THE ALEXANDER—SPANIER COHOMOLOGY AS A PART OF CYCLIC COHOMOLOGY

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Abstract. Basing on a sheaf $\mathcal{O}$ with a fixed section 1 on a manifold M we introduce the notions of the de Rham, cyclic and Hochschild cohomological complexes of the Alexander—Spanier type for M with coefficients in $\mathcal{O}$. We show when these complexes are quasi-isomorphic to the usual cohomology of M and how to build cocycles for these complexes basing on cocycles for M. If $\mathcal{O}$ is a sheaf of algebras with a trace on the ring $\mathcal{A}$ of global sections, we construct mappings from these complexes to the corresponding cohomology of $\mathcal{A}$. In the case of the ring of pseudodifferential operators these mappings are isomorphisms if we consider cyclic or Hochschild complexes.

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0. Introduction

In the last couple of years there was a big progress in construction of cocycles for non-commutative algebras with local multiplication. In fact the first results in this direction were achieved a long time ago, when there appeared a description of cohomology of algebras of differential or pseudodifferential operators ([**?BryGet**], [**?Wod**]). However, these description were nonconstructive, so the first sign of the progress was the description of one particular Lie-algebraic cocycle of the Lie algebra of pseudodifferential operators with a use of the symbol for $\log \partial$[**?KheKra**].

It was a very easy task to pinpoint the topological origin of the Khesin—Kravchenko construction, and it seems now that the generalization of this construction is a common knowledge between specialists. The description of the cohomology obtained in the "ancient" papers [**?BryGet**], [**?Wod**] shows that there is a tight connection between the cohomology of the support of the algebra and the cohomology of the algebra itself. So the generalizations assign to a topological cocycle of some kind an algebraic cocycle. The best candidates for that are Čech cohomology and de Rham cohomology.

The discussion below has two targets: to give the simplest examples of the cocycles we will obtain later and to provide the reader with euristics why these cocycles are in the best cases nontrivial. We do not restrict ourselves to be *absolutely* correct with the second target, therefore the reader who needs proofs should skip all the vague arguments like "if some conditions of non-degeneracy are satisfied … ". However, even in this section any construction of cocycles is still correct, hence even the most demanding reader *can* get something if he will not skip to the section 1.

0.1. A construction of 2-cocycles. Let us give a construction of a 2-cocycle as an example. Consider a manifold M over a field k and a sheaf $\mathcal{O}$ of associative algebras with units on M. Let $\mathcal{A}$ be $\Gamma(M, \mathcal{O})$, and suppose that there is a trace on the algebra $\mathcal{A}$, i.e., a linear functional $\operatorname{Tr} : \mathcal{A} \to k$ such that

$$\operatorname{Tr}(ab - ba) = 0$$

for any two elements $a, b \in \mathcal{A}$. The best example would be the sheaf $\mathcal{D}$ of differential operators, however, this sheaf allows only trivial trace $\operatorname{Tr} a = 0$. We will explain how to correct this deficiency later, when we use *pseudodifferential operators.*

We can consider (though approximately) a differential operator or a pseudodifferential operator as a function on a cotangent bundle. In the same way the trace on pseudodifferential operators is an analogue of integration of functions on a symplectic manifold. Therefore the reader should now imagine that there is some noncommutative deformation of the sheaf of functions on a manifold, and that the *integration* of functions deforms to a non-trivial trace on this algebra. Or, if the reader is too recalcitrant, he should consider instead *any* sheaf of algebras with a global trace. What we want to do is to construct a morphism from $H^1(M,k)$ to $H^2_{\mathrm{Lie}}(\mathcal{A},k)$. As we see below, in good cases this morphism is an isomorphism.

We stole the following innocent statement from [?**KhesKra91Coc**] (though it is present there only virtually): let $X \in \mathcal{A}$ and $c_1 \colon \mathcal{A} \to k$ given by

$$c^1 \colon A \mapsto \operatorname{Tr} X \cdot A \tag{0.1}$$

be a 1-cochain for $\mathcal{A}$ (here we consider, say, cochain complex for the Lie algebra that correspond to $\mathcal{A}$). Then we can rewrite a coboundary of c^1 Label equ0.3,

$$dc^1 \colon \mathcal{A} \otimes \mathcal{A} \to k \colon \ (A,B) \mapsto \operatorname{Tr} X \cdot [A,B]$$

as

$$dc^1(A,B) = \operatorname{Tr}[X,A] \cdot B. \tag{0.2}$$

Let us note that we can represent any 1-cochain on $\mathcal{A}$ in the form (0.1) if the trace on $\mathcal{A}$ is "sufficiently nondegenerate". Therefore under this condition of nondegeneracy *any* 2-*coboundary* for $\mathcal{A}$ can be written in the form (0.2). Moreover, the cochain (0.2) is remarkable by its *locality* property: let us call a 2-cochain c^2 local if there is a mapping Label equ0.6,

$$\mathcal{X} \colon \mathcal{O} \to \mathcal{O} \colon \Gamma(U,\mathcal{O}) \ni \varphi \mapsto \mathcal{X}(\varphi) \in \Gamma(U,\mathcal{O})$$

such that $c^2(A,B) = \operatorname{Tr} \mathcal{X}(A) \cdot B$.

It is clear that on local cochains the closeness is a *local property*: if we have a covering $\mathfrak{U}$ of M and a set of closed local cochains on $\mathcal{O}|_U$, $U \in \mathfrak{U}$, that are *restrictions*[1] of some global cochain, then this cochain is also closed. The following step we want to do now is to construct a local cochain that does not correspond to any global section X. By the locality property it is closed, and since it does not correspond to any section, it cannot be a coboundary. Therefore it is a nontrivial cocycle!

Moreover, we want to do it for an arbitrary class in $H^1(M,k)$. We want here to consider a geometric realization of this cocycle as the intersection index with an (coorientable) hypersurface $H \subset M$. Consider a pair of tubular neighborhoods U_1, U of H such that $\overline{U}_1 \subset U$ and a section X_1 on U that is identically 0 near one

[1]I.e., a local cochain coincides with the global on local sections with compact support, and such sections for different U generate the set of global sections.

boundary of U and identically 1 near another. Let X_2 be a 0 section on $M \smallsetminus \overline{U}_1$. The sections $X_{1,2}$ define by (0.2) two local cochains on their domains,[2] and these cochains "coincide" on the intersection of these domains. As we explained it above, that determines a cochain *on* M, and in order this cochain to be a coboundary, the section X_1 should extend to the entire M as a local constant (i.e., as a section in $k \subset \mathcal{O}$). We can write this cochain as

$$c^2(M, X_{1,2}):\ (A, B) \mapsto \operatorname{Tr} \mathcal{X}(A) \cdot B, \quad \mathcal{X}(A) \stackrel{\text{def}}{=} [X_{1,2}, A].$$

In the definition of $\mathcal{X}$ we should take a different function X_1 or X_2 depending on the region of M we are currently in—the result $\mathcal{X}(A)$ does not depend on the choice anywhere a choice is possible.

If H divides M into two parts, then X_1 can be extended into one part as 0 and into another as 1. However, in this case H represent a trivial cohomology class. Therefore we constructed a promised mapping

$$H^1(M, k) \to H^2_{\text{Lie}}(\mathcal{A}, k).$$

The cheating in this construction is the choice of the section X_1. If $\mathcal{O}$ is indeed the isomorphic as a sheaf to the sheaf of functions, and M is a C^∞-manifold, then there is no problem in providing such a section. Otherwise the notion of such a section is correctly defined (since $\mathcal{A}$ is an algebra with unity, there is a constant subsheaf $k \subset \mathcal{O}$, so there is a notion of section being locally 0 or locally 1), but to find it we need some additional "nice" properties of the sheaf $\mathcal{O}$, like $\mathcal{O}$ being soft.

