathbb{B}^1_\rho\right)^d$ is connected as well. $\square$

This finishes the proof of Lemma 8.1. $\square$

Amplification 8.4. *Consider a* 1-*form* α *on* $\mathbb{B}_r^d$, *and a* 1-*form* $\widetilde{\alpha}$ *with components* $\widetilde{\alpha}_k = \kappa_k \alpha_k$, $k = 1, \dots, d$; *here* κ_k *are arbitrary numbers, some of which are non-0. Consider a foliation* $\mathcal{F}$ *on* $\mathbb{B}_r^d$ *of codimension 1. Suppose that* $\widetilde{\alpha}(x)$ *is normal to* L_x *for any* $x \in \mathbb{B}_r^d$; *here* L_x *is the leaf of* $\mathcal{F}$ *which passes through* x. *Let* $a_k \stackrel{\text{def}}{=} \alpha_k(0, \dots, 0) \neq 0$ *for* $1 \leq k \leq d$.

There are numbers $D, E > 0$ *(which depend on* d *only) such that for* $0 < \rho < r/D$

1. *if* α *satisfies* (8.1) *with* $P = 2$ *in* $\mathbb{B}_r^d$, *then* $\mathbb{B}_\rho^d$ *is strictly* $\mathcal{F}$*-convex;*
2. *if* α *satisfies* (8.1) *with* $P = 4$ *in* $\mathbb{B}_r^d$, *then* $\left(\mathbb{B}_\rho^1\right)^d$ *is strictly* $\mathcal{F}$*-convex;*

Proof. Proceed similarly to the proof of Lemma 8.1. One may assume that $\max_k |a_k| = 1$. Let $A = \min_k |a_k|$. With the stronger conditions of the amplification one can ensure that $|\alpha_k - a_k|/A$ is sufficiently small in $\mathbb{B}_r^d$. Then the condition (8.1) give absolute bounds on derivatives of α_k, both from above and from below.

Multiplying κ_k by an appropriate constant, we may assume that $\max_k |\kappa_k| = 1$. Then given an estimate (8.1) for α, we can estimate $\sum_{k=1}^d |\widetilde{\alpha}_k|^2$ from below, and $\sum_{\substack{k=1\\l=1}}^d \left|\frac{\partial \widetilde{\alpha}_k}{\partial x_l}\right|^2$ from above in $\mathbb{B}_r^d$, loosing 2 units in P. In particular, $\widetilde{\alpha}$ satisfies (8.1) with $P = 0$ or $P = 2$. ∎

Apply the obtained results to the nonlinear wave equation. Consider the following condition on a function w defined on a subset $V \subset \mathbb{V}^3$, $0 \in V$:

$$\sum_{\substack{k=1\\l=1}}^{3} \left|\frac{\partial^2 w}{\partial x_k \partial x_l}\right|^2 \leq \frac{1}{E \Delta^P r^2} \sum_{k=1}^{3} \left|\frac{\partial w}{\partial x_k}\right|^2, \tag{8.3}$$

here E, P and r are numbers, and $\Delta = \Delta\left(w_{x_1}(0,0,0), w_{x_2}(0,0,0), w_{x_3}(0,0,0)\right)$.

Theorem 8.5. *There are numbers* $E, D > 0$ *such that given a non-degenerate solution* $w(x,y,z)$ *of Equation* (0.2) *which satisfies* (8.3) *in a ball* $\mathbb{B}_r^3$, *then there is a neighborhood* U *of (0,0,0) which is strictly* $\mathcal{F}_\bullet$*-convex w.r.t. the Veronese web* $\mathcal{F}_\bullet$ *which corresponds to* w; *here one can take*

1. $U = \mathbb{B}_\rho^3$ *if* $P = 2$, $0 < \rho < r/D$;
2. $U = \left(\mathbb{B}_\rho^1\right)^3$ *if* $P = 4$, $0 < \rho < r/D$.

Proof. Obviously, any ball or polydisk is $\mathcal{F}$-convex for 3 exceptional foliations $\{x = \text{const}\}$, $\{y = \text{const}\}$, $\{z = \text{const}\}$ of the web. Other foliations of the web are given by $\left\{v_{(\widetilde{A},\widetilde{B},\widetilde{C})} = \text{const}\right\}$; here $v_{(\widetilde{A},\widetilde{B},\widetilde{C})}$ is a non-degenerate solution of (0.5). Application of Amplification 8.4 finishes the proof. ∎

This theorem allows one to explicitly construct the twistor transform of the Veronese web $\mathcal{F}_\bullet$ associated to w. Consider the set U of the theorem, then the manifold with points enumerating leaves of all the foliations $\mathcal{F}_\lambda|_U$, $\lambda \in \mathbb{P}^1$, is the twistor transform of $\mathcal{F}_\bullet$.

Given an abstract Veronese web $\mathcal{F}_\bullet$, by Lemma 2.6 one can describe this web by a function $w(x,y,z)$ which, by Theorem 3.8, satisfies (0.2) for appropriate (A, B, C). Thus one can apply the theorem above to construct the twistor transform of $\mathcal{F}_\bullet$.

9. Sectional coordinates

Recall that a *submersion* is a smooth mapping of manifolds $f\colon M \to N$ such that $df|_m\colon \mathcal{T}_m M \to \mathcal{T}_{f(m)} N$ is surjective for any $m \in M$.

Lemma 9.1. *Consider a complex manifold* $\mathfrak{T}$ *with a submersion* π *onto a manifold* Λ *and a submanifold* $S \subset \mathfrak{T}$ *of codimension* r *such that* $\pi|_S$ *is a diffeomorphism. Given a covering* $\{V_i\}$ *of* $\mathfrak{T}$ *by Stein submanifolds, there is an open subset* $U \supset S$ *and identifications of* $U \cap \pi^{-1}(V_i)$ *with* $V_i \times S_i$, $S_i \subset \mathbb{C}^r$, $S_i \ni 0$; *these identifications intertwine* π *with the projections* $V_i \times S_i \to V_i$, *and send* $S \cap \pi^{-1}(V_i)$ *to* $V_i \times \{0\}$.

Proof. Suppose that $r = 1$. Consider any function $\widetilde{s}_i$ on a neighborhood of $S_i \stackrel{\text{def}}{=} S \cap \pi^{-1}(V_i)$ such that the vertical derivative of $\widetilde{s}_i$ on S_i does not vanish. Put $s_i \stackrel{\text{def}}{=} \widetilde{s}_i - \widetilde{s}_i \circ \Sigma_m \circ \pi$. Then (π, s_i) gives the required identification of a neighborhood of S_i with a subset of $V_i \times \mathbb{C}$.

The existence of such a function $\widetilde{s}_i$ follows from the fact that a neighborhood of $\pi^{-1}(V_i) \cap S$ is Stein if V_i is Stein. Indeed, any bundle over a Stein manifold with a fiber isomorphic to a disk $\mathbb{B}_\varepsilon^1$ is Stein [2, 1].

In the case $r > 1$ one needs to consider d functions $\widetilde{s}_{i,k}$ instead of one, and replaces $\mathbb{B}_\varepsilon^1$ by $\mathbb{B}_\varepsilon^d$ (using results of [11].) ∎

Remark 9.2. These "abstract nonsense" arguments allow the following construction: given a twistor transform $\mathfrak{T}$ of a complex-analytic Veronese web $M \ni m$, cover $\mathbb{P}^1$ by two disks $V_{1,2}$, and glue a neighborhood of Σ_m from two domains isomorphic to $V_i \times \mathbb{B}_\varepsilon^1$ (with π compatible with projections to V_i). The gluing function g is going to be a mapping $V_1 \times \mathbb{B}_\varepsilon^1 \ni (\lambda, t) \mapsto (\lambda, g(\lambda, t)) \in V_2 \times \mathbb{B}_\varepsilon^1$, with $g(\lambda, t)$ defined on $(V_1 \cap V_2) \times \mathbb{B}_\varepsilon^1$. In particular, the function g determines the germ of $\mathfrak{T} \to \mathbb{P}^1$ near Σ_m up to isomorphism. Later, in Theorem 13.12, we will see that this implies that the germ of the Veronese web near m is determined by g up to isomorphism.

However, if $\mathfrak{T}$ is a twistor transform one can achieve the same result without applying the heavy machinery of complex analysis. One can explicitly construct the required coordinate systems on open subsets of $\mathfrak{T}$.

Definition 9.3. Consider a submanifold γ of a manifold M equipped with a web $\mathcal{F}_\bullet$ with a twistor transform $\mathfrak{T} \xrightarrow{\pi} \Lambda$. Say that an open subset $U \subset \mathfrak{T}$ is *compatible* with γ, if for any $m \in \gamma$ and any $\lambda \in \pi(U)$ the leaf of $\mathcal{F}_\lambda$ passing through m is in U.

Obviously, a γ-compatible open subset $U \subset \mathfrak{T}$ is diffeomorphic to $\pi(U) \times \gamma$. In other words, such a subset defines a *local trivialization* of the bundle π. It is clear that γ and $V \stackrel{\text{def}}{=} \pi(U)$ determine U uniquely.

In the rest of this section we assume that $\mathcal{F}_\bullet$ is a Veronese web. As Lemma 13.13 will show, for Veronese webs the normal bundles to sections of π are not trivializable, thus in this case $\pi(U)$ cannot coincide with $\mathbb{P}^1$.

Continue assuming that $\mathfrak{T}$ is not a germ, but a bona fide manifold.

Lemma 9.4. *Consider a point m on a Veronese web $\mathcal{F}_\bullet$ on M and a curve γ passing through m. Let $V_{m,\gamma} \subset \mathbb{P}^1$ consist of points λ such that γ is not tangent to $L_\lambda(m)$ at m; here $L_\lambda(m)$ is the leaf of $\mathcal{F}_\lambda$ which passes through m. Let an open subset $V \subset \mathbb{P}^1$ be compactly included into $V_{m,\gamma}$. Then there is a neighborhood γ_1 of m in γ and a compatible with γ_1 subset $U \subset \mathfrak{T}$ with $\pi(U) = V$.*

Proof. If $\lambda_0 \in V_{m,\gamma}$, there is a neighborhood V of λ_0 and a neighborhood W of m such that for $\lambda \in V$ the leaves of $\mathcal{F}_\lambda$ are not tangent to γ at any point of W, and each leaf intersects $\gamma \cap W$ in at most one point. Since $\bar{V} \subset \mathbb{P}^1$ is compact, one can decrease W so that this condition is satisfied for any $\lambda \in V$. Taking $\gamma_1 = \gamma \cap W$, and U to consists of leaves of $\mathcal{F}_\lambda$, $\lambda \in V$, which intersect γ finishes the proof. □

Lemma 9.5. *The subset $V_{m,\gamma} \subset \mathbb{P}^1$ of Lemma 9.4 is open, depends on $\mathcal{T}_m\gamma$ only, and $\mathbb{P}^1 \smallsetminus V_{m,\gamma}$ consists of at most $\dim M - 1$ points. Given any subset $Z \subset \mathbb{P}^1$ of at most $\dim M - 1$ points and $m \in M$, one can find a curve γ passing through m such that $V = \mathbb{P}^1 \smallsetminus Z$. Different possible directions $\mathcal{T}_m\gamma$ correspond 1-to-1 to different ways of assigning multiplicities to points of Z with the total being $\dim M - 1$.*

Proof. The statements of this lemma concern one tangent space $\mathcal{T}_m M$ only. The tangent spaces $\mathcal{T}_m L_\lambda(m) \subset \mathcal{T}_m M$ are orthogonal complements to directions $\boldsymbol{n}_m(\lambda)$ in $\mathcal{T}_m^* M$. Thus $V_{m,\gamma}$ is determined by $\mathcal{T}_m\gamma$ and the image of the curve $\boldsymbol{n}_m\colon \mathbb{P}^1 \to \mathbb{P}\mathcal{T}_m^* M$. This is a Veronese curve, and any two such curves are isomorphic. Thus we may replace $\mathcal{T}_m^* M$ by an arbitrary vector space S with a Veronese curve.

Take S to be the symmetric power $\operatorname{Sym}^{d-1}\mathbb{V}^2$, $\dim S = d$, and let the Veronese curve consists of $(d-1)$st powers of elements of $\mathbb{V}^2$. Then S^* can be identified with homogeneous polynomials of degree $d-1$ of two variables (two coordinates on $\mathbb{V}^2$), thus $\mathcal{T}_m\gamma \subset \mathcal{T}_m M = S^*$ provides such a polynomial p up to a constant.