We can consider $X_{1,2}$ as a (global) section of the sheaf $\mathcal{O}/k$. Then the discussion above can be rewritten in one phrase: the mapping

$$\mathcal{X}\colon \mathcal{O} \to \mathcal{O}\colon A \to [X, A]$$

is correctly defined even in the case when X is not an element of $\mathcal{A} = \Gamma(M, \mathcal{O})$, but an element of $\Gamma(M, \mathcal{O}/k)$, and the sequence

$$0 \to k = \Gamma(M, k) \to \mathcal{A} = \Gamma(M, \mathcal{O}) \to \Gamma(M, \mathcal{O}/k) \to H^1(M, k) \to H^1(M, \mathcal{O}) \to \dots$$

is exact. However, as the following generalization shows, this abstraction is too concrete to sustain useful modifications.

0.2. 2-cocycles for pseudodifferential symbols. As we will see later (when we give a precise definition of a pseudodifferential symbol on a circle), this ring is a ring of global section over a product of two circles: one ordinary, another infinitesimal. This manifold has 2-dimensional space H^1, therefore we can construct two 2-cocycles. However, this two 2-cocycles correspond to different geometrical objects (since the radii of the circles are so different), therefore we need two slightly different constructions.

[2]More precise, on the rings of global sections with compact support on their domains.

Example 0.1. Consider the sheaf of pseudodifferential symbols on a circle S^1. We consider them as "functions" $\varphi(x,\xi)$ on the cotangent bundle T^*S^1. In fact these functions are just asymptotic expansions when $\xi \to \infty$, so they are defined on the infinitesimal neighborhood of the infinity in the cotangent bundle. There are two classes in H^1 of this manifold: one corresponds to a hypersurface $x = \text{const}$, another one to

$$\xi = \text{very-very big const}.$$

Consider a first one of these two classes and the corresponding function X_1. We can suppose that X_1 depends only on x, and that it has a "jump" near the point $x = 0$. Now we want to expand X_1 to be as near as it is possible to a function on a circle, i.e., a function with period 1. This function (where defined) is 0 if $x < -c$, is 1 if $x > c$. Let us extend it as 0 on the interval $-1 + c < x < -c$ and as 1 on the interval $c < x < 1 - c$. Now this function is already non-periodic, but it satisfies the relation

$$X_1(x+1) = X_1(x) + 1$$

instead. Moreover, we can uniquely extend it to a function $\widetilde{X_1}$ on the entire line leaving this relation true. However, since for any function $A(x,\xi)$ with period 1 in x the expression

$$\left[\widetilde{X_1}, A\right]$$

is periodic with period 1, we can still apply the formula (0.2) and get a 2-cocycle

$$(A,B) \mapsto \operatorname{Tr}\left[\widetilde{X_1}, A\right] \cdot B.$$

(And we do not need to know the precise law of multiplication for pseudodifferential operators, the only thing we need to know is the translation-invariance of this multiplication.)

However, we can still simplify this formula a lot. Let as note that an addition of a periodical function to $\widetilde{X_1}$ results in changing this cocycle by a coboundary, as the formula (0.2) shows. Therefore we can substitute the function x instead of $\widetilde{X_1}(x)$, since $\widetilde{X_1}(x) - x$ is a periodical function. We result in the following formula for a cocycle:

$$(A(x,\xi), B(x,\xi)) \mapsto \operatorname{Tr}[x,A] \cdot B = -\operatorname{Tr}\frac{\partial A}{\partial \xi} \cdot B.$$

Example 0.2. To deal with the second case is a little bit more tricky, especially since we cannot formulate precisely what we mean by "very-very big const". Let us proceed first as in the first example. Consider a hypersurface $\xi = \text{const}$ and a corresponding function X_1. The big problem is that the functions we consider should also have good symmetry properties. In the previous example they should have been invariant with respect to translation in x, here they should have a good decomposition with respect to the action of expansions in ξ, as the definition of a psudodifferential symbol shows.

One way to circumvent this is to consider a family of surfaces that are "approximately invariant" with respect to expansions in ξ, say

$$\xi = \text{const} \cdot \alpha^k, \quad k \in \mathbb{Z},\ \alpha > 1.$$

The corresponding function X_1 is locally constant away from these surfaces and has a "jump" 1 near any one of them. This modification is in direct analogy with the step from a locally defined function X_1 to an "almost periodical" function $\widetilde{X}_1$.

This function X_1 satisfies the property

$$X_1\left(\alpha^k \xi\right) = X_1(\xi) + k$$

of "almost-invariance" with respect to a discrete group of expansions. If we consider instead of a discrete family of hypersurfaces a "continuous family", or if we take the limit $\alpha \to 1$ with the corresponding scaling of X_1, we get a function

$$X_1(\xi) = \log \xi.$$

If the reader believes what was discussed so far, he should understand now that the formula

$$(A, B) \mapsto \operatorname{Tr}[\log \xi, A] \cdot B$$

is correct, defines a cocycle for Lie algebra of pseudodifferential operators, and that this cocycle cannot be a coboundary (since $\log \xi$ is not a pseudodifferential symbol). Moreover, it should be clear that the classes of two defined cocycles are linearly independent, since no linear combination of x and $\log \xi$ is simultaneously periodic and a sum of homogeneous in ξ functions.

Remark 0.1. The second cocycle has certain advantages comparing to the first. While the first cocycle is trivial after restriction on the ring of differential operators, the second one gives (the only nontrivial) 2-cocycle for this ring. This is a reason why the much simpler first cocycle was missed so far—and while it is discovered, the discussed in this paper theory becomes almost obvious.

We want to note also that though it is possible to *consider* the second cocycle on differential operators only, to *define* it we need pseudodifferential symbols.

0.3. 3-cocycles and 4-cocycles. Here we want to construct a generalization of the above construction to higher codimensions. Again, we want to begin with constructions of (local) cochains and coboundaries.

Call an n-cochain c on $\mathcal{A}$ a *local cochain* if

$$c\,(A_1, \dots, A_n) = 0 \text{ if } \bigcap_{i=1}^{n} \operatorname{Supp} A_i = \varnothing.$$

Suppose that the sheaf of algebras $\mathcal{O}$ is isomorphic to a sheaf of functions on M. In this case such a cochain is just a skew-symmetric generalized function with a support on a diagonal in M^n. Locally we can write any such function (i.e., a functional on the space of functions) as a linear combination of the terms

$$A_1 \otimes \dots \otimes A_n \mapsto \operatorname{Tr} \operatorname{Alt} \mathcal{D}_1 A_1 \cdot \ldots \cdot \mathcal{D}_n A_n,$$

and

$$A_1 \otimes \dots \otimes A_n \mapsto \operatorname{Tr} \operatorname{Alt} \mathcal{D}_1 A_1 \cdot \ldots \cdot \mathcal{D}_{n-1} A_{n-1} \cdot f_0 A_n,$$

where $\mathcal{D}_i$ are differential operators without a term of degree 0, and f is a function on M. Now suppose that the product on $\mathcal{O}$ is a deformation of the commutative product on the sheaf of functions with respect to a non-degenerate Poisson structure. In this case we can write the operator $\mathcal{D}_i$ as a composition of vector fields, i.e., of Poisson brackets with functions on M. We can see that in this case we can write any local cochain as

$$A_1 \otimes \dots \otimes A_n \mapsto \operatorname{Tr} \operatorname{Alt} \left[f_1^1, \left[f_1^2, \left[\dots, \left[f_1^{k_1}, A_1\right]\right]\right]\right] \cdot \ldots \cdot \left[f_{n-1}, \left[\dots \left[f_{n-1}^{k_{n-1}}, A_{n-1}\right]\right]\right] \cdot f_0 A_n,$$

or as the analogous expression without f_0. Now we can write any commutator as a difference of products, therefore any such function can be written as

$$A_1 \otimes \dots \otimes A_n \mapsto \operatorname{Tr} \operatorname{Alt} f_1 \cdot A_1 \cdot f_2 \cdot A_2 \cdot \ldots \cdot f_n \cdot A_n.$$

Therefore we obtained a general formula for local cocycles, and we can write a general formula for local coboundaries (all under the above assumptions). If we avoid the question of a local cochain being a coboundary, but of non-local cochain only, then to construct a non-trivial cocycle we can try to find a local coboundary that is not a global coboundary. To do this we need to fix a geometrical realization of a class of cohomology on M, say a submanifold in M.

Suppose that codimension is 2. Let X_1, X_2 be two functions on M. Consider a cochain

$$c^2_{\{X_i\}}\,(A_1, A_2) = \operatorname{Tr} \operatorname*{Alt}_{\sigma,\tau \in \mathfrak{S}_2} X_{\sigma_1} \cdot A_{\tau_1} \cdot X_{\sigma_2} \cdot A_{\tau_2}.$$

Then we can write a coboundary of this cochain as

$$dc^2_{\{X_i\}}(A_1, A_2, A_{n+1}) = \operatorname{Tr} \operatorname*{Alt}_{\sigma\in\mathfrak{S}_2,\tau\in\mathfrak{S}_3} \left(\frac{1}{3} [X_{\sigma_1}, A_{\tau_1}] \cdot [X_{\sigma_2}, A_{\tau_2}] \cdot A_{\tau_3} + \frac{1}{12} [A_{\tau_1}, A_{\tau_2}] \cdot [X_{\sigma_1}, X_{\sigma_2}] \cdot A_{\tau_3} \right). \tag{0.3}$$

Suppose that codimension is 3. Let X_i, $i = 1, \dots, 3$, be functions on M. Consider a cochain Label equ0.10,

$$c^3_{\{X_i\}}(A_1, A_2, A_3) = \operatorname{Tr} \operatorname*{Alt}_{\sigma,\tau\in\mathfrak{S}_3} X_{\sigma_1} \cdot A_{\tau_1} \cdot X_{\sigma_2} \cdot A_{\tau_2} \cdot X_{\sigma_3} \cdot A_{\tau_3}.$$

Then we can write a coboundary of this cochain as

$$dc^3_{\{X_i\}}(A_1, \dots, A_4) = \operatorname{Tr} \operatorname*{Alt}_{\sigma\in\mathfrak{S}_3,\tau\in\mathfrak{S}_4} \left(\frac{1}{4} [X_{\sigma_1}, A_{\tau_1}] \cdot \dots \cdot [X_{\sigma_3}, A_{\tau_3}] \cdot A_{\tau_4} + \frac{1}{16} [X_{\sigma_1}, A_{\tau_1}] \cdot [A_{\sigma_2}, A_{\sigma_3}] \cdot [X_{\tau_2}, X_{\tau_3}] \cdot A_{\tau_4} + \frac{1}{16} [X_{\tau_1}, X_{\tau_2}] \cdot [A_{\sigma_1}, A_{\sigma_2}] \cdot [X_{\sigma_3}, A_{\tau_3}] \cdot A_{\tau_4} \right). \tag{0.4}$$

Now we want to show that (at least in some particular cases) we can use these two formulae for generation of cocycles, and we can hope that in reasonable cases these cocycles should be non-trivial. We see that in a formula for a local coboundary in the codimension 2 and 3 any occurence of X_i is in the form Label equ0.11,

$$[X_i, \text{something}].$$

Therefore if we know X up to a (locally defined) constant only, we can still use these formulae and we get a cocycle. If we cannot find global X_i with the specified non-constant part, then there is a big hope that this cocycle is non-trivial.

Now consider a submanifold S of codimension n in M and let us try to repeat the above construction in these conditions. One particular case is when this submanifold is a complete intersection in its neighborhood. We mean that we can construct hypersurfaces H_i, $i = 1, \dots, n$, in a neighborhood of S such that M is a transversal intersection of H_i. Now let X_i be the functions with a change 1 in a narrow neighborhood of H_i and locally constant far from it. Consider the right-hand sides of the formulae (0.3)–(0.4). They define some $(n+1)$-cochains of $\mathcal{A}$. Indeed, though X_i are defined only in a neighborhood of S, but the function under the trace sign is non-zero only in a smaller neighborhood. Therefore we can extend it everywhere as 0 and take the trace.

In the same way as above what we get is a cocycle (since locally it looks as a coboundary). If the class of S in $H^n(M, k)$ is non-trivial, there is a big hope that we get a non-trivial cochain.

Example 0.3. Let us combine the two discussed above examples of cocycles to construct a 3-cocycle for pseudodifferential symbols. We get the following formula:

$$(A, B, C) \mapsto \operatorname{Tr}\left(\frac{\partial A}{\partial \xi} \cdot [\log \xi, B] \cdot C - [\log \xi, A] \cdot \frac{\partial B}{\partial \xi} \cdot C\right).$$

This cocycle corresponds to the intersection of the plane $x = \mathrm{const}$ with the plane $\xi = \mathrm{const}$, i.e., to a cohomological class of a point.

0.4. Higher dimensions. In the case codim > 3 we do not know if we can write a differential of a local cochain in a form similar to (0.3)–(0.4). However, it is not necessary. Let X_i, $i = 1, \dots, n$, be functions on M. Consider a cochain

$$c^n_{\{X_i\}}(A_1, \dots, A_n) = \operatorname{Tr} \operatorname*{Alt}_{\sigma,\tau \in \mathfrak{S}_n} X_{\sigma_1} \cdot A_{\tau_1} \cdot \dots \cdot X_{\sigma_n} \cdot A_{\tau_n}.$$

Then we can write a coboundary of this cochain as

(0.5)
$$dc^n_{\{X_i\}}(A_1, \dots, A_n, A_{n+1}) = \pm \operatorname{Tr} \operatorname*{Alt}_{\sigma \in \mathfrak{S}_n, \tau \in \mathfrak{S}_{n+1}} A_{\tau_1} \cdot X_{\sigma_1} \cdot A_{\tau_2} \cdot \dots \cdot X_{\sigma_n} \cdot A_{\tau_{n+1}}.$$

Now it is very easy to see that if $X_1 = \mathrm{const}$, then the alternation vanishes. Therefore we can substitute a section of $\mathcal{O}/k$ instead of X in this formula, therefore any argument above is still applicable. Again under some non-degeneracy conditions any cochain can be written as a linear combination of such, therefore there is a hope to write down a cocycle that is locally of the same form. What does the word "locally" mean here? We can see that if any one of X_i vanishes in a neighborhood of some point, then the expression under the trace sign vanishes there. Therefore we can consider a function $X_1 \otimes \dots \otimes X_n$ on $M \times \dots \times M$: Label equ0.12,

$$X_1 \otimes \dots \otimes X_n (m_1, \dots, m_n) = X_1(m_1) \dots X_n(m_n).$$

This function uniquely determines the corresponding cochain, moreover, the above remark on locality shows that it is sufficient to know this function in a neighborhood of the diagonal. So "locally" means exactly this consideration in a neighborhood of the diagonal.