It is easy to check that $\lambda \in V_{m,\gamma} \subset \mathbb{P}^1 = \mathbb{P}\mathbb{V}^2$ iff p does not vanish at the points of $\mathbb{V}^2$ in the direction of λ. There are at most $\deg p = d-1$ such directions, and given such directions with appropriate multiplicities, one can find a polynomial $p \in S^*$ which vanishes at these points. □

Now we can implement the program outlined in Remark 9.2:

Corollary 9.6. *Given $m \in M$, one can find two curves γ_1, γ_2 passing through m and two open subsets $U_1, U_2 \subset \mathfrak{T}$ compatible with γ_1, γ_2 correspondingly such that $U_1 \cup U_2$ is a neighborhood of the section $\Sigma_m \subset \mathfrak{T}$.*

Proof. Indeed, one can find γ_1, γ_2 such that $\mathbb{P}^1 \smallsetminus V_{m,\gamma_1}$ is contained in a small neighborhood of 0, and $\mathbb{P}^1 \smallsetminus V_{m,\gamma_1}$ is contained in a small neighborhood of ∞. To finish

the proof, note that $\Sigma_m \cap \pi^{-1}(\pi U) \subset U$ for any subset $U \subset \mathfrak{T}$ which is compatible with a curve γ passing through m. □

Consider two curves as in Corollary 9.6. Let $V_1 = \pi U_1$, $V_2 = \pi U_2$. Then $U_1 \simeq V_1 \times \gamma_1$, $U_2 \simeq V_2 \times \gamma_2$, thus identifications of γ_1 and γ_2 with $\mathbb{B}^1_\varepsilon$ lead the gluing function $g(\lambda, t)$ as in the beginning of this section. The other way to look at g is to consider it as a family of gluings $\widehat{g}_\lambda\colon \gamma_1 \to \gamma_2$, $\lambda \in V_1 \cap V_2$.

Describe these gluings $\widehat{g}_\lambda$ in geometric terms. This description does not mention $\mathfrak{T}$ as a manifold, thus one need not assume that $\mathfrak{T}$ exists as a manifold.

Corollary 9.7. *Given a point m_0 on a Veronese web M, one can find two curves γ_1, γ_2 passing through m_0, a neighborhood $W \subset M$ of m_0, and two open subsets $V_1, V_2 \subset \mathbb{P}^1$ such that*

1. *For any $\lambda \in V_j$, $j = 1, 2$, and any $m \in \gamma_j$ the leaf of $\mathcal{F}_\lambda|_W$ which passes through m intersects γ_j at exactly one point m and is transversal to γ_j;*
2. *For any $\lambda \in V_1 \cap V_2$, and any $m \in \gamma_1$ the leaf of $\mathcal{F}_\lambda|_W$ which passes through m intersects γ_2; denote the (unique) point of intersection by $\widehat{g}_\lambda(m)$;*
3. *$V_1 \cup V_2 = \mathbb{P}^1$.*

The germ near $(V_1 \cap V_2) \times \{m\}$ of the function $\widehat{g}_\bullet\colon (V_1 \cap V_2) \times \gamma_1 \to \gamma_2$ uniquely determines the germ of the twistor transform $\mathfrak{T}$ of M near the section Σ_{m_0} and the germ of $\mathcal{F}_\bullet$ near m. For any $0 < \varepsilon < 1$ one can ensure that $V_1 \supset \{z \mid |z| > \varepsilon\}$, $V_2 \supset \{z \mid |z| < 1/\varepsilon\}$.

10. Explicit construction of the gluing function

In conditions of Corollary 9.7 identify a neighborhood of m_0 in γ_1 with $\mathbb{B}^1_\varepsilon$, and a neighborhood of m_0 in γ_2 with a subset of $\mathbb{C}$. This would make the gluing function $g(\lambda, t)$ into a function $(V_1 \cap V_2) \times \mathbb{B}^1_\varepsilon \to \mathbb{C}$. A different choice of identifications would lead to $\widetilde{g}(\lambda, t) = f(g(\lambda, F(t)))$ for appropriate invertible functions $f(z)$, $F(z)$.

Describe $g(\lambda, t)$ in terms of the function $w(x, y, z)$ which identifies the Veronese web. Later, in Appendix 15, we will see that the gluing function should depend only on the restriction of w and first derivatives of w to an appropriate surface. Here we prove this only in the case of surfaces of a special form.

Theorem 10.1. *Consider a complex-analytic non-degenerate solution $w(x, y, z)$ of the nonlinear wave equation (0.2) defined in a neighborhood of (0,0,0). Fix $0 < r < 1$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{P}^1$, $|\lambda_{1,2}| < r$, $|\lambda_3| > 1/r$. Let $Y(x)$ be any function such that $Y'(x) = dY/dx$ is nowhere 0, and $Y(0) = 0$. Consider the following family of ODEs with a parameter μ on a function $z(x)$:*

$$\frac{dz}{dx} = \frac{Aw_x(x, Y(x), z)}{\mu C w_z(x, Y(x), z)} - \frac{Bw_y(x, Y(x), z)}{(\mu - 1) C w_z(x, Y(x), z)} Y'(x);$$

Let $g_\mu(t)$ be $z(0)$; here $z(x)$ is the solution of this equation with the initial data $z(t) = 0$. Then for any $\varepsilon_1 > 0$ one can find an appropriate $\delta > 0$ so that the function $g_\mu(t)$ is correctly defined if $|\mu| > \varepsilon_1$, $|\mu - 1| > \varepsilon_1$, and $|t| < \delta$.

Consider ε such that $r < \varepsilon < 1$. Define a surface $\widetilde{\mathfrak{T}}$ by gluing $\mathbb{B}^1_{1/\varepsilon} \times \mathbb{B}^1_\delta$ and $\left(\mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon\right) \times \mathbb{C}$ via $\mathbb{B}^1_{1/\varepsilon} \times \mathbb{B}^1_\delta \ni (\lambda, t) \mapsto (\lambda, \widetilde{g}(\lambda, t)) \in \left(\mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon\right) \times \mathbb{C}$, $\varepsilon < |\lambda| < 1/\varepsilon$, $|t| < \delta$; here $\widetilde{g}(\lambda, t) = g_\mu(t)$, $\mu = (\lambda_1 : \lambda_2 : \lambda_3 : \lambda)$, and δ corresponds to ε_1 such that $|\mu| > \varepsilon_1$ and $|\mu - 1| > \varepsilon_1$ if $|\lambda| > \varepsilon$. Since $g_\mu(0) \equiv 0$, $\widetilde{\mathfrak{T}}$ has a section $\widetilde{\Sigma}_{(0,0,0)} = \{(\lambda, 0)\}$. Coordinates λ glue into a projection $\mathfrak{T} \to \mathbb{P}^1$.

Suppose that

$$\left|\frac{Q\lambda_1 - \lambda_2}{Q - 1}\right| > 1/\varepsilon, \qquad Q = Y'(0)\frac{Bw_y(0,0,0)}{Aw_x(0,0,0)}. \tag{10.1}$$

Then the germ of $\widetilde{\mathfrak{T}}$ near $\widetilde{\Sigma}_{(0,0,0)}$ is isomorphic to the germ of the twistor transform $\mathfrak{T}$ of the Veronese web associated[4] *to $w(x, y, z)$ near $\Sigma_{(0,0,0)}$.*

Proof. Consider the 3-dimensional Veronese web M associated to $w(x, y, z)$ such that the foliations $\{x = \text{const}\}$, $\{y = \text{const}\}$, $\{z = \text{const}\}$ are associated to $\lambda = \lambda_1$, $\lambda = \lambda_2$, $\lambda = \lambda_3$. Take $m_0 = (0, 0, 0)$, γ_2 to be the z-axis. Then the subset V_{m_0,γ_2} (in notations of Lemma 9.4) is $\mathbb{P}^1 \smallsetminus \{\lambda_1, \lambda_2\} \supset \mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon$, since γ_2 is an intersection of a leaf of $\mathcal{F}_{\lambda_1}$ and of a leaf of $\mathcal{F}_{\lambda_2}$. Similarly, for a curve γ in xy-plane the subset $V_{m,\gamma}$ is $\mathbb{P}^1 \smallsetminus \{\lambda_3, \lambda(m)\}$; here $\lambda(m) = \lambda_1$ for the curves $x = \text{const}$ in xy-plane, $\lambda(m) = \lambda_2$ for the curves $y = \text{const}$ in xy-plane. It is clear that for a curve with any other direction $\lambda(m) \neq \lambda_1$ and $\lambda(m) \neq \lambda_2$. In particular, it is so for the curve γ_1 given by $y = Y(x)$. Thus $V_{m_0,\gamma_1} \cup V_{m_0,\gamma_2} = \mathbb{P}^1$. Thus γ_1, γ_2 satisfy conditions of Corollary 9.7, thus one can glue the twistor transform $\mathfrak{T}$ from two open subsets, one being a bundle over V_{m_0,γ_1}, another over V_{m_0,γ_2}.

Moreover, $V_{m_0,\gamma_1} \supset \mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon$, and if $|\lambda(m_0)| > 1/\varepsilon$, then $V_{m_0,\gamma_2} \supset \mathbb{B}^1_{1/\varepsilon}$. In such a case $\mathfrak{T}$ can be glued from two open subsets, one being a bundle over $\mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon$, another over $\mathbb{B}^1_{1/\varepsilon}$. To describe $\mathfrak{T}$, it is enough to describe the gluing function $g(\lambda, t)$, $\varepsilon < |\lambda| < 1/\varepsilon$, for small t. Taking z as the coordinate on γ_2 and x as the coordinate on γ_1, one can describe this gluing function in the following way: take a point $m = (t, Y(t), 0)$ on γ_1, find the leaf of $\mathcal{F}_\lambda$ which passes through m, and intersect this leaf with γ_2. Then $g(\lambda, t)$ is the z-coordinate of the point of intersection.

Consider the surface N given by the equation $y = Y(x)$. The foliation $\mathcal{F}_\lambda$ can be described by the equations $v(x, y, z) = \text{const}$; here the derivative of v is given by Corollary 3.6. The curves cut out by this foliation on N have both $(-dY/dx, 1, 0)$ and (v_x, v_y, v_z) as normal vectors. Thus these curves are tangent to directions

$$(p_3(\lambda) w_z, p_3(\lambda) w_z dY/dx, -p_1(\lambda) w_x - p_2(\lambda) w_y dY/dx)$$

[4] As in Remark 3.9.

(notations as in (3.1)). One can easily check that the ODE of the theorem describes xz-projections of these curves for $\mu = (\lambda_1 : \lambda_2 : \lambda_3 : \lambda)$. Thus $g(\lambda, t) = \widetilde{g}(\lambda, t)$.

The only thing to prove is $|\lambda(m_0)| > 1/\varepsilon$. In fact $\lambda(m_0) = \frac{Q\lambda_1 - \lambda_2}{Q-1}$. To check this, it is enough to find the intersection of the leaf of $\mathcal{F}_\lambda$ through (0,0,0) with $z = 0$. As above, the direction of this curve is given by $(-v_y, v_x, 0) = (-p_2(\lambda) w_y, p_1(\lambda) w_x, 0)$. Again, it is easy to check that this agrees with (10.1). ☐

Remark 10.2. Obviously, the condition (10.1) is satisfied in $Y'(0)$ is inside a non-empty disk in $\mathbb{CP}^1$. In fact, there is a canonical choice of $Y(x)$ which automatically satisfies (10.1). Indeed, the condition $\lambda(m) = \lambda_3$ gives a direction field on xy-plane, take an integral curve of this direction field. Explicitly,

$$\frac{dY}{dx} = \frac{Aw_x(x, Y, 0)}{Bw_y(x, Y, 0)}, \qquad Y(0) = 0. \tag{10.2}$$

Remark 10.3. If ε with the properties required in the theorem does not exist, by decreasing δ one can ensure that the set of values of λ for which (10.1) does not hold is in a small disk D which does not contain λ_1 and λ_2. If there is a circle on $\mathbb{P}^1$ which separates $\{\lambda_1, \lambda_2\}$ from λ_3 and D, then one can use this circle instead of $\{|z| = 1\}$ in Theorem 10.1.

If there is no such circle, then $\frac{BY'(0)w_y(0,0,0)}{Bw_y(0,0,0)Y'(0) - Aw_x(0,0,0)}$ is real and is between 0 and 1. In particular, by a projective transform of $\mathbb{P}^1$ one can make $\lambda_1, \lambda_2, \lambda_3$ real, and the disk D centered on the real axis between λ_1 and λ_2. If additionally $w(x, y, z)$ is real for real x, y, z, and A, B, C are real, then the real (A, B, C)-equation is hyperbolic near (0,0,0) w.r.t. the surface $y = Y(x)$. Thus this case is of special interest.