The only problem now is what to do with the case of when S is not a local intersection. In less demanding cohomological theories we could consider a decomposition of unity. To do this in our case we should put some cut-off functions in the formula (0.5). However, there are too many places to "put a horse into", therefore it is not so easy to do this in such a way that the result will remain closed. Another problem is that we have too many degrees of freedom: we can get a mapping of cohomology groups, but this mapping is too far away from the "cohomological dream", when we have mapping of complexes themselves.

0.5. The appearing of Alexander—Spanier theory. One of the possible constructions is the use of Alexander—Spanier theory as a source for the initial cocycle on M. Consider the construction of a 2-cocycle basing on a section of $\mathcal{O}/k$. This section is essentially a closed 1-form on M, if $\mathcal{O}$ is the sheaf of functions. In fact we can write the basic element $[X, A]$ from (0.2) as

$$X \cdot A \cdot 1 - 1 \cdot A \cdot X.$$

In both terms A is in between, therefore we just consider the action of the element $1 \otimes X - X \otimes 1 \in \mathcal{A} \otimes \mathcal{A}$ on $A \in \mathcal{A}$ with respect to the usual left-right action. Now come two crucial observations: if we change X by a constant, the element $1 \otimes X - X \otimes 1$ does not change, and we need to know $1 \otimes X - X \otimes 1$ only on a neighborhood of a diagonal in $M \times M$ (we consider $\mathcal{A} \otimes \mathcal{A}$ as sections of $\mathcal{O} \boxtimes \mathcal{O}$ on $M \times M$). Indeed, if an element of $\mathcal{A} \otimes \mathcal{A}$ is zero in a neighborhood of the diagonal, it acts as 0 on $\mathcal{A}$. Hence this element of $\mathcal{A} \otimes \mathcal{A}$ (i.e., a section of $\mathcal{O} \boxtimes \mathcal{O}$ on $M \times M$) is correctly defined in a neighborhood of a diagonal if X is defined up to a locally constant section.

Therefore we come to the following construction: basing on a section $X \in \Gamma(M, \mathcal{O}/k)$ we get a section $1 \otimes X - X \otimes 1$ of $\mathcal{O} \boxtimes \mathcal{O}$ in a neighborhood of diagonal in $M \times M$. However, this section is just a representation of dX in the Alexander—Spanier complex. What remains to do is to find a more natural place for B from (0.2) and construct a generalization to the case of cocycles of higher order (this is a definition of "strange pairing").

So the topic of this article is a strange observation that while there is a big ambiguity in a construction of the mapping from the, say, Čech complex to a cyclic complex, this ambiguity is washed out if we start with an Alexander—Spanier complex. That means that, in fact, all the ambiguity is lying in the step from the Čech complex to the Alexander—Spanier one.

We remind here several useful mapping (including ambiguities) from various topological complexes to the Alexander—Spanier one and construct a *canonical* mapping from the latter complex to the cocyclic complex. (This in fact gives us also a mapping to the Hochschild complex and Serre—Hochschild one.) A remarkable property of this mapping is that it *does not depend* on the structure of the algebra, only on sheaf-theoretical structure of the corresponding sheaf.

We also show that the described set of cocycles give the entire cohomology of the corresponding algebra in cases when this cohomology is known.

I am indebted to a lot of people for fruitful discussions and inestimable help, among them I. M. Gelfand, A. Goncharov, D. Kazhdan, B. Khesin, M. Kontsevich, O. Kravchenko, H. McKean, A. Radul, B. Tsygan. Another approach to what is done here is contained in the resent works of B. Tsygan. In these papers the cyclic cohomology is connected with the Atiyah—Singer theorem of index.

These papers together with what is written here suggest that it is interesting to try to rewrite some "standard" proof of this theorem using the Alexander—Spanier cohomology instead of the usual one.

1. Alexander—Spanier cohomology

Label h1

If you have a differential manifold M, usually there is a lot of different ways to describe the same object: the cohomology of M. You can write a lot of different complexes that are all quasi-isomorphic. In various geometrical situations you can apply the complex that you feel is more suitable for it.

However, there is one particular type of complex that appears very rare if you need a geometrical description of cohomology. I mean the *Alexander—Spanier complex*, applications of which are usually met in hard topological papers. Here I want to show that (quite unanticipated) it is very useful in descriptions of highly geometrical objects: cyclic cohomology, that are just a non-commutative analogue of the de Rham cohomology.

1.1. Alexander—Spanier complex. Consider a topological space M and the vector space $\mathcal{A}$ of (say, continuous) functions on M. Let

$$\underbrace{\mathcal{A} \hat{\otimes} \mathcal{A} \hat{\otimes} \dots \hat{\otimes} \mathcal{A}}_{n \text{ times}} = \mathcal{A}^{\hat{\otimes} n}$$

be the space of functions[3] on M^n. We can consider the inclusion

$$\underbrace{\mathcal{A} \otimes \mathcal{A} \otimes \dots \otimes \mathcal{A}}_{n \text{ times}} = \mathcal{A}^{\otimes n} \subset \mathcal{A}^{\hat{\otimes} n}$$

of the space of functions *of finite rank* into this space. Let me remind you that a function of rank 1 is just a function of the form

$$f(m_1, m_2, \dots, m_n) = f_1(m_1) f_2(m_2) \dots f_n(m_n),$$

and a function of rank k can be represented as a linear combination of such functions. Let $\Lambda^k \mathcal{A} \subset \widehat{\Lambda}^k \mathcal{A}$ denote the spaces of skewsymmetric functions on M^n of finite rank and of any type correspondingly. This vector spaces form two complexes, if we consider the operation of exterior multiplication by $1 \in \mathcal{A}$

$$\wedge 1 \colon f_1 \wedge f_2 \wedge \dots \wedge f_n \mapsto f_1 \wedge f_2 \wedge \dots \wedge f_n \wedge 1 \colon \Lambda^k \mathcal{A} \to \Lambda^{k+1} \mathcal{A}$$

as a differential of degree 1. We can extend this operation on $\widehat{\Lambda}^k \mathcal{A}$ if we note that this operation can be written as

$$f(x_1, x_2, \dots, x_k) \mapsto df(x_1, x_2, x_k, \dots, x_{k+1}) = \sum_i (-1)^{k+1-i} f(x_1, x_2, \dots, \widehat{x}_i, \dots, x_{k+1}).$$

[3]Here the completed tensor product $\hat{\otimes}$ is *by definition* what is written above. Since we do not need this notion below, we skip the discussion of this notion.

Remark 1.1. The geometrical realization of the bigger complex is as following: Call an n-tuple of points in M considered up to an alternation a simplex in a manifold. There is a natural operation of taking a boundary in the vector space spanned by simplices. Now we can consider a skewsymmetric function on M^n as a function on the set of simplices. It is easy to understand that the differential above is exactly the combinatorial differential on the simplicial complex.

At last, let M_Δ be diagonal subset in M^n, $\Delta\colon M_\Delta \hookrightarrow M^n$ denote the inclusion and $\Delta^*\left(\Lambda^k\mathcal{A}\right) \subset \Delta^*\left(\widehat{\Lambda}^k\mathcal{A}\right)$ denote the spaces of *germs* of skewsymmetric continuous functions at a neighborhood of the diagonal (of finite rank and arbitrary correspondingly).