In such a case it is hard to describe $\mathfrak{T}$ by representing $\mathbb{P}^1$ as a union of two disks, but one can glue $\mathfrak{T}$ using the same function $g_\mu(t)$ if one covers $\mathbb{P}^1$ by two regions of more complicated form. For example, consider small disks $D_{1,2,3}$ centered at $\lambda_{1,2,3}$, consider a contour L which goes along the line $\operatorname{Im}\lambda = 0$ with the exceptions of going around D_1 and D_2 from above, and around D and D_3 from below. The function $g(\lambda, t)$ is still correctly defined for λ near L, thus one can describe $\mathfrak{T}$ by gluing neighborhoods of the regions above L and below L.

Note that for the values of $\lambda \in L$ which are on the real axis the function $g(\lambda, t)$ can be defined in terms of solving a real ODE.

11. Equipped twistor transforms and infinitesimal families

For a mapping $\pi\colon M \to N$ denote by $\Gamma(N, \pi)$ the set of *sections* of π, i.e., of right inverse mappings to π.

Definition 11.1. Given a web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on M with the twistor transform $\mathfrak{T} \xrightarrow{\pi} \Lambda$, consider the family $\{\Sigma_m\}_{m\in M}$ of sections of π. The *equipped twistor transform* of $\mathcal{F}_\bullet$ is the mapping $\mathfrak{T} \xrightarrow{\pi} \Lambda$ together with a family of sections $\{\Sigma_m\}_{m\in M}$.

Given such a structure $(\mathfrak{T}, \Lambda, \pi, M, \Sigma_\bullet)$, and $\lambda \in \Lambda$, consider a mapping $\boldsymbol{s}_\lambda : M \to \pi^{-1}(\lambda) : m \mapsto \Sigma_m(\lambda)$. If this structure comes from an equipped twistor transform, this mapping is a submersion.

Lemma 11.2. *Consider a submersion $\mathfrak{T} \xrightarrow{\pi} \Lambda$ together with a family of sections $\{\Sigma_m\}_{m\in M}$ parameterized by a manifold M. If the rank of differential $d\boldsymbol{s}_\lambda|_m$ of the mapping $\boldsymbol{s}_\lambda$ does not depend on m and λ, then M is equipped with a canonically defined web structure $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$, the leaf L_{λ,m_0} of $\mathcal{F}_\lambda$, $\lambda \in \Lambda$, which passes through $m_0 \in M$ consists of points $m \in M$ such that $\Sigma_m(\lambda) = \Sigma_{m_0}(\lambda)$.*

If $(\mathfrak{T}, \Lambda, \pi, M, \Sigma_\bullet)$ is a twistor transform of a web $\widetilde{\mathcal{F}}_\bullet$ on M, then $\widetilde{\mathcal{F}}_\bullet = \mathcal{F}_\bullet$.

Proof. Indeed, mappings with constant rank of the differential are submersions onto their images, thus preimages of points are foliations on M. The other statements are obvious. $\square$

It is clear that in the conditions of the lemma if $\boldsymbol{s}_\lambda$ is not of maximal possible rank (i.e., is not a submersion), then $\mathfrak{T}' = \bigcup_\lambda \operatorname{Im} \boldsymbol{s}_\lambda$ is a submanifold of $\mathfrak{T}$, and $(\mathfrak{T}', \Lambda, \pi', M, \Sigma_\bullet)$ is the twistor transform of $\mathcal{F}_\bullet$; here $\pi' = \pi|_{\mathfrak{T}'}$.

Lemma 11.2 shows that one can reconstruct a web on M by its equipped twistor transform $(\mathfrak{T}, \Lambda, \pi, M, \Sigma_\bullet)$. In fact in many cases to reconstruct the web one needs much less data than $(\mathfrak{T}, \Lambda, \pi, M, \Sigma_\bullet)$. Later, in Section 13, we explain when the same information is contained in $(\mathfrak{T}, \Lambda, \pi)$, at least if one considers M up to isomorphism. Illustrate this by several weaker statements.

Suppose that the mapping $\Sigma_\bullet : M \to \Gamma(\Lambda, \pi) : m \mapsto \Sigma_m$ is injective, in other words, $\mathcal{F}_\bullet$ is separating. In such cases M as a set is identified with $\operatorname{Im} \Sigma_\bullet$. In fact $\Gamma(\Lambda, \pi)$ has a natural topology, and if Σ is a homeomorphism on its image, then the topology on M can be also reconstructed basing on $\operatorname{Im} \Sigma_\bullet \subset \Gamma(\Lambda, \pi)$. In such a case if we are interested in $(M, \mathcal{F}_\bullet)$ up to homeomorphism, it may be reconstructed given $(\mathfrak{T}, \Lambda, \pi, \operatorname{Im} \Sigma_\bullet)$.

One should expect that the same argument will work for diffeomorphisms as far as the differential of $\Sigma_\bullet$ is injective. However, in general $\Gamma(\Lambda, \pi)$ is not finite-dimensional, thus this question is a little bit more subtle. However, it is relatively easy to describe what is an individual tangent space to $\Gamma(\Lambda, \pi)$. This tangent space is going to be the target of the differential of $\Sigma_\bullet$.

Definition 11.3. Given a section Σ of submersion $\pi : \mathfrak{T} \to \Lambda$, the *tangent space* to $\Gamma(\Lambda, \pi)$ at Σ is the vector space $\Gamma(S, \mathcal{N}S)$, $S = \operatorname{Im} \Sigma$. Call elements of $\Gamma(S, \mathcal{N}S)$ *infinitesimal deformations*. Given a family $\{\Sigma_m\}_{m\in M}$ of sections of π, the *infinitesimal family* of $\{\Sigma_m\}$ at $m_0 \in M$ is the naturally defined mapping $d\Sigma|_{m_0} : \mathcal{T}_{m_0} M \to \Gamma(\Sigma_{m_0}, \mathcal{N}\Sigma_{m_0})$. Say that a family $\{\Sigma_m\}$ is *immersive* if $d\Sigma|_m$ is a monomorphism for any $m \in M$.

Describe what is $d\Sigma|_m$ and what is the geometric meaning of this definition. To define $d\Sigma|_m$, it is enough to consider the case $\dim M = 1$. A *smooth* 1*-parametric family* σ_t, $t \in T \subset \mathbb{V}^1$, *of sections* of π is a mapping $\sigma : \Lambda \times T \to \mathfrak{T}$ such that $\pi \circ \sigma$ coincides with the projection $p_1 : \Lambda \times T \to \Lambda$. Given σ and $t \in T$, consider the derivatives $d\sigma|_{(\lambda,t)}$ at points of $\Lambda \times \{t\}$. Clearly, $d\sigma|_{(\lambda,t)}$ maps $\mathcal{T}_\lambda \Lambda \oplus \mathcal{T}_t \mathbb{V}^1$ to $\mathcal{T}_{\sigma(\lambda,t)} \mathfrak{T}$. It can be split into a direct sum of a mapping $d\sigma|_{(\lambda,t)}^{(1)} : \mathcal{T}_\lambda \Lambda \to \mathcal{T}_{\sigma(\lambda,t)} \mathfrak{T}$ and $d\sigma|_{(\lambda,t)}^{(2)} : \mathcal{T}_t \mathbb{V}^1 \to \mathcal{T}_{\sigma(\lambda,t)} \mathfrak{T}$.

Note that the condition $\pi \circ \sigma = p_1$ determines some components of $d\sigma|_{(\lambda,t)}$. Indeed, consider $S = \operatorname{Im} \sigma(\bullet, t)$. It is a submanifold of $\mathfrak{T}$. Given $\lambda \in \Lambda$, the vector space $\mathcal{T}_{\sigma(\lambda,t)} \mathfrak{T}$ can be decomposed into a direct sum of tangent spaces to $\pi^{-1}(\lambda)$ and to S. Denote components of $v \in \mathcal{T}_{\sigma(\lambda,t)} \mathfrak{T}$ in this decomposition by v^{vert} and v^{hor}. In particular, the mappings $d\sigma^{(1)}$, $d\sigma^{(2)}$ can be further subdivided into $d\sigma^{(1)\mathrm{vert}}$, $d\sigma^{(2)\mathrm{vert}}$, $d\sigma^{(1)\mathrm{hor}}$, $d\sigma^{(2)\mathrm{hor}}$. It is clear that given two families σ and $\widetilde{\sigma}$, if vertical components of $d\sigma$ and $d\widetilde{\sigma}$ coincide, then $d\sigma$ and $d\widetilde{\sigma}$ coincide. Moreover, $d\sigma^{(1)\mathrm{vert}}$ obviously vanishes. In particular, the only "interesting" part of differential of σ is $d\sigma^{(2)\mathrm{vert}}$.

On the other hand, the vertical component of $v \in \mathcal{T}_{\sigma(\lambda,t)} \mathfrak{T}$ can be also naturally identified with an element of the quotient by the vector subspace of horizontal sections $\mathcal{T}_{\sigma(\lambda,t)} \mathfrak{T} / \mathcal{T}_{\sigma(\lambda,t)} S = \mathcal{N}_{\sigma(\lambda,t)} S$, i.e., with a normal vector to S at $\sigma(\lambda, t)$. Since $d\sigma^{(2)\mathrm{vert}}$ sends $\delta t \in \mathcal{T}_{t_0} \mathbb{V}^1$ to a normal vector to S at $\sigma(\lambda, t)$ for each $\lambda \in \Lambda$, it associates to δt a section of the normal bundle $\mathcal{N}S$.

The following statement is obvious:

Lemma 11.4. *The equipped twistor transform of a web is immersive iff the web is separating.*

It is clear that for an immersive family Σ_m, $m \in M$, the mappings $\boldsymbol{s}_\lambda$, $\lambda \in \Lambda$, separate points on small open subsets of M (even infinitesimally). Thus the structure of the manifold on M is reconstructed from the mapping of the set M to $\Gamma(\Lambda, \pi)$.

Corollary 11.5. *Consider a weakly separating and separating web $\mathcal{F}_\bullet$ on M. Then $\mathcal{F}_\bullet$ can be reconstructed up to a diffeomorphism by the twistor transform $\mathfrak{T} \xrightarrow{\pi} \Lambda$ of $\mathcal{F}_\bullet$ together with the subset $\mathcal{M} \subset \Gamma(\Lambda, \pi)$ consisting of sections which correspond to points of M.*

12. Kodaira–Spencer deformation of a section

In the classification of complex-analytic Veronese webs the central role is played by the following corollary[5] of Kodaira–Spencer deformation theory (for example, see [8]).

Definition 12.1. Say that a vector bundle E over a topological space Λ is *cohomologically trivial* if $H^k(\Lambda, E) = 0$ for $k > 0$.

[5]Since one-dimensional Cauchy–Riemann equations are not overdetermined, in the case $\dim \Lambda = 1$ we are most interested in Kodaira–Spencer deformation theory can be replaced by an argument involving an implicit function theorem (in normed spaces).

Theorem 12.2. *Consider an n-dimensional complex manifold $\mathfrak{T}$ equipped with a surjective submersion $\pi\colon \mathfrak{T} \to \Lambda$, and with a section $\Sigma\colon \Lambda \to \mathfrak{T}$ of the projection π. Let $S = \operatorname{Im}\Sigma$, suppose that $\mathcal{N}S$ is cohomologically trivial, and Λ is compact. Then there is a connected complex manifold M, a mapping $\sigma\colon \Lambda \times M \to \mathfrak{T}$, and a neighborhood U of S in $\mathfrak{T}$ such that*

1. *$\pi \circ \sigma$ coincides with the projection $\Lambda \times M \to \Lambda$;*
2. *for any section s of $\pi|_U$ there is unique $m \in M$ such that $s = \sigma|_{\Lambda\times\{m\}}$; denote by $m_0 \in M$ the point which corresponds to $s = \Sigma$;*
3. *the infinitesimal family[6] $d\sigma|_{m_0}\colon \mathcal{T}_{m_0}M \to \Gamma(S, \mathcal{N}S)$ is a bijection.*

Remark 12.3. To translate to the usual formulation of deformation theory, instead of deforming the mapping Σ, one should deform the submanifold S. Then the first condition on σ disappears (is just gives a normalization by identifying the deformed submanifold with Λ), the second one identifies M with the moduli set of those submanifolds in $U \subset \mathfrak{T}$ which project 1-to-1 to Λ. The fact that the set M can be equipped with a structure of a manifold is the most nontrivial part of the statement. If $\mathfrak{T}$ is in fact a total space of a vector bundle $\mathcal{E}$ over Λ, then this statement is trivial, with $M = \Gamma(\mathbb{P}^1, \mathcal{E})$.

Additionally, the existence of the projection on Λ (thus of retraction on S) removes all the bulkiness from the statement on a deformation of an arbitrary submanifold, since one does not need to consider the deformation of the the complex structure on S.

Remark 12.4. One should interpret the last statement of the theorem as the fact that any infinitesimal deformation is a infinitesimal family of *an actual* 1*-parameter deformation* of S. Compare this with Definition 11.3.

In our discussion we are most interested in the case $\dim\mathfrak{T} = 2$, $\Lambda = \mathbb{P}^1$. Then $\mathcal{N}S$ is a line bundle, thus is isomorphic to $\mathcal{O}(d-1)$ with $d \geq 0$, and $\dim M = d$. In fact we need a particular case $d = 3$, but for some time we are going to discuss the general case of arbitrary d, $\mathfrak{T}$ and Λ.

Definition 12.5. Say that a mapping $\pi\colon \mathfrak{T} \to \Lambda$ of complex manifolds is a *disk bundle* if $\mathfrak{T}$ is a manifold with C^0-boundary, $\dim\mathfrak{T} = \dim S + 1$, for any $\lambda \in \Lambda$ there is a neighborhood $U \ni \lambda$ such that $\pi|_{\pi^{-1}U}$ is homeomorphic to the projection $p_1\colon U \times D \to U$; here D is $\{z \in \mathbb{C} \mid |z| \leq 1\}$.

Proposition 12.6. *In the conditions of Theorem 12.2 assume that* $\dim\mathfrak{T} = 2$, $\Lambda = \mathbb{P}^1$*, and that π is a disk bundle. Consider two curves $\gamma_{1,2} \subset \mathfrak{T}$ such that restrictions $\pi|_{\gamma_1}$ and $\pi|_{\gamma_2}$ are bijections. Suppose that $d = \deg(\mathcal{N}\gamma_1) + 1$, and γ_1 intersects γ_2 in $\geq d$ points. Then $\gamma_1 = \gamma_2$.*

Proof. Suppose $\gamma_1 \neq \gamma_2$. Let $X_1, \dots, X_k$ be the points of intersection of γ_1 and γ_2. Let $\widetilde{\mathfrak{T}}$ be blow-up of $\mathfrak{T}$ at these points (make repeated blow-ups if needed to remove all the points of intersection). Removing proper preimages of $\pi^{-1}\pi(X_i)$, $i = 1, \dots, k$, from $\widetilde{\mathfrak{T}}$, we obtain a manifold $\widetilde{\widetilde{\mathfrak{T}}}$ with a mapping $\widetilde{\pi}$ to $\mathbb{P}^1$ such that preimages of points of $\mathbb{P}^1$ are disks, with the exception of the points $\pi(X_i)$, preimages of which are isomorphic to $\mathbb{P}^1 \smallsetminus \{\bullet\} \simeq \mathbb{C}$. Cutting out far-away points of $\mathbb{C}$ together with an appropriate neighborhood on $\widetilde{\widetilde{\mathfrak{T}}}$, one may ensure that the resulting manifold is a disk bundle over $\mathbb{P}^1$.

[6] See Definition 11.3.

Each blow-up decreases the degree of the normal bundle by 1, thus we reduced the statement to the case $d < 0$, and $\gamma_1 \cap \gamma_2 = \varnothing$. Show that this leads to contradiction.

Indeed, topological bundles with the fibers being oriented disks are isomorphic iff their boundaries are isomorphic as bundles with a fiber being oriented circles. In turn, any such bundle is isomorphic to a spherical bundle of a line bundle over $\mathbb{P}^1$, which is determined by its degree up to an isomorphism. We conclude that the topological bundle $\mathfrak{T} \to \mathbb{P}^1$ is isomorphic to a neighborhood of 0-section in the total space of $\mathcal{O}(-n)$, $n > 0$. However, $\mathcal{O}(-n)$ has no continuous nowhere-0 sections: indeed, such a section would give a trivialization of the spherical bundle of $\mathcal{O}(-n)$, thus, due to arguments given above, to an isomorphism of $\mathcal{O}(-n)$ with $\mathcal{O}(0)$. □

Remark 12.7. The condition of being a disk bundle is very essential. For example, suppose that $\mathfrak{T}$ is an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$ with π being the projection on the first $\mathbb{P}^1$. It is easy to find such an $\mathfrak{T}$ which contains both the "constant" section $x \mapsto 0$ of π, and the id-section $x \mapsto x$. Moreover, for most points of $\mathbb{P}^1$ the preimage in $\mathfrak{T}$ can be made a disk. Thus a topological argument is required indeed.

The next step is to provide a way to find the subset U of Theorem 12.2 if all we new is the family σ.

Proposition 12.8. *In the conditions of Theorem 12.2 assume that* $\dim\mathfrak{T} = 2$, $\Lambda = \mathbb{P}^1$*, and that π is a disk bundle. Let $\deg(\mathcal{N}S) = d-1$, $\{\lambda_1, \dots, \lambda_d\} \subset \mathbb{P}^1$ be a set of d distinct points. Let $B_k = \pi^{-1}\lambda_k$, U_k be an open subset of B_k, $k = 1, \dots, d$. Let $\widetilde{\sigma}$ be a mapping $\mathbb{P}^1 \times M \to \mathfrak{T}$ such that $\pi \circ \widetilde{\sigma}$ is the projection $\mathbb{P}^1 \times M \to \mathbb{P}^1$. Suppose that for any collection $X = \{X_k\}_{k=1}^{d} \subset \mathfrak{T}$ such that $X_k \in U_k$ there is $m_X \in M$ such that $\operatorname{Im}(\widetilde{\sigma}_{m_X}) \cap B_k = X_k$, $k = 1, \dots, d$.*

Let $U = \mathfrak{T} \smallsetminus \left(\bigcup_k (B_k \smallsetminus U_k)\right)$ (in other words, narrow fibers B_k over X_k to become U_k). Then for any curve $\gamma \subset U$ which projects isomorphically to $\mathbb{P}^1$ there is $m \in M$ such that $\gamma = \operatorname{Im}\widetilde{\sigma}_m$.

Proof. Take $X_k = \gamma \cap B_k$, and apply Proposition 12.6 to γ and $\operatorname{Im}(\widetilde{\sigma}_{m_X})$. □

Remark 12.9. This proposition provides a way to check that a given family $\widetilde{\sigma}$ and $U \subset \mathfrak{T}$ may work as the family σ from Theorem 12.2. Note that given $\widetilde{\sigma}$ which satisfies the last condition of Theorem 12.2, it is always possible to find the subsets U_k with the required properties. Indeed, if $\mathfrak{T}$ is an open subset of the total space of $\mathcal{O}(d-1)$, then this follows from the fact that a section of $\mathcal{O}(d-1)$ is uniquely determined by values in d different points (compare with Legendre interpolation formula, or Vandermond determinant). In general one needs to apply the implicit function theorem to the mapping $m \mapsto \Sigma_m = \operatorname{Im} \widetilde{\sigma}|_{\Lambda\times\{m\}}$.

Moreover, for the manifolds we are going to consider here (twistor transforms of Veronese webs) we can provide an explicit description of the family σ and of subsets U_k.

Corollary 12.10. *In the conditions of Theorem 8.5, consider the twistor transform $\mathfrak{T} \xrightarrow{\pi} \mathbb{P}^1$ of the $\mathcal{F}_\bullet$-convex subset $\left(\mathbb{B}^1_\rho\right)^3$. Let Σ_m be the section of $\mathfrak{T}$ corresponding to $m \in \left(\mathbb{B}^1_\rho\right)^3$. Then the mapping $\widetilde{\sigma}(m,\lambda) \stackrel{\text{def}}{=} \Sigma_m(\lambda)$, $m \in \left(\mathbb{B}^1_\rho\right)^3$, $\lambda \in \mathbb{P}^1$, satisfies the conditions of Proposition 12.8 with $U_k = B_k$.*

Proof. Let $\lambda_{1,2,3}$ be the values of λ which correspond to exceptional foliations $\{x = \text{const}\}$, $\{y = \text{const}\}$, $\{z = \text{const}\}$ of the web. Then B_k, $k = 1,2,3$, are naturally identified with $\mathbb{B}^1_\rho$. A choice of $X_k \in U_k = B_k$, $k = 1,2,3$, corresponds to a choice of 3 leaves of these 3 foliations, or, in other words, to a choice of coordinates x, y, z such that $|x|, |y|, |z| < \rho$. Put $m = (x,y,z) \in \left(\mathbb{B}^1_\rho\right)^3$, then Σ_m passes through X_k, $k = 1,2,3$. ◻

13. Airy webs

Consider a smooth web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on M. Suppose that the twistor transform $\mathfrak{T} \xrightarrow{\pi} \Lambda$ of $\mathcal{F}_\bullet$ is well-defined as a manifold.

Definition 13.1. Say that a smooth web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ is *strictly airy* if for any smooth section Σ of $\mathfrak{T} \xrightarrow{\pi} \Lambda$ there is a point $m \in M$ such that $\Sigma = \Sigma_m$. A web is *airy* if any point has a neighborhood U such that $\mathcal{F}_\bullet|_U$ is strictly airy.

Remark 13.2. This definition requires some modifications if only the germ of $\mathfrak{T} = \mathfrak{T}_{\mathcal{F}_\bullet}$ near $\Sigma_{m_0} \subset \mathfrak{T}$ is well-defined; here $m_0 \in M$. In such a case consider a family $\boldsymbol{\sigma}_\bullet \colon \Lambda \times T \to \mathfrak{T}$ of sections of π parameterized by (a germ of) a manifold T, and such that $\sigma_{t_0} = \Sigma_{m_0}$ for the base point $t_0 \in T$. We would require that there is a family $\boldsymbol{p}_\bullet \colon T \to M$ of points of M such that $\sigma_t = \Sigma_{p_t}$ for $t \in T$ near t_0.

Remark 13.3. It should be clear that airy webs exist only in complex-analytic situation, otherwise the set of sections is not a finite-dimensional manifold. Moreover, it is reasonable to conjecture that Λ cannot be a Stein manifold if $\dim \Lambda > 0$.

The principal property of airy webs is the following immediate corollary of Corollary 11.5:

Theorem 13.4. *Consider a weakly separating and separating strictly airy web. Locally such a web is uniquely determined (up to a local diffeomorphism) by its twistor transform $\mathfrak{T} \xrightarrow{\pi} \Lambda$.*

Proposition 13.5. *In the conditions of Theorem 12.2 suppose that global sections of the vector bundle $\mathcal{N}S$ span any fiber of $\mathcal{N}S$. Then a neighborhood M_1 of m_0 in M is equipped with a web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ of codimension equal to $\operatorname{codim}\Sigma$. This web is separating and airy.*

Proof. Deduce the first statement from Lemma 11.2. It is enough to calculate $\operatorname{rk} \boldsymbol{ds}_\lambda|_m$. The condition on global sections is equivalent to $\operatorname{rk} \boldsymbol{ds}_\lambda|_{m_0} = \operatorname{codim}\Sigma$, thus all we need to show is that this rank does not change if we move to a nearby section Σ_m of π.

Consider $\mathcal{N}S$ as a sheaf of $\mathcal{O}_S$-modules. For $P \in S$ denote by $\mathcal{N}S(-P)$ the sheaf of $\mathcal{O}_S$-modules with local sections being sections of $\mathcal{N}S$ which vanish at P. By the Grauert semicontinuity theorem [6], the Euler characteristic $\sum(-1)^k \dim H^k(S, \mathcal{N}S(-P))$ of $\mathcal{N}S(-P)$ does not change when P changes, and the individual terms $\dim H^k(S, \mathcal{N}S(-P))$ are semicontinuous from above. Similar results hold for $\dim H^k(\Sigma_m, \mathcal{N}\Sigma_m(-P))$ considered as functions of $m \in M$ and $P \in \Sigma_m$.

Consider the exact sequence of sheaves $0 \to \mathcal{N}S(-P) \to \mathcal{N}S \xrightarrow{v_P} \overline{\mathcal{N}_P S} \to 0$; here $\overline{\mathcal{N}_P S}$ is the skyscraper sheaf with the fiber over P being $\mathcal{N}_P S$. Since the mapping v of taking the value at P is surjective on global sections, the cohomological long exact sequence shows that $H^k(S, \mathcal{N}S(-P)) = 0$ for $k \geq 1$ and $P \in S$. This implies $H^k(\Sigma_m, \mathcal{N}\Sigma_m(-P)) = 0$ for $k > 1$ if $m \approx m_0$, thus $\dim H^0(\Sigma_m, \mathcal{N}\Sigma_m(-P))$ does not depend on $m \approx m_0$ and $P \in \Sigma_m$. Now a consideration of the long exact sequence for $0 \to \mathcal{N}\Sigma_m(-P) \to \mathcal{N}\Sigma_m \xrightarrow{v_P} \overline{\mathcal{N}_P\Sigma_m} \to 0$ shows that v_P is surjective for $m \approx m_0$ and $P \in \Sigma_m$. This implies that $\boldsymbol{ds}_\lambda|_m$ is a surjection.