Definition 1.1. The Alexander—Spanier complex $\mathrm{AS}(\mathcal{A})$ consists of the vector spaces $\Delta^*\left(\widehat{\Lambda}^k\mathcal{A}\right)$ (or $\Delta^*\left(\Lambda^k\mathcal{A}\right)$). The differential in this complex is the image of the differential in the complex $\left(\widehat{\Lambda}^k\mathcal{A}, \wedge 1\right)$ (or $\left(\Lambda^k\mathcal{A}, \wedge 1\right)$).

Remark 1.2. To get a geometrical description of this complex we should call an n-tuple of nearby points on M a simplex.

Remark 1.3. In what follows we use primarily the smaller complex. However, it is known that in nice situations the inclusion of the smaller complex into the bigger is a quasi-isomorphism.

1.2. A case with an arbitrary sheaf. Let us consider instead of the vector space $\mathcal{A}$ of functions on M the corresponding sheaf $\mathcal{O}$ of vector spaces over M. We can easily see that the definition of the complex $\left(\Lambda^k\mathcal{A}, \wedge 1\right)$ in fact does not depend on anything but the sheaf structure of $\mathcal{O}$ and the global section 1 of this sheaf. So we are going to rewrite this definition using only these data.

Definition 1.2. Let $\mathcal{O}$ be a sheaf of vector spaces over M. Denote as $\mathcal{O}^{\boxtimes n}$ the *exterior tensor product* of the sheaf $\mathcal{O}$ with itself. This sheaf over M^n is defined by the following rule:

$$\Gamma\left(U_1 \times \cdots \times U_n, \mathcal{O}^{\boxtimes n}\right) = \Gamma\left(U_1, \mathcal{O}\right) \otimes \cdots \otimes \Gamma\left(U_n, \mathcal{O}\right).$$

It is clear that the symmetric group $\mathfrak{S}_n$ is acting on M^n and on the sheaf $\mathcal{O}^{\boxtimes n}$. Denote as $\mathrm{Alt}\,\mathcal{O}^{\boxtimes n}$ the subsheaf of skewsymmetric sections (i.e., sections φ on $U \subset M^n$ such that for any $s \in \mathfrak{S}_n$ the section $s\varphi$ satisfies the relation $s\varphi|_{sU\cap U} = (-1)^s \varphi|_{sU\cap U}$). For any fixed global section of $\mathcal{O}$ (call it 1) the sheafes $\mathrm{Alt}\,\mathcal{O}^{\boxtimes n}$ form a natural complex with the exterior product by 1 as a differential.

Let us denote by $\Lambda^k\mathcal{O}$ the sheaf $\Delta^*\left(\mathrm{Alt}\,\mathcal{O}^{\boxtimes k}\right)$. Sheaves $\Lambda^k\mathcal{O}$ form a natural complex for any fixed global section of the sheaf $\mathcal{O}$. A section of $\Lambda^k\mathcal{O}$ over $U \subset M$ is a skewsymmetric section of $\mathcal{O}^{\boxtimes k}$ over a small neighborhood of $\Delta\left(U\right) \subset M^k$. Let $C^k_{\mathrm{AS}}\left(\mathcal{O}\right) = \Lambda^{k+1}\mathcal{O}$, $k \geq 0$.

1.3. Realization of Alexander—Spanier cocycles. Here we are going to give several examples of mappings from some complexes calculating the cohomology of M to the Alexander—Spanier complex. These constructions give us a possibility to provide explicit formulae for cocycles in case we need one.

Label cs1

Case 1.1. Let M be covered by open subsets U_i. Let σ_i be a unity decomposition for the covering $\{U_i\}$.

Consider a Čech cochain $c_{i_0 i_1 \dots i_n}$ for $\{U_i\}$. Let us associate to c the following Alexander—Spanier cochain:

$$f(x_0, \dots, x_n) = \sum_{i_0, \dots, i_n} \sigma_{i_0}(x_0) \dots \sigma_{i_n}(x_n) c_{i_0 \dots i_n}. \tag{1.1}$$

It is easy to see that this mapping from the Čech complex to the Alexander—Spanier complex is compatible with differentials. Label equ1.10,

A chain from the cosimplicial complex is a function on the set of embedded simplices. To construct a chain in the Alexander—Spanier complex we need only to assosiate with an $(n+1)$-tuple of nearby points on M an embedded simplex (or a linear combination thereof). To proceed in this way we need a further structure on M.

Label cs2

Case 1.2. M is a Riemannian manifold.

In this case given two nearby points $m_1, m_2 \in M$ we can consider a geodesic arc $\mathcal{S}^1(m_1 m_2)$ with ends in this points. Given a point $m \in M$ and a subset $V \subset M$ we can construct

$$\operatorname{Arc}(m, V) = \bigcup_{v \in V} \mathcal{S}^1(mv).$$

Let us associate (using induction) to the ordered $(n+1)$-tuple $(m_0, \dots, m_n)$ of points of M a simplex

$$\mathcal{S}^n(m_0, \dots, m_n) = \operatorname{Arc}\left(m_0, \mathcal{S}^{n-1}(m_1, \dots, m_n)\right)$$

in M. Taking the antisymmetrization of this map, we associate to the $(n+1)$-tuple $(m_0, \dots, m_n)$ a linear combination

$$\frac{1}{(n+1)!} \sum_{s \in \mathfrak{S}_{n+1}} (-1)^s \mathcal{S}(m_{s_0}, \dots, m_{s_n})$$

of imbedded simplices in M. It is easy to see that this mapping is compatible with taking a boundary.

Now given an n-form ω we can integrate it over this linear combination of simplices. It is easy to see that the resulting skew-symmetric function on M^{n+1} is closed if ω is closed.

Label cs3

Case 1.3. Let M be covered by subsets U_i with an identification of U_i with an open convex subset in an affine space. Let σ_i be a unity decomposition for the covering $\{U_i\}$. Let ω be a differential k-form on M.

In this case we can proceed as in the previous one. If ω has a support in one of subsets U_i we can define the following Alexander—Spanier cochain in U_i: to $k+1$ given points in U_i we associate the integral of ω over the oriented convex hull of this points. We can extend this function to the entire M (more precise, to the neighborhood of the entire M in M^n) to get a cochain on M. Now we can apply this construction to the forms $\sigma_i\omega$.

1.4. The analogues for the cases of cyclic and Hochschild complexes. We will see below that the discussed above complex is adopted to the case of cohomology of Lie algebra. Here we introduce two other complexes adopted to calculations of cyclic and Hochschild cohomology.

Definition 1.3. Let $\mathcal{O}$ be a sheaf of vector spaces over M with a marked section 1. Consider the following differential in the graded sheaf $\bigoplus_n \mathcal{O}^{\boxtimes n+1}$:

$$d\left(f_0 \boxtimes \cdots \boxtimes f_n\right) = (-1)^{n+1}\, 1 \boxtimes f_0 \boxtimes \cdots \boxtimes f_n \\ + (-1)^n\, f_0 \boxtimes 1 \boxtimes \cdots \boxtimes f_n + \cdots + f_0 \boxtimes \cdots \boxtimes f_n \boxtimes 1, \qquad d^2 = 0.$$

Let $C^\bullet_{\mathrm{HAS}}\left(\mathcal{O}\right) = \left(\Gamma\left(M, \Delta^*\left(\mathcal{O}^{\boxtimes \bullet+1}\right)\right), \Delta^* d\right)$, $\bullet \geq 0$. Call this complex a Hochschild—Alexander—Spanier complex for $\mathcal{O}$.