By Lemma 11.2, a neighborhood of $\Sigma_{m_0} \subset \mathfrak{T}$ is the twistor transform of a web on an open subset $M_1 \subset M$, $M_1 \ni m_0$. Since $d\sigma|_{m_0}$ (and, by similar arguments, $d\sigma|_m$ for any $m \in M_1$) is an injection, this web is separating.

Prove airiness. One can find a neighborhood U of S in $\mathfrak{T}$ such that $\Sigma_m \subset U$ implies $m \in M_1$. Indeed, let $U_1 = \bigcup_{m\in M_1} \Sigma_m$. It is a neighborhood of S, thus one can apply Theorem 12.2 to U_1 instead of $\mathfrak{T}$. Obviously, the resulting neighborhood $U \subset U_1$ of S satisfies the requirement above. Let $M_2 = \{m \in M_1 \mid \Sigma_m \subset U\}$. Now any section Σ of $U \to \Lambda$ has a form $\Sigma = \Sigma_m$ for $m \in M_1$. Obviously, this implies also $m \in M_2$. On the other hand, a section of a twistor transform of M_2 induces a section of $U \to \Lambda$, thus the restriction of the web on M_2 is airy. ◻

By Definition 2.1, given a smooth web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on M, each point $m \in M$ induces a vector bundle $\boldsymbol{n}_m$ over Λ, the fiber over λ being $\boldsymbol{n}_m(\lambda)$. Obviously,

Lemma 13.6. *Consider the section Σ_m of the twistor transform $\mathfrak{T} \xrightarrow{\pi} \Lambda$ of $\mathcal{F}_\bullet$. Then $\mathcal{N}\Sigma_m \simeq \pi^*\boldsymbol{n}^*_m$.*

Proposition 13.7. *Consider a complex-analytic separating web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on M with compact Λ. Suppose that $\boldsymbol{n}_m$ is cohomologically trivial for any $m \in M$. Then there is a manifold $M' \supset M$ with an airy separating web $\{\mathcal{F}'_\lambda\}_{\lambda\in\Lambda}$ on it such that for any $\lambda \in \Lambda$ and any leaf L of $\mathcal{F}_\lambda$ there is a leaf L' of $\mathcal{F}'_\lambda$ such that $L = M \cap L'$. The germ of M' near M is canonically defined.*

Proof. Due to canonicity of M' it is enough to prove this statement locally on M. Thus we may assume that $\mathcal{F}_\bullet$ is weakly separating and separating. Consider the twistor transform of $\mathcal{F}_\bullet$. Since $\boldsymbol{n}_m$ may be identified with $\mathcal{N}\Sigma_m$, Proposition 13.5 is applicable. (The condition on global sections is automatically satisfied if $\mathfrak{T}$ is a twistor transform.) This provides a construction of M' and $\mathcal{F}'_\bullet$. □

Remark 13.8. This explains the choice of the term *airy*: it reasonable to imagine that M' is obtained from M by "blowing out" M. Here leaves of foliations $\mathcal{F}_\lambda$ work as "walls of microscopic air cells" in M. If M' is a Veronese web, and we consider just enough foliations $\mathcal{F}_{\lambda_k}$, $k = 1, \dots, K$, to uniquely determine $\mathcal{F}_\bullet$ (so $K = \dim M' + 1$) then after blow-out each cell becomes a tiny simplex in M'. Before "expansion" each cell is folded into a polytop of smaller dimension.

Proposition 13.7 immediately implies

Theorem 13.9. *Consider a complex-analytic separating web $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ on a connected manifold M such that $\boldsymbol{n}_m(\lambda)$ is a cohomologically trivial vector bundle over Λ. Then $\mathcal{F}_\bullet$ is airy iff $\dim M = \dim \Gamma(\Lambda, \boldsymbol{n}_m)$ for one (then any) $m \in M$.*

Remark 13.10. Note that if $\Lambda = \mathbb{P}^1$, then $\boldsymbol{n}_m(\lambda)$ is automatically cohomologically trivial (since by definition this vector bundle is induced from a Grassmannian).

Remark 13.11. In Section 15 we provide a somewhat inverse construction to Proposition 13.7: given M', we introduce a class of submanifolds $M \subset M'$ which are equipped with a web having the same twistor transform.

The arguments above give a more detailed proof of one of the principal results of [4]:

Theorem 13.12. *Complex-analytic Veronese webs are airy and are (uniquely up to a local diffeomorphism) locally determined by their twistor transform.*

This a direct corollary of

Lemma 13.13. *Consider a Veronese web $\{\mathcal{F}_\lambda\}_{\lambda\in\mathbb{P}^1}$ on a d-dimensional manifold M. Then $\boldsymbol{n}_m \simeq \mathcal{O}(d-1)$ for any $m \in M$.*

Proof. It is enough to show that $\mathcal{L} \simeq \mathcal{O}(-d+1)$; here the line bundle $\mathcal{L}$ is induced by the Veronese inclusion $j\colon \mathbb{P}^1 \to \mathbb{P}^{d-1}$ from the tautological line bundle on $\mathbb{P}^{d-1}$ (which is isomorphic to $\mathcal{O}(-1)$). The fiber of $\mathcal{L}$ over $\lambda \in \mathbb{P}^1$ is the 1-dimensional subspace of $\mathbb{V}^d$ corresponding to $j(\lambda)$.

A linear function l on $\mathbb{V}^d$ induces a section of $\mathcal{L}^*$, zeros of this section correspond to points on $\operatorname{Ker} l \cap \operatorname{Im} j$. Thus it is enough to construct a hyperplane in $\mathbb{P}^{d-1}$ which transversally intersects $\operatorname{Im} j$ in $d-1$ points. Since all the Veronese inclusions are projectively isomorphic, it is enough to consider one given by $(x : y) \mapsto (x^{d-1} : x^{d-2}y : \dots : xy^{d-2} : y^{d-1})$. Let $\Pi_{k=1}^{d-1}(t-k) = t^{d-1} + \sum_{k=0}^{d-2} a_k t^k$. Then the functional l with coordinates $(1, a_{d-2}, \dots, a_0)$ satisfies the condition above. □

In fact Veronese webs coincide (in complex-analytic situation) with separating airy smooth webs of codimension 1 with $\Lambda = \mathbb{P}^1$. One can also classify arbitrary separating airy smooth webs with $\Lambda = \mathbb{P}^1$, the result coincides with *Kronecker webs* as defined in [15].

14. Non-linear Riemann problem

Consider $0 < \varepsilon < 1$, $\delta > 0$, and a complex-analytic function $g(\lambda, t)$ defined for $\varepsilon < |\lambda| < 1/\varepsilon$ and $|t| < \delta$. Assume that for any given λ, $\varepsilon < |\lambda| < 1/\varepsilon$, the function $g(\lambda, t)$ is invertible. Glue domains $\left(\mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon\right) \times \mathbb{C}$ and $\mathbb{B}^1_{1/\varepsilon} \times \mathbb{B}^1_\delta$ together by gluing $(\lambda, t_+) \in \mathbb{B}^1_{1/\varepsilon} \times \mathbb{B}^1_\delta$ to $(\lambda, t_-) = (\lambda, g(\lambda, t_+)) \in \left(\mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon\right) \times \mathbb{C}$ for $\varepsilon < |\lambda| < 1/\varepsilon$ and $|t| < \delta$. The result is a 2-dimensional complex manifold $\mathfrak{T}$ with a surjective submersive mapping $\pi\colon \mathfrak{T} \to \mathbb{P}^1$ given by $(\lambda, t) \mapsto \lambda$.

Lemma 14.1.
1. *If $g(\lambda, 0) \equiv 0$, then π has a section given by $\lambda \mapsto (\lambda, 0)$;*
2. *Sections Σ of π can be identified with pairs of functions $\sigma_+ : \mathbb{B}^1_{1/\varepsilon} \to \mathbb{B}^1_\delta$ and $\sigma_-\colon \left(\mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon\right) \to \mathbb{C}$ such that $\sigma_-(\lambda) = g(\lambda, \sigma_+(\lambda))$ if $\varepsilon < |\lambda| < 1/\varepsilon$;*
3. *Given a section Σ of π associated to a pair $\sigma_\pm$, the degree of the normal bundle of Σ in $\mathfrak{T}$ is given by $-\operatorname{ind} \frac{\partial g}{\partial t}(\lambda, \sigma_+(\lambda))$. Here $\operatorname{ind}\varphi(\lambda) = \frac{1}{2\pi i}\oint_{|\lambda|=1} \frac{d\varphi(\lambda)}{\varphi(\lambda)}$.*

Proof. The first two statements are obvious. On the other hand, the normal bundle of Σ is canonically trivialized on $|\lambda| < 1/\varepsilon$ and on $|\lambda| > \varepsilon$, with the gluing function being $\frac{\partial g}{\partial t}(\lambda, \sigma_+(\lambda))$. To calculate the degree of a line bundle $\mathcal{L}$ it is enough to construct a section τ_+ in $|\lambda| \leq 1$ and a section τ_- in $|\lambda| \geq 1$. Suppose that $\tau_\pm(\lambda)$ have no zeros on $|\lambda| = 1$. Then $\tau_0 = \tau_-(\lambda)/\tau_+(\lambda)$ is a well-defined function on the unit circle with values in $\mathbb{C}^\times$, and $\deg \mathcal{L} = n_+ + n_- - \operatorname{ind} \tau_0$; here $n_\pm$ are numbers of zeros of $\tau_\pm$ (inside and outside of the unit circle correspondingly). In our case $n_+ = n_- = 0$, and $\tau_0 = \frac{\partial g}{\partial t}(\lambda, \sigma_+(\lambda))$. □

Theorem 14.2. *Consider a manifold K and a function $g(\lambda, t, \kappa)$, $\kappa \in K$, which depends analytically on parameters and such that for any given κ the function satisfies the condition in the beginning of this section. Suppose that for $\kappa_0 \in K$ one has $g(\lambda, 0, \kappa_0) \equiv 0$, and suppose that $\operatorname{ind} \frac{\partial g}{\partial t}(\lambda, 0, \kappa_0) = 1$. Then there exists $0 < \delta_1 < \delta$ and a neighborhood $K_1 \ni \kappa_0$, $K_1 \subset K$, such that for any $\kappa \in K_1$ the conditions*

$$\sigma_{-,\kappa}(\lambda) = g(\lambda, \sigma_{+,\kappa}(\lambda), \kappa) \text{ for } \varepsilon < |\lambda| < 1/\varepsilon, \qquad |\sigma_{+,\kappa}(\lambda)| < \delta_1 \text{ for } |\lambda| < 1/\varepsilon,$$

uniquely determine analytic functions $\sigma_{+,\kappa}(\lambda)$ defined for $|\lambda| < 1/\varepsilon$, and $\sigma_{-,\kappa}(\lambda)$ defined for $|\lambda| > \varepsilon$. Functions $\sigma_{\pm,\kappa}(\lambda)$ depend analytically on κ.

Proof. Glue domains $\mathbb{B}^1_{1/\varepsilon} \times \mathbb{B}_\delta \times K$ and $\left(\mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon\right) \times \mathbb{C} \times K$ together by gluing (λ, t, κ) to $(\lambda, g(\lambda, t, \kappa), \kappa)$ for $\varepsilon < |\lambda| < 1/\varepsilon$, $|t| < \delta$, and $\kappa \in K$. Denote the resulting manifold by $\mathfrak{T}$, denote by $\pi\colon \mathfrak{T} \to \mathbb{P}^1$ the mapping $(\lambda, t, \kappa) \mapsto \lambda$, by Π the natural projection $\mathfrak{T} \to K$, and by Σ the section of π given by $\lambda \mapsto (\lambda, 0, \kappa_0)$. Consider the normal bundle $\mathcal{N}\Sigma$ of Σ inside $\mathfrak{T}$. Let $\mathcal{N}^{(0)}\Sigma$ be the normal bundle of Σ inside $\Pi^{-1}(\kappa_0)$.

We know that $\deg \mathcal{N}^{(0)}\Sigma = -1$. On the other hand, $\mathcal{N}\Sigma/\mathcal{N}^{(0)}\Sigma$ is isomorphic to $\Pi^{*}\mathcal{T}_{\kappa_0}K$, thus is a trivial vector bundle over Σ. Thus both $\mathcal{N}^{(0)}\Sigma$ and $\mathcal{N}\Sigma/\mathcal{N}^{(0)}\Sigma$ are cohomologically trivial.

The exact sequence $\dots \to H^k\left(\Sigma, \mathcal{N}^{(0)}\Sigma\right) \to H^k\left(\Sigma, \mathcal{N}\Sigma\right) \to H^k\left(\Sigma, \mathcal{N}\Sigma/\mathcal{N}^{(0)}\Sigma\right) \to \dots$ shows that $\mathcal{N}\Sigma$ is also cohomologically trivial, and $\Gamma\left(\Sigma, \mathcal{N}\Sigma\right) \simeq \mathcal{T}_{\kappa_0}K$. In other words, the Kodaira–Spencer theory (Theorem 12.2) is applicable, and there is an mapping $\Sigma_{\bullet}\colon \mathbb{P}^1 \times M \to \mathfrak{T}$ and $m_0 \in M$, such that $\operatorname{Im}\Sigma_{m_0} = \Sigma$, and the associated infinitesimal family $\delta\Sigma_m\colon \mathcal{T}_{m_0}M \to \Gamma\left(\Sigma, \mathcal{N}\Sigma\right)$ is a bijection.

On the other hand, $\Pi \circ \Sigma_m\colon \mathbb{P}^1 \to K$ is a deformation of a constant mapping to a point $\kappa_0 \in K$, thus is a constant mapping itself. Denote the image-point of this constant mapping by $\kappa(m)$. It is clear that the derivative of $m \mapsto \kappa(m)$ coincides with the composition $\mathcal{T}_{m_0}M \to \Gamma\left(\Sigma, \mathcal{N}\Sigma\right) \to H^k\left(\Sigma, \mathcal{N}\Sigma/\mathcal{N}^{(0)}\Sigma\right) \simeq \mathcal{T}_{\kappa_0}K$, thus $\kappa(m)$ is a local diffeomorphism. Thus we can identify M with an open subset of K. We obtain a family of mappings $\Sigma_\kappa : \mathbb{P}^1 \to \mathfrak{T}$, $\kappa \in M \subset K$, such that $\Pi \circ \Sigma_\kappa$ is the constant mapping to $\kappa \in K$. In other words, Σ_κ is a section of $\pi|_{\Pi^{-1}\kappa}$, thus induces a pair of functions $\sigma_{\pm,\kappa}(\lambda)$.

This shows existence of solutions $\sigma_{\pm,\kappa}$, as well as the analytic dependence on parameters. Uniqueness follows from the other parts of Kodaira–Spencer theory (Theorem 12.2). ☐

Using Definition 0.5, one can restate Theorem 14.2 in the following way:

Corollary 14.3. *Consider a function $g(\lambda, t)$ defined for $\varepsilon < |\lambda| < 1/\varepsilon$ and $|t| < \delta$, such that $g(\lambda, 0) \equiv 0$ and $\operatorname{ind}\frac{\partial g}{\partial t}(\lambda, 0) = 1$. (Obviously, $\mathfrak{R}(g) = 0$.) Consider an analytic family $g_\kappa(\lambda, t)$, $\varepsilon < |\lambda| < 1/\varepsilon$, $|t| < \delta$, $\kappa \in U \subset \mathbb{C}^n$, such that $g_0 = g$. Then there is a neighborhood U_1 of 0 in U such that $\mathfrak{R}(g_\kappa)$ is defined for $\kappa \in U_1$ and $\mathfrak{R}(g_\kappa)$ depends smoothly on $\kappa \in U_1$.*

Remark 14.4. Since $\mathfrak{R}(g)$ does not change when ε increases, it is clear that $\sigma_+(\mu)$ for $|\mu| < 1/\varepsilon$ can be written in terms of $\mathfrak{R}$. For example, if $|\mu| < 1$, then $\sigma_+(\mu) = \mathfrak{R}\left(g\left(\frac{\lambda-\mu}{\mu\lambda-1}, t\right)\right)$.

Similarly, one can calculate $\sigma_-(\mu)$ by considering the inverse function for $g(\lambda, t)$ in t (with λ being a parameter) instead of $g(\lambda, t)$.

Consider now what changes if one takes the gluing functions $g(\lambda, t)$ with $g(\lambda, 0) \equiv 0$ and non-positive values of $\operatorname{ind}\frac{\partial g}{\partial t}(\lambda, 0)$ (as opposed to $\operatorname{ind} = 1$). In such a case $\deg \mathcal{N}\Sigma = d$ is non-negative, thus there is a $(d+1)$-parametric family of sections of π. By Proposition 12.8 we expect that a section is determined by its values at $d+1$ different points of $\mathbb{P}^1$.

Let us write the formula for the section in terms of g and $\mathfrak{R}$. Use notations of Definition 0.6.

Proposition 14.5. *Suppose that k numbers $\lambda_1, \dots, \lambda_k$ satisfy $0 < |\lambda_l| < 1$, m numbers $\mu_1, \dots, \mu_m$ satisfy $|\lambda_l| > 1$. Consider a pair of functions satisfying*

$$\sigma_-(\lambda) = g\left(\lambda, \sigma_+(\lambda)\right), \qquad |\sigma_+(\lambda)| < \delta \text{ for } |\lambda| < 1/\varepsilon, \tag{14.1}$$

and conditions $\sigma_+(\lambda_l) = a_l$, $l = 1, \dots, k$, $\sigma_-(\mu_l) = b_l$, $l = 1, \dots, m$. Suppose that $\operatorname{ind}\frac{\partial g}{\partial t}(\lambda, 0) = 1 - k - m$. Then $\sigma_+(0) = \mathfrak{R}\left(\mathcal{G}_{\Lambda\mathrm{M},\{a_i\}\{b_i\}}\right)$.

Proof. Indeed, one can write

$$\sigma_+(\lambda) = \widetilde{\sigma}_+(\lambda)\,F_+(\lambda) + \sum_{l=1}^{k} a_l F_{+,l}(\lambda), \qquad \sigma_-(\lambda) = \widetilde{\sigma}_-(\lambda)\,F_-(\lambda) + \sum_{l=1}^{m} b_l F_{-,l}(\lambda).$$

Then $\sigma_-(\lambda) = g\left(\lambda, \sigma_+(\lambda)\right)$ can be rewritten as $\widetilde{\sigma}_-(\lambda) = \mathcal{G}_{\Lambda\mathrm{M},\{a_i\}\{b_i\}}\left(\lambda, \widetilde{\sigma}_+(\lambda)\right)$, and $\sigma_+(0) = \widetilde{\sigma}_+(0)$. The only thing one needs to prove is that $\operatorname{ind}\frac{\partial \mathcal{G}_{\Lambda\mathrm{M},\{a_i\}\{b_i\}}}{\partial t} = 1$, which follows from $\operatorname{ind}F_+(\lambda) = k$, $\operatorname{ind}F_-(\lambda) = -m$. ☐

Proof of Theorem 0.7 . Correctness follows from Proposition 14.5. Show that w is non-degenerate. Suppose that $\partial w/\partial x = 0$ for some value of (x, y, z). Recall that $w(m)$, $m = (x, y, z)$, is the value of $\sigma(0, m)$; here $\sigma(\lambda, m)$ is a Kodaira–Spencer family of sections of $\mathfrak{T}$, and x, y, z are $\sigma(\lambda_{1,2,3}, m)$. If $\partial w/\partial x = 0$, this would mean that there is a one-parametric family of sections such that the infinitesimal family is non-vanishing, but infinitesimal family vanishes for $\lambda \in \{0, \lambda_2, \lambda_3\}$. However, by Lemma 13.13, the infinitesimal family is a section of $\mathcal{O}(2)$, thus cannot vanish at 3 distinct points.

Show that w is $(\lambda_1, \lambda_2, \lambda_3, 0)$-admissible. Glue domains $\mathbb{B}^1_{1/\varepsilon} \times \mathbb{B}_\delta$ and $\left(\mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon\right) \times \mathbb{C}$ together by gluing $(\lambda, t_+) \in \mathbb{B}^1_{1/\varepsilon} \times \mathbb{B}_\delta$ to $(\lambda, t_-) = \left(\lambda, g(\lambda, t_+)\right) \in \left(\mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon\right) \times \mathbb{C}$ for $\varepsilon < |\lambda| < 1/\varepsilon$, $|t_+| < \delta$. Call the resulting 2-dimensional manifold $\mathfrak{T}$. It is equipped with a projection π to $\mathbb{P}^1$ and a section $S = \{(\lambda, 0)\}$ of this projection. As in the proof of Lemma 14.1, one can show that degree of $\mathcal{N}S$ is 2. Since $\deg \mathcal{N}S \geq 0$, $\mathcal{N}S$ is cohomologically trivial, and fibers of $\mathcal{N}S$ are generated by global sections. By Proposition 13.5, a neighborhood U of S in $\mathfrak{T}$ is a twistor transform of a web of codimension 1 on a manifold M. Since $\dim \Gamma(S, \mathcal{N}S) = 3$, $\dim M = 3$. Again, $\deg \mathcal{N}S = 2$ implies that $\boldsymbol{n}_m(\lambda)$ of this web spans a quadratic cone in $\mathcal{T}^*_m M$, thus this web is a Veronese web.

Taking a point $u \in U$, $\pi(u) = \lambda \in \mathbb{P}^1$, gives a leaf of the foliation $\mathcal{F}_\lambda$ on M. Since fibers of U over λ_1, λ_2, λ_3 and $\lambda_4 = 0$ are identified with subsets of $\mathbb{C}$ by the construction of $\mathfrak{T}$, this gives 4 functions x, y, z, W on M, each constant on leaves of $\mathcal{F}_{\lambda_{1,2,3,4}}$. We may assume that $W = W(x, y, z)$ for an appropriate function W defined in a neighborhood of $0 \in \mathbb{C}^3$. By definition, the latter function is $(\lambda_1, \lambda_2, \lambda_3, 0)$-admissible.

On the other hand, a point $m \in M$ induces a section of π. A section of π which is close to S is determined by two functions σ_+ and σ_- which satisfy (14.1). By

Proposition 14.5 this section is determined by $x = \sigma_+(\lambda_1)$, $y = \sigma_+(\lambda_2)$, and $z = \sigma_-(\lambda_3)$, moreover, $\sigma_+(0) = w(x, y, z)$. We conclude that $w(x, y, z) = W(x, y, z)$.

This implies the first statement of the theorem. The second statement is a direct corollary of the first one.

Given (A, B, C), find $\lambda_{1,2,3,4} \in \mathbb{P}^1$ as in Remark 3.9. By a projective transform of $\mathbb{P}^1$ one can make $\lambda_4 = 0$, and $\lambda_3 = \infty$. By transformations $\lambda \mapsto c\lambda$ one can make $\lambda_{1,2}$ arbitrarily small. After this a transformation $\lambda \to \frac{\lambda}{1+\lambda/N}$ with $N \gg 0$ would produce a triple $\lambda_{1,2,3}$ with desired properties.

Given a non-degenerate solution $\widehat{w}(x, y, z)$ of the (A, B, C)-equation and $\lambda_{1,2,3}$ as found above, consider the 3-dimensional Veronese web defined by Theorem 3.8. Let $\widetilde{\mathfrak{T}}$ is the twistor transform of this web. Then $\widetilde{\pi}\colon \widetilde{\mathfrak{T}} \to \mathbb{P}^1$ is (locally near $\Sigma_{(0,0,0)}$) isomorphic to $\pi\colon \mathfrak{T} \to \mathbb{P}^1$; here $\mathfrak{T}$ is glued using the function $g(\lambda, t)$ defined in Theorem 10.1. It is clear that $\operatorname{ind} g = -\deg \mathcal{N}\Sigma_{(0,0,0)} = -2$.

Functions x, y, z, $\widehat{w}(x, y, z)$ on the manifold of this Veronese web define local coordinates on the fibers $\widetilde{\pi}^{-1}(\lambda)$, $\lambda \in \{\lambda_1, \lambda_2, \lambda_3, 0\}$, thus on $\pi^{-1}(\lambda)$. Denote these coordinates by the same symbols x, y, z, $\widehat{w}$. Investigate how these coordinates are related to coordinates t_+, t_- on two pieces of $\mathfrak{T}$ it is glued of.

Recall that Theorem 10.1 defined the coordinate t_- on $\left(\mathbb{P}^1 \smallsetminus \bar{\mathbb{B}}^1_\varepsilon\right) \times \mathbb{C}$ by taking z-coordinates of the intersection point of leaves of the foliations with γ_2, which is the z-axis. The leaves of the foliation $\mathcal{F}_{\lambda_3}$ are $z = \text{const}$, thus the coordinates z and t_- on $\pi^{-1}\{\lambda_3\}$ coincide, and there is no translation of the argument z of the function $\widehat{w}$. The coordinate t_+ over $\mathbb{B}^1_{1/\varepsilon}$ is induced by taking x-coordinate of the intersection point of leaves with γ_1. Thus the coordinates x and t_+ on $\pi^{-1}(\lambda_1)$ coincide, and there is no translation of the argument x. Similarly, the coordinates y and t_+ on $\pi^{-1}(\lambda_2)$ differ by the transformation $y = Y(t_+)$; here $y = Y(x)$, $z = 0$ are the equations of the curve γ_1.