Definition 1.4. Let $\mathcal{O}$ be a sheaf of vector spaces over M with a marked section 1. Consider the following differential in the graded sheaf $\bigoplus_n \mathcal{O}^{\boxtimes n+1}$:

$$d_a\left(f_0 \boxtimes \cdots \boxtimes f_n\right) = (-1)^n\, f_0 \boxtimes 1 \boxtimes \cdots \boxtimes f_n + \dots \\ - f_0 \boxtimes \cdots \boxtimes 1 \boxtimes f_n + f_0 \boxtimes \cdots \boxtimes f_n \boxtimes 1, \qquad d_a^2 = 0.$$

Let $C^\bullet_{\mathrm{aHAS}}\left(\mathcal{O}\right) = \left(\Gamma\left(M, \Delta^*\left(\mathcal{O}^{\boxtimes \bullet+1}\right)\right), \Delta^* d_a\right)$, $\bullet \geq 0$. Call this complex an “acyclic” Hochschild—Alexander—Spanier complex for $\mathcal{O}$.

Remark 1.4. In what follows we are not so rigorous and use often the notation $\otimes$ instead of $\boxtimes$.

Definition 1.5. Consider a product $V^{\otimes k} \otimes V^{\otimes l} \to V^{\otimes k+l}$ defined by the following rule: to define the image of

$$\left(f_1 \otimes \cdots \otimes f_k\right) \otimes \left(g_1 \otimes \cdots \otimes g_l\right)$$

consider all the decomposition of the set $\{1,\dots,k+l\}$ into two subsets of k and l elements. Insert the elements f_i on the places of the first subset and the element g_j on the places in the second subset in the expression

$$\underbrace{\bullet\otimes\cdots\otimes\bullet}_{k+l \text{ times}}$$

preserving the order in both sets of elements. Now sum the resulting elements with signs corresponding to the the substitution being even or odd. Call this associative product a *shuffle product*.

Definition 1.6. Consider the action of $\mathbb{Z}_n$ in $V^{\otimes n}$ (here V is a vector space) by

$$v_1\otimes\cdots\otimes v_n \overset{t}{\mapsto} (-1)^{n+1}\, v_2\otimes\cdots\otimes v_n\otimes v_1.$$

Call the space of invariants of this action $(V^{\otimes n})^{\mathbb{Z}_n}$ the cyclic n-th power of V. It is clear that the shuffle product sends cyclic powers into cyclic. Let $\mathbb{Z}_{n+1}$ acts in this way on C^n_{HAS}, and consider the corresponding space of invariants $(C^n_{\mathrm{HAS}})^{\mathbb{Z}_n}$. Consider a mapping of shuffle product with 1:

$$\wedge 1\colon\ (C^n_{\mathrm{HAS}})^{\mathbb{Z}_n} \to \left(C^{n+1}_{\mathrm{HAS}}\right)^{\mathbb{Z}_{n+1}}.$$

Since the shuffle product is associative, the square of the mapping $\wedge 1$ vanishes. Call this complex the cyclic Alexander—Spanier complex and denote it $C^\bullet_{\mathrm{cAS}}(\mathcal{O})$.

Remark 1.5. Until this moment we considered (say) the exterior power of a vector space as a subspace in the tensor power. However, the usual definition presents this space as a quotient of the tensor power, and the difference becomes apparent if we consider not vector spaces in char $=0$, but modules over a ring—to take an antisymmetrization, we should be able to divide by $n!$. The same is applicable to the cyclic case.

All the definitions given here allow a modification to this case. Say, in the formula (1.1) we should take a summation over ordered $(n+1)$-tuples instead. In the definition of the shuffle product for the cyclic case we should make the following modification: in multiplication

$$(a_0\otimes a_1\otimes\cdots\otimes a_n)\,(b_0\otimes\cdots\otimes b_m) = a_0\otimes X$$

we put X being the shuffle product of $a_1\otimes\cdots\otimes a_n$ and the *cyclization* of $b_0\otimes\cdots\otimes b_m$

$$b_0\otimes\cdots\otimes b_m + (-1)^m\, b_1\otimes\cdots\otimes b_0 + b_2\otimes\cdots\otimes b_1 + \cdots + (-1)^{m^2}\, b_m\otimes\cdots\otimes b_{m-1}.$$

It is easy to see that the old definition coincides with the cyclization of this product with some integer constant. This constant can be non-invertible, and in this case this difference becomes important. Everywhere below where we use the politically correct language of quotients we denote this (quotient) complex as C_{qaAS}. There is a natural mapping $C^{\bullet}_{\text{cAS}} \hookrightarrow C^{\bullet}_{\text{qcAS}}$. It is compatible with differentials if we multiply a differential in $C^{\bullet}_{\text{qcAS}}$ by the grading of its image.

The following proposition can be proved by a simple calculation:

Proposition 1.1. *Consider a natural mapping π of projection from the cyclic power of a vector space into the exterior power, the projection π_1 from the tensor power to the cyclic power, and the cyclization mapping* Cycl *from the cyclic power into the tensor power. Then the following mappings commute with differentials, therefore are mappings of complexes:*

$$\begin{array}{ccccc} C^{\bullet}_{aHAS}(\mathcal{O}) & \xrightarrow{\pi_1} & C^{\bullet}_{qcAS}(\mathcal{O}) & \xrightarrow{\text{Cycl}} & C^{\bullet}_{HAS}(\mathcal{O}) \\ & & \pi\big\downarrow & & \\ & & C^{\bullet}_{AS}(\mathcal{O}) & \xrightarrow{\text{Alt}} & C^{\bullet}_{cAS}(\mathcal{O}) \end{array}$$

if we multiply the differential in the complex $C^{\bullet}_{AS}(\mathcal{O})$ by the gradings of its image.

Proposition 1.2. *For a soft sheaf $\mathcal{O}$ the "acyclic" Hochschild—Alexander—Spanier complex is acyclic indeed, the Hochschild—Alexander—Spanier and Alexander—Spanier complexes are quasi-isomorphic to the complex of cohomology of M with coefficients in k, and if $k \supset \mathbb{Q}$ the cyclic Alexander—Spanier complex is quasi-isomorphic to a direct sum of an infinite number of such complexes with non-negative even shifts.*

Proof. Fix a mapping from $\mathcal{A} = \Gamma(M, \mathcal{O})$ to k that sends $1 \in \mathcal{A}$ to $1 \in k$. Let us construct a *homotopy* for the complex $\left(\mathcal{A}^{\boxtimes n+1}, d_a\right)$:

$$s \cdot f_0 \otimes \cdots \otimes f_n = \varphi(f_0) f_1 \otimes \cdots \otimes f_n, \qquad s \cdot f_0 = 0.$$

It is easy to check that $sd_a + d_a s = \text{id}$ indeed, therefore the complex is acyclic. Fix a point $m \in M$ and consider a local section φ of $\Delta^*\left(\mathcal{O}^{\boxtimes n+1}\right)$ over $U \subset M$. Lessening U we can suppose that φ corresponds to a section of $\mathcal{O}^{\boxtimes n+1}$ over U^{n+1}. Changing M to U in the discussion above we get a local homotopy. This means that for any closed local section we can find a section on a smaller subset such that the boundary of the latter section is the former. Therefore the differential d_a on the *complex of sheaves* $\mathcal{C}^{\bullet}_{\text{aHAS}}$ is acyclic.

Now the complex of vector spaces $C^{\bullet}_{\text{aHAS}}$ is the complex of global sections of this complex of sheaves $\mathcal{C}^{\bullet}_{\text{aHAS}}$. We can consider a bicomplex

$$C^*\left(M, \mathcal{C}^{\bullet}_{\text{aHAS}}\right),$$

columns of which compute the cohomology of the sheaves $\mathcal{C}^{\bullet}_{\text{aHAS}}$. We have seen that the rows are exact, therefore the row spectral sequence gives the total complex associated with this bycomplex being also exact.

If the sheaf $\mathcal{O}$ is soft or satisfies some other nice cohomological properties, then $\mathcal{C}^\bullet_{\mathrm{aHAS}}$ is also soft (or whatever), therefore the columns of the bicomplex are acyclic in degree ≥ 1. Now the column spectral sequence gives the acyclity of the complex

$$H^0\left(M, \mathcal{C}^\bullet_{\mathrm{aHAS}}\right) = C^\bullet_{\mathrm{aHAS}}.$$

Consider now the complex $C^\bullet_{\mathrm{HAS}}$. The same homotopy as above satisfies

$$sd_a + d_a s = \mathrm{id}$$

in degree ≥ 1, and if $f \in \mathcal{A}$

$$\left(sd_a + d_a s\right) f = f - \varphi\left(f\right) \cdot 1.$$

Therefore the mapping $\left(\mathcal{A}^{\boxtimes n+1}, d\right) \to k$ given by φ if $n = 0$ and 0 otherwise is a quasi-isomorphism. Hence the analogues inclusion $k \to \left(\mathcal{A}^{\boxtimes n+1}, d\right)$ is also a quasi-isomorphism. Repeating this argument on the level of sheaves, we get that the complex of sheaves $\mathcal{C}^\bullet_{\mathrm{HAS}}$ is quasi-isomorphic to its constant subsheaf k.

To get information about the complex of global sections of this complex of sheaves consider again the bicomplex. Again the row spectral sequence gives a quasi-isomorphism of the total complex of this bicomplex with the cohomology of the rows, i.e., the complex $C^*\left(M, k\right)$. Again, if $\mathcal{O}$ has nice topological properties, the total complex is quasi-isomorphic to its first row, i.e., $C^\bullet_{\mathrm{HAS}}$.

Consider now the complex $\Lambda^{k+1}\left(\mathcal{O}\right)$. Here we can construct the homotopy

$$s \cdot f_0 \wedge \cdots \wedge f_n = \sum_k \left(-1\right)^k \varphi\left(f_k\right) f_1 \wedge \cdots \wedge \widehat{f_k} \wedge \cdots \wedge f_n, \quad s \cdot f_0 = 0.$$

It is easy to see that $ds + sd = \mathrm{id}$ if $n > 0$ and $\left(ds + sd\right) f = f - \varphi\left(f\right) 1$. Therefore the same argument as above shows that $C^\bullet_{\mathrm{HAS}}$ is also quasi-isomorphic to $C^*\left(M, k\right)$.

Consideration of $C^\bullet_{\mathrm{cAS}}$ is a little bit more tricky. We use an analogue of the construction from [?**LodQuill84Cyc**]. Consider a bicomplex

$$\mathcal{C}^\bullet_{\mathrm{HAS}} \xrightarrow{1-t} \mathcal{C}^\bullet_{\mathrm{aHAS}} \xrightarrow{N} \mathcal{C}^\bullet_{\mathrm{HAS}} \xrightarrow{1-t} \mathcal{C}^\bullet_{\mathrm{aHAS}} \xrightarrow{N} \dots \tag{1.2}$$

Here t is the action of $\mathbb{Z}_{n+1}$ on $\mathcal{C}^n_{\mathrm{HAS}} = \mathcal{C}^n_{\mathrm{aHAS}}$, N is equal to $1 + t + t^2 + \cdots + t^n$ on $\mathcal{C}^n_{\mathrm{HAS}}$. It is easy to check the conditions of bicomplex for this system of mappings. Now the rows are acyclic in all the terms but the first, the homology in the first term are exactly $\mathcal{C}^\bullet_{\mathrm{cAS}}$. Now the row spectral sequence shows that the complex $\mathcal{C}^\bullet_{\mathrm{cAS}}$ is quasi-isomorphic to the total complex of this bicomplex. Label equ1.20,

From the other side, the column spectral sequence shows that the total complex is quasi-isomorphic to the complex

$$k \to 0 \to k \to 0 \to k \to \dots$$

of constant sheaves, or a direct sum of constant sheaves k in even degrees. □

Remark 1.6. We can consider an analogue of (1.2)

$$\dots \xrightarrow{N} \mathcal{C}^{\bullet}_{\mathrm{HAS}} \xrightarrow{1-t} \mathcal{C}^{\bullet}_{\mathrm{aHAS}} \xrightarrow{N} \mathcal{C}^{\bullet}_{\mathrm{HAS}} \xrightarrow{1-t} \mathcal{C}^{\bullet}_{\mathrm{aHAS}}.$$

The rows are quasi-isomorphic to $\mathcal{C}^{\bullet}_{\mathrm{qcAS}}$, the columns to

$$\dots \to k \to 0 \to k \to 0.$$

However, this bicomplex is in a "wrong" quadrant, therefore we should not (and do not) have the isomorphisms of cohomology. Anyway, consideration of the homotopy s for $\mathcal{C}^{\bullet}_{\mathrm{aHAS}}$ leads to a mapping $B\colon \mathcal{C}^{\bullet}_{\mathrm{HAS}} \to \mathcal{C}^{\bullet}_{\mathrm{qaAS}}[-1]$, $B = \pi_1 \circ s \circ (1-t)$:

$$f_0 \otimes f_1 \otimes f_2 \otimes \dots \otimes f_n \mapsto (\varphi(f_0) f_1 - \varphi(f_1) f_0) \otimes f_2 \otimes \dots \otimes f_n.$$

It is easy to see that this mapping is compatible with differentials. We use it below in the exact sequence relating cyclic and Hochschild Alexander—Spanier cohomology.

In the proof of the proposition we have seen that the cyclic complex is always quasi-isomorphic to the total complex of the bicomplex (1.2). In the bicomplex (1.2) there is a remarkable periodicity operation S: the translation on two columns to the right. It commutes with the differentials, therefore it results in an operation in cohomology. The remarkable fact is that we can express this operation on the quasi-isomorphic complex C_{cAS}.

Definition 1.7. Let the shift S send the class of $a_0 \otimes \dots \otimes a_n$ in C^n_{qcAS} into the class of

$$\sum_{0 \le k \le l \le n} (2(l-k) - n - 1)\, a_0 \otimes \dots \otimes a_k \otimes 1 \otimes a_{k+1} \otimes \dots \otimes a_l \otimes 1 \otimes a_{l+1} \otimes \dots \otimes a_n$$

in C^{n+2}_{qcAS}.

Proposition 1.3. *The operation of shift is correctly defined and commutes with differential. If $k \supset \mathbb{Q}$, then S is quasi-isomorphic to the operation of translation on two columns in* (1.2). *The image* $\operatorname{Im} S$ *is the kernel of the cyclization* Cycl, *moreover, the sequence*

$$\dots \xrightarrow{\mathrm{Cycl}} C^{n+1}_{HAS} \xrightarrow{B} C^{n}_{qcAS} \xrightarrow{S} C^{n+2}_{qcAS} \xrightarrow{\mathrm{Cycl}} C^{n+2}_{HAS} \xrightarrow{B} C^{n+1}_{qcAS} \to \dots$$

is exact.

Remark 1.7. We see that if $k \supset \mathbb{Q}$ and $\mathcal{O}$ is soft, $C^{*}_{\mathrm{cAS}}(\mathcal{O})$ is quasi-isomorphic to $C^{*}(M, k[S])$ as $k[S]$-module. This mapping is given by the inclusion of the constant sheaf $k[S]$ into $\mathcal{C}^{\bullet}_{\mathrm{qcAS}}$:

$$1 \mapsto 1 \in \mathcal{O} = \mathcal{C}^{0}_{\mathrm{qcAS}}, \qquad S^k \mapsto S^k \cdot 1 = \mathrm{const} \cdot \underbrace{1 \otimes \dots \otimes 1}_{2k+1 \text{ times}} \in \mathcal{C}^{2k}_{\mathrm{qcAS}}.$$

Remark 1.8. We have seen that the differential sends a skewsymmetric element of $C^{\bullet}_{\mathrm{qcAS}}(\mathcal{O})$ to a skewsymmetric element, therefore the Alexander—Spanier complex is a subcomplex of a cyclic Alexander—Spanier complex. Moreover, a differential sends a cyclically symmetric element of $C^{\bullet}_{\mathrm{HAS}}(\mathcal{O})$ to a cyclically symmetric element, therefore the cyclic complex is in turn a subcomplex of the Hochschild complex. Therefore the above constructions of Alexander—Spanier cocycles gives in fact cyclic and Hochschild Alexander—Spanier cocycles. The application of the mapping S allows to construct in this way any class of the cocycle in the case of soft $\mathcal{O}$ and $k \supset \mathbb{Q}$.

2. Complexes in algebraic situation

2.1. Definitions of complexes. Let K be a commutative ring over $\mathbb{Q}$. We use here several complexes associated with an associative algebra A over K.

Definition 2.1. The Hochschild homological complex consists of vector spaces $CH_k(A) = A^{\otimes k+1}$ with the differential

$$d\colon f_0 \otimes \cdots \otimes f_k \mapsto \sum_l (-1)^l f_0 \otimes \cdots \otimes (f_l \cdot f_{l+1}) \otimes \cdots \otimes f_k + (-1)^k (f_k \cdot f_0) \otimes f_1 \otimes \cdots \otimes f_k.$$

The acyclic Hochschild complex differs from this one only by the absence of the last term in differential. The cyclic complex CC_* consists of coinvariant "in" the Hochschild complex with respect to the following action of $\mathbb{Z}_{k+1}$ on $A^{\otimes k+1}$:

$$t\colon f_0 \otimes \cdots \otimes f_k \mapsto (-1)^k f_1 \otimes \cdots \otimes f_k \otimes f_0.$$

(It is easy to see that the above differential sends indeed coinvariants $\left(A^{\otimes k+1}\right)_{\mathbb{Z}_{k+1}}$ into coinvariants $\left(A^{\otimes k}\right)_{\mathbb{Z}_k}$.)

In the same way we can consider the corresponding dual cohomological complexes.

We can also consider the corresponding to A Lie algebra $\mathrm{Lie}(A)$ (this algebra coincides with A as a vector space and has commutator as a Lie operation) and homological and cohomological complexes $C^{\mathrm{Lie}}_*(\mathrm{Lie}(A))$ and $C^*_{\mathrm{Lie}}(\mathrm{Lie}(A))$.

This definition has a big resemplance with the definitions of corresponding objects in the topological situation. As then, we have some maps between these complexes, however not any map extends to the topological situation.

Definition 2.2. The mapping of shift S sends the class of $f_0 \otimes \cdots \otimes f_k$ in CC_k into the class of

$$\sum_l (3-k) f_0 \otimes \cdots \otimes (f_l \cdot f_{l+1} \cdot f_{l+2}) \otimes \cdots \otimes f_k + \sum_{l+1<m} (2(m-l)-k+1) f_0 \otimes \cdots \otimes (f_l \cdot f_{l+1}) \otimes \cdots \otimes$$

in CC_{k-2}. The mapping B sends the class of $f_0 \otimes \cdots \otimes f_k$ in CC_k into the element

$$\sum_i (-1)^{ik} 1 \otimes f_i \otimes \cdots \otimes (f_k \cdot f_0) \otimes \cdots \otimes f_{i-1} + \sum_i (-1)^{(i+1)m} f_i \otimes \cdots \otimes f_{i-1} \otimes 1$$

of CH_{k+1}.

The mappings S and B commute with differentials, therefore define an exact sequence of cohomologies

$$\cdots \to HH_{k+1} \to HC_k \to HC_{k-2} \to HH_{k-1} \to \dots .$$

2.2. The Lie algebra complex and the cyclic complex. We can consider any given associative algebra A as a Lie algebra $\mathrm{Lie}\,(A)$ with the commutator operation. Consider the inclusion of the homological Serre—Hochschild complex for $\mathrm{Lie}\,(A)$ to the homological cyclic complex for A that sends $X_1 \wedge \cdots \wedge X_n \in \Lambda^n \mathfrak{g}$ to the corresponding element of $\mathfrak{g}^{\otimes n}/\mathbb{Z}_n$. It is easy to see that differential of these two complexes are compatible (up to a factor 2), hence there is a corresponding mapping of homologies:

$$H_*^{\mathrm{Lie}}\left(\mathrm{Lie}\,(A)\right) \to HC_*\left(A\right)$$

and of cohomologies

$$HC^*\left(A\right) \to H^*_{\mathrm{Lie}}\left(\mathrm{Lie}\,(A)\right).$$

2.3. The Hochschild complex and the cyclic complex. In the same way as above we can consider a projection from the Hochschild complex to the cyclic complex, that is (by definition) compatible with differentials. Together with the mapping from the previous section we get a diagram

$$\begin{array}{ccc} C_*^{\mathrm{Lie}}\left(A\right) & \longrightarrow & CC_*\left(A\right) \\ & & \Vert \\ CH_*\left(A\right) & \longrightarrow & CC_*\left(A\right). \end{array}$$

We defined above three pairings of these complexes with complexes $\left(C_*^{\mathrm{Lie}}\left(A\right), \wedge 1\right)$, $\left(CC_*\left(A\right), \wedge 1\right)$ and $\left(CH_*\left(A\right), m\left(1\right)\right)$. It is easy to see that there exists a dual diagram to the previous diagram:

$$\begin{array}{ccc} \left(C_*^{\mathrm{Lie}}\left(A\right), \wedge 1\right) & \xleftarrow{\alpha} & \left(CC_*\left(A\right), \wedge 1\right) \\ & & \Vert \\ \left(CH_*\left(A\right), m\left(1\right)\right) & \xleftarrow{\beta} & \left(CC_*\left(A\right), \wedge 1\right). \end{array}$$

The mappings α and β are projection and symmetrization correspondingly.

2.4. A case with a commutative ring. Suppose that the ring A in the above situation is commutative. In this case it is possible to compute the cohomology explicitly at least in the case when A is smooth in the algebraic-geometrical case.

The simplest possible answer is in the situation of Lie algebra homologies. The differential in the homological complex vanishes, therefore

$$H_*^{\mathrm{Lie}}(A) = \Lambda^* A.$$

The situation with Hochschild homology is also very simple. If A is a space of functions on the manifold M, define Ω_A^* as the space of differential forms on M. It is possible to define this space in terms of A itself, but we do not need such complications, therefore