Finally, the coordinate $\widehat{w}$ on $\pi^{-1}(0)$ is given by taking the value of $\widehat{w}(x, y, z)$ on the leaf of $\mathcal{F}_0$. The leaf which corresponds to a given value of t_+ passes through the point $(t_+, Y(t_+), 0)$, thus the corresponding value of $\widehat{w}$ is $\widehat{w}(t_+, Y(t_+), 0)$. $\square$

Remark 14.6. Consider the case when $w(x, y, z)$ is real for real values of x, y, z, and $\lambda_{1,2,3}$ are real. In such a case it is a meaningful question to reconstruct w basing on the Cauchy data on a hypersurface w.r.t. which the linearization is hyperbolic. As we have seen in Remark 10.3, the nonlinear Riemann problem with gluing data provided on a neighborhood of a circle is not enough to treat such a problem. One should be able to treat gluing data on more general regions.

However, the results of [13, 14] suggest that providing the gluing data of some kind *on the real axis alone* should provide enough information. Note that this gluing data should be more general than one we consider here, since a literal application of our arguments leads to a function $\frac{\partial g}{\partial t}$ with zeros and/or poles on the real axis.

15. Appendix on transversal sections of webs

Some statements of this section are stated in complex-analytic case only. To restate them in real-analytic case is straightforward. Additionally, there is a C^∞-treatment of some of these statements as well, see [13, 14].

Definition 15.1. Say that a submanifold $N \subset M$ is *transversal* to the web $\mathcal{F}_\bullet$ on M if it is transversal to any leaf of any foliation of the web.

If $\operatorname{codim} \mathcal{F}_\bullet = r$, then the usual count of dimensions shows that there are transversal varieties with dimensions down to $r + \dim \Lambda$. However, one should not expect them to exist is smaller dimensions, for example, Lemma 9.5 shows that there are no curves transversal to a complex-analytic Veronese web.

Obviously, a web $\mathcal{F}_\bullet$ on M cuts out a smooth web $\mathcal{F}_\bullet^{[N]}$ on a transversal submanifold $N \subset M$. Call this web the *transversal section* of $\mathcal{F}_\bullet$ by N. By definition of transversality, the germs of twistor transforms of $\mathcal{F}_\bullet$ and of $\mathcal{F}_\bullet^{[N]}$ near $m \in N$ coincide. This implies

Theorem 15.2. *Consider a separating airy web $\mathcal{F}_\bullet$ on M and a transversal to $\mathcal{F}_\bullet$ submanifold $N \subset M$. Then the transversal section web $\mathcal{F}_\bullet^{[N]}$ on N determines the germ of M near N and the web $\mathcal{F}_\bullet$ on this germ (uniquely up to diffeomorphisms $M \to \widetilde{M}$ which preserve N).*

Remark 15.3. Note that Theorem 10.1 and taken together with Theorem 13.12 imply a particular case of this statement: the Veronese web is locally determined by its restriction on the surface N given by $y = Y(x)$ (in terms of Theorem 10.1).

On the other hand, classification of $\mathcal{F}_\bullet^{[N]}$ up to diffeomorphism can be much easier than classification of $\mathcal{F}_\bullet$, since leaves of $\mathcal{F}_\bullet^{[N]}$ have a smaller dimension. If $\dim N = r + \dim \Lambda$, and $\dim \Lambda = 1$, then leaves of $\mathcal{F}_\bullet^{[N]}$ have dimension 1. But to specify a foliation on N of codimension $\dim N - 1$ is exactly the same as to specify a direction $\boldsymbol{d}_n \in \mathbb{P}\mathcal{T}_n N$ in a tangent space at every point $n \in N$, there is *no* integrability condition involved (as, for example, one in Lemma 3.3). A family of foliations induces a family of directions $\boldsymbol{d}_n(\lambda)$. (Note that $\boldsymbol{d}_n(\lambda)$ is a direction in a tangent space, not in a cotangent space, as is $\boldsymbol{n}_n(\lambda)$ in the case of webs of codimension 1.)

If Λ is a compact complex curve, then a mapping $\boldsymbol{d}\colon \Lambda \to \mathbb{P}\mathcal{T}_n N$ of given degree in general position is uniquely determined by images of P points on Λ for an appropriate $P > 0$. The standard arguments of algebraic geometry of curves show that for $\boldsymbol{d}_n$ one should expect this for $P \geq r + 2 + \frac{g+(r+1)(d-r-1)}{r}$; here g is the genus of Λ, $d = \dim M$. Additionally, if this inequality is an equality, then the images of these P points in $\mathbb{P}\mathcal{T}_n N$ should be expected to be arbitrary. Moreover, in the case $g = 0$, $r = 1$ these expectations can be easily checked to be true, as far as among these P images no more than $d - 1$ glue into any point of $\mathbb{P}^1$. Additionally, the condition of general position can be removed, if one allows the degree of the mapping $\boldsymbol{d}\colon \mathbb{P}^1 \to \mathbb{P}^1$ to drop. This leads to

Theorem 15.4. *Consider a complex-analytic Veronese web $\mathcal{F}_\bullet$ on M and a transversal surface $N \subset M$, $\dim N = 2$, $\dim M = d$. Consider $2d-1$ distinct points $\lambda_1, \dots, \lambda_{2d-1} \in \mathbb{P}^1$. Then the germ of $\mathcal{F}_\bullet$ near N is determined (uniquely up to a diffeomorphism preserving N) by $2d-1$ foliations $\mathcal{F}^{(k)} = \mathcal{F}^{[N]}_{\lambda_k}$, $k = 1, \dots, 2d-1$, of codimension 1 on N. The foliations $\mathcal{F}^{(k)}$ on N can be taken arbitrarily with the restriction that at any point of N no more than $d-1$ foliation have any given tangent direction, and there is no mapping $\boldsymbol{f}\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree less than $d-1$ such that $\boldsymbol{f}(\lambda_k) = \mathcal{T}_n\mathcal{F}^{(k)}$, $k = 1, \dots, 2d-1$.*

By a choice of coordinates on N one can take last two of these foliations to be $\{x = \text{const}\}$, $\{y = \text{const}\}$, the rest to be $\{w_k(x,y) = \text{const}\}$. Thus the collection $(M, N, \mathcal{F}_\bullet)$ (up to the same transformations as in the theorem) is determined by $2d-3$ functions on N.

The next step is to use the freedom in the choice of N to reduce the number of parameters. Accidentally, a proper choice of N also allows to ensure that no mapping like $\boldsymbol{f}$ exists.

Recall Lemma 9.5: given a Veronese web on M, a choice of a subset $T \subset \mathbb{P}^1$ with multiplicities and the total count $\dim M - 1$ determines a direction at each point of M. In particular, given T and $m_0 \in M$, there is a canonically defined curve $\gamma_{m_0,T} \ni m_0$ on M (one with the prescribed directions). Taking another subset T', one can put a curve $\gamma_{m,T'}$ through every point m of γ. Taken together, these curves $\gamma_{m,T'}$, $m \in \gamma_{m_0,T}$ sweep a surface $N_{m_0,T,T'}$ in M.

One can check that if $T \cap T' = \varnothing$, then $N_{m,T,T'}$ is transversal to $\mathcal{F}_\bullet$ at m, thus in a neighborhood of m. A proof of the following statement is straightforward:

Lemma 15.5. *Suppose that N is transversal to a Veronese web $\mathcal{F}_\bullet$ and contains a curve $\gamma_{m,T}$. Then γ is a leaf of $\mathcal{F}^{[N]}_\lambda$ for any $\lambda \in T$.*

In particular, for $N = N_{m,T,T'}$ the foliations $\mathcal{F}^{[N]}_\lambda$, $\lambda \in T'$, coincide. Since T' contains $d-1$ points, this condition ensures that no mapping $\boldsymbol{f}$ of degree smaller than $d-1$ can exist. Additionally, the leaves of the foliations $\mathcal{F}^{[N]}_\lambda$, $\lambda \in T$, which pass through m coincide. This leads to the following

Corollary 15.6. *Consider a complex-analytic surface N, a point $n \in N$, $2d-1$ distinct points λ_k on $\mathbb{P}^1$, and $2d-1$ foliations $\widetilde{\mathcal{F}}_k$, $k = 1, \dots, P$, on N. Let γ_k be the leaf of $\widetilde{\mathcal{F}}_k$ through n. Suppose that $\widetilde{\mathcal{F}}_k = \widetilde{\mathcal{F}}_{k'}$ if $1 \leq k, k' \leq d-1$, $\gamma_k = \gamma_{k'}$ if $d \leq k, k' \leq 2d-2$, that for any fixed k, $d \leq k \leq 2d-1$, the foliations $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_k$, $d \leq k \leq 2d-1$, have distinct directions at any point of N, and that directions of the foliations $\widetilde{\mathcal{F}}_k$, $d \leq k \leq 2d-1$, are not all the same at any point of N. Then there is a complex-analytic Veronese web $(M, \mathcal{F}_\bullet)$ of dimension d and an embedding $f\colon N \hookrightarrow M$ such that $\operatorname{Im} f = N_{f(n),T,T'}$, and that f identifies the foliations $\widetilde{\mathcal{F}}_k$ on N with foliations $\mathcal{F}^{[\operatorname{Im} f]}_{\lambda_k}$ on $\operatorname{Im} f$; here $T = \{\lambda_d, \dots, \lambda_{2d-2}\}$, $T' = \{\lambda_1, \dots, \lambda_{d-1}\}$. The germ of $(M, \mathcal{F}_\bullet)$ near $\operatorname{Im} f$ is determined uniquely up to isomorphism.*

This is a geometric local classification of complex-analytic Veronese webs: given $M \ni n$ and a Veronese web on M, $N = N_{n,T,T'}$ is canonically defined, thus foliations $\mathcal{F}^{[N]}_{\lambda_k}$ are canonically defined. These foliations satisfy the conditions of the corollary, and allow reconstruction of the web on M. In addition to the restriction that M is defined only as a germ near N, there is another direction of locality in this result: N can be embedded into M, not included into M.

One can make appropriate modifications to this statements if some of the points λ_k, $1 \leq k \leq d-1$ or $d \leq k \leq 2d-2$ can collide. In any case, count the number of parameters in this representation. It is enough to specify $\widetilde{\mathcal{F}}_1$ and $\widetilde{\mathcal{F}}_k$, $d \leq k \leq 2d-1$, one can suppose that $\widetilde{\mathcal{F}}_1$ is $\{x = \text{const}\}$, $\widetilde{\mathcal{F}}_d$ is $\{y = \text{const}\}$. Then one can write $\widetilde{\mathcal{F}}_k$ as $\{w_k(x,y) = \text{const}\}$, $d+1 \leq k \leq 2d-1$. Assume that the point n is given by $x = y = 0$. As in [4, 5, 3], one can normalize $w_{2d-1}(x,y) = x + y + xy\, u_{2d-1}(x,y)$. Additionally, one can normalize $w_k(x,y)$ by $w_k(0,y) = y$. Since $\frac{dw_k(x,0)}{dx} = 0$, one can write $w_k(x,y) = y + xy\, u_k(x,y)$. The functions $u_k(x,y)$ are defined uniquely up to a transformation

$$\widetilde{u}_k(x,y) = C^{-1} u_k(Cx, Cy), \qquad d+1 \leq k \leq 2d-1.$$

Thus an analytic d-dimensional Veronese web on M near a point $m \in M$ is locally uniquely determined by $d-1$ functions $u_k(x,y)$ up the transformation above.

Note the similarity of this description with the Turiel classification of Veronese webs [13, 14]. In fact what we did above is just a geometric reformulation of this result. Unfortunately, our approach works in an analytic situation only, and does not imply the C^∞-case of the Turiel classification.

Remark 15.7. In the case of arbitrary separating airy webs and $\dim N = r + \dim \Lambda$, here $r = \operatorname{codim} \mathcal{F}_\bullet$, it is not feasible to describe transversal sections of webs by specifying several foliations on N, since these foliations should satisfy too many conditions. However, if $\operatorname{codim} \mathcal{F}_\bullet = 1$ (so there are no integrability conditions on $\mathcal{F}^{[N]}_\bullet$) one can make a substitution. The mapping $\boldsymbol{n}_n\colon \Lambda \to \mathcal{T}^*_n N$ induces a line bundle $\mathcal{L}_n = \boldsymbol{n}^*_n \mathcal{O}(1)$ over Λ, and an inclusion $\iota_n\colon \mathcal{T}_n N \hookrightarrow \Gamma(\Lambda, \mathcal{L}^*_n)$.

Suppose that Λ is a compact curve, then the latter space is finite-dimensional. Since ι_n (up to multiplication by a constant) determines $\boldsymbol{n}_n$, it is enough to provide enough information to describe $\mathcal{L}_n$ and ι_n. Since we are free to multiply ι_n by a constant, it is enough to know $\mathcal{L}_n$ up to isomorphism. We can see that to describe $\mathcal{F}^{[N]}_\bullet$, it is enough to describe the degree δ of $\mathcal{L}_\bullet$, provide the mapping $\boldsymbol{l}_\bullet\colon N \to \operatorname{Pic}^\delta(\Lambda)$ which sends n to the class of $\mathcal{L}_n$ inside the Picard variety, the mapping $\tau\colon N \to \operatorname{Gr}_2(\Gamma(\Lambda, l_n)) : n \mapsto \operatorname{Im} \iota_n \subset \Gamma(\Lambda, l_n)$, and an identification of $\mathcal{T}_n N$ with the 2-dimensional vector subspace described by $\tau(n)$ up to a constant. In general position, given l and τ, one needs to know $\mathcal{F}^{[N]}_\lambda$ for 3 values of λ to provide such an identification.

If $\Lambda = \mathbb{P}^1$, then δ is a number, and $\operatorname{Pic}^\delta(\Lambda)$ has one point only. It is clear that $\mathcal{L}_n \simeq \mathcal{O}(\dim M - 1)$, so it is enough to provide 3 foliations on N, and a mapping

$N \to \mathrm{Gr}_2\left(\mathbb{V}^{\dim M}\right)$. It is easy to see that this data is equivalent to the data of Theorem 15.4.

Consider now a transversal submanifold N to a web $\mathcal{F}_\bullet$, and a submanifold $\gamma \subset N$ of codimension $\operatorname{codim} \mathcal{F}_\bullet$. Let $m \in \gamma$. If $\mathcal{T}_m\gamma \subset \mathcal{T}_m N$ is in general position, then $\mathcal{T}_m\gamma$ is transversal to $\mathcal{T}_m\mathcal{F}_\lambda$ for $\lambda \in \Lambda \smallsetminus Z$, here Z is a proper analytic subset of Λ. Consequently, a small neighborhood of m in γ is transversal to $\mathcal{F}_\lambda$ for λ in an open subset $U \subset \Lambda$. Reducing M to a neighborhood of m, we obtain the corresponding sectional coordinate system on $\mathfrak{T}$. Given two such submanifolds γ_1, γ_2, $m \in \gamma_1 \cap \gamma_2$ we obtain two subsets $U_{1,2} \subset \Lambda$, and the corresponding local identifications $g_\lambda \colon \gamma_1 \to \gamma_2$, $\lambda \in U_1 \cap U_2$. This identification are obtained in the same way as in Section 9, the principal difference being that the whole construction is performed on N instead of M.

Taking enough γ_k to cover Λ, the corresponding pairwise gluing functions determine the germ of $\mathfrak{T}$ near Σ_m up to isomorphism, thus the germ of $\mathcal{F}_\bullet$ near m up to isomorphism (assuming $\mathcal{F}_\bullet$ is airy). Note that to construct g_λ, we need to find a leaf of $\mathcal{F}_\lambda^{[N]}$ which passes through a given point of γ_1, and find the intersection of this leaf with γ_2. Obviously, to do this it is enough to solve some ordinary differential equations.

Consequently, the construction of Theorem 10.1 can be generalized to arbitrary webs.

16. Appendix on computational complexity of the nonlinear Riemann transform

Continue using notations of Section 14. Consider not the mapping $\mathfrak{R} \colon g(\lambda, t) \mapsto \sigma_+(0)$, but a more general mappings $\widetilde{\mathfrak{R}}_\pm \colon g(\lambda, t) \mapsto \sigma_\pm(\lambda)$. Let us introduce operators solving the linear Riemann problem: given a function $\varphi(\lambda)$ defined for $\varepsilon < |\lambda| < 1/\varepsilon$, define functions $\mathbb{H}_+\varphi$ and $\mathbb{H}_-\varphi$ by the conditions $\varphi(\lambda) = \lambda\mathbb{H}_+\varphi(\lambda) + \mathbb{H}_-\varphi(\lambda)$ and the conditions that $\mathbb{H}_+\varphi(\lambda)$ and $\mathbb{H}_-\varphi(\lambda)$ can be holomorphically extended on $|\lambda| < 1/\varepsilon$ and $|\lambda| > \varepsilon$ correspondingly. Similarly, if $\varphi(\lambda)$ is nowhere 0, and $\operatorname{ind}\varphi = 0$, define $\mathbb{M}_+\varphi$ and $\mathbb{M}_-\varphi$ by $\varphi(\lambda) = \mathbb{M}_+\varphi(\lambda)/\mathbb{M}_-\varphi(\lambda)$ and the conditions that $\mathbb{M}_+\varphi(\lambda)$ and $\mathbb{M}_-\varphi(\lambda)$ can be holomorphically extended on $|\lambda| < 1/\varepsilon$ and $|\lambda| > \varepsilon$ correspondingly, these extensions are nowhere 0, and $\mathbb{M}_+\varphi(0) = 1$.

Uniqueness of $\mathbb{M}_+\varphi$ and $\mathbb{M}_-\varphi$ is obvious, existence follows from the theory of the linear Riemann problem—or, what is the same, classification of line bundles over $\mathbb{P}^1$. Uniqueness of $\mathbb{H}_+\varphi$ and $\mathbb{H}_-\varphi$ is obvious, existence follows from existence of $\log \mathbb{M}_+ e^\varphi$ and $\log \mathbb{M}_- e^\varphi$.

Lemma 16.1. *Denote by S^1 the circle $|\lambda| = 1$. Then $\mathbb{H}_\pm\varphi|_{S^1}$ is uniquely determined by $\varphi|_{S^1}$. This induces two linear operators on real-analytic complex-valued functions on S^1. These operators can be extended to continuous linear operators in Sobolev spaces $H^s(S^1)$ for any $s \in \mathbb{R}$.*

Proof. The first statement is obvious, since $\varphi|_{S^1}$ uniquely determines φ. The second statement follows from $\mathbb{H}_+\lambda^k = c_k\lambda^{k-1}$ and $\mathbb{H}_-\lambda^k = c'_k\lambda^{k-1}$ with c_k and c'_k being 0 or 1, and from the fact that $\left(\lambda^k\right)_{k\in\mathbb{Z}}$ is an orthogonal basis in the Sobolev spaces $H^s(S^1)$. □

Denote the continuations of operators $\mathbb{H}_\pm$ into $H^s(S^1)$ by the same symbols. Similarly, if $s > 1/2$, then the mappings $\mathbb{M}_\pm$ can be considered as continuous mappings from an open subset of $H^s(S^1)$ into $H^s(S^1)$. Indeed, if $s > 1/2$, then $\varphi \mapsto e^\varphi$ is a continuously differentiable mapping $H^s(S^1) \to H^s(S^1)$ with an open image.

Lemma 16.2. *Consider $\varepsilon, \delta > 0$ and a family $g_\kappa(\lambda, t)$, $\kappa \in K$, of functions such that $\mathfrak{R}_{\varepsilon\delta}(g_\kappa)$ is well-defined for any $\kappa \in K$. Let $\sigma_{\pm,\kappa}(\lambda) = \widetilde{\mathfrak{R}}_\pm(g_\kappa)$. Then*

$$\frac{\partial}{\partial\kappa}\widetilde{\mathfrak{R}}_\pm(g_\kappa) = a_{\pm,\kappa}^{-1}\frac{\partial b_{\pm,\kappa}}{\partial\kappa},$$

here

$$a_{\pm,\kappa}(\lambda) = \mathbb{M}_\pm\left(\lambda^{-1}\frac{\partial g_\kappa}{\partial t}(\lambda, \sigma_{+,\kappa}(\lambda))\right), \qquad b_{\pm,\kappa}(\lambda) = \mathbb{H}_\pm\left(a_{-,\kappa}(\lambda)\, g_\kappa(\lambda, \sigma_{+,\kappa}(\lambda))\right).$$

Proof. Fix $\kappa_0 \in K$. We may assume that $\dim K = 1$, for example, $K = \mathbb{B}^1_r$. Let $\sigma_\pm = \widetilde{\mathfrak{R}}_\pm(g_{\kappa_0})$, $\delta_\pm = \frac{d}{d\kappa}\widetilde{\mathfrak{R}}_\pm(g_\kappa)|_{\kappa_0}$. Then

$$\frac{\partial g_{\kappa_0}}{\partial t}(\lambda, \sigma_+(\lambda))\,\delta_+(\lambda) + \frac{\partial g_\kappa}{\partial\kappa}|_{\kappa_0}(\lambda, \sigma_+(\lambda)) = \delta_-(\lambda).$$

Since $\operatorname{ind}\frac{\partial g}{\partial t} = 1$, one can write $\frac{\partial g_{\kappa_0}}{\partial t}(\lambda, \sigma_+(\lambda))$ as $\lambda a_+(\lambda)/a_-(\lambda)$, here $a_+(\lambda)$ and $a_-(\lambda)$ have invertible holomorphic continuations into $|\lambda| < 1/\varepsilon$ and $|\lambda| \geq \varepsilon$ correspondingly. Similarly, write

$$a_-(\lambda)\frac{\partial g_\kappa}{\partial\kappa}|_{\kappa_0}(\lambda, \sigma_+(\lambda)) = \lambda B_+(\lambda) + B_-(\lambda),$$

here $B_+(\lambda)$ and $B_-(\lambda)$ have holomorphic continuations into $|\lambda| < 1/\varepsilon$ and $|\lambda| \geq \varepsilon$ correspondingly. Then $\sigma_+(\lambda) = a_+^{-1}B_+$, $\sigma_-(\lambda) = a_-^{-1}B_-$. Since operators $\mathbb{H}_\pm$ are linear and continuous in an appropriate topology, it is easy to see that $B_\pm = \frac{db_{\pm,\kappa}}{d\kappa}|_{\kappa_0}$. □

Consider a vector space $V = H^s(S^1) \times H^s(S^1)$, $s \geq 1/2$. Denote the element of V by (σ_+, σ_-). In the conditions of Lemma 16.2 suppose that $\dim K = 1$. Define a mapping $v_\kappa \colon U \to V \colon (\sigma_+, \sigma_-) \mapsto (\delta_+, \delta_-)$, here U is an appropriate open subset of V, and $\delta_\pm(\lambda) = a_{\pm,\kappa}(\lambda)^{-1} B_{\pm,\kappa}(\lambda)$,

$$a_{\pm,\kappa}(\lambda) = \mathbb{M}_\pm\left(\lambda^{-1}\frac{\partial g_\kappa}{\partial t}(\lambda, \sigma_+(\lambda))\right), \qquad B_{\pm,\kappa}(\lambda) = \mathbb{H}_\pm\left(a_-(\lambda)\frac{dg_\kappa}{d\kappa}(\lambda, \sigma_+(\lambda))\right).$$

Since one can take value of elements of $H^s(S^1)$, $s > 1/2$, at points, it makes sense to require that $|\sigma_+(\lambda)| < \delta$ for any λ, thus $v_\kappa(\sigma_+, \sigma_-)$ is indeed well-defined on an open subset of $H^s(S^1)$. Moreover, v_κ is Lipschitz on $K \times \mathbb{B}$, here $\mathbb{B}$ is any ball in $H^s(S^1)$.

Corollary 16.3. *Given a family $g_\kappa(\lambda, t)$, $\kappa \in K$, $\dim K = 1$, of functions as in Definition 0.5, one can define a Lipschitz family v_κ of vector fields on an open subset U of $H^s(S^1) \times H^s(S^1)$ such that if $\widetilde{\mathfrak{R}}_\pm(g_\kappa)$ makes sense for any $\kappa \in K$, then the curve $\left(\widetilde{\mathfrak{R}}_+(g_\kappa), \widetilde{\mathfrak{R}}_-(g_\kappa)\right)$ is an integral curve of the ODE $\frac{d\Phi(\kappa)}{d\kappa} = v_\kappa(\Phi)$.*

Now Lipschitz ODEs in Banach spaces enjoy most of the properties of finite-dimensional ODEs, and are not harder to solve. We conclude that in the setting of Corollary 14.3 one can calculate $\mathfrak{R}(g_\kappa)$ by solving a Lipschitz ODE in a Hilbert space. In particular, Theorem 0.7 reduces solution of Cauchy problem for the nonlinear wave equation to solution of such an ODE.

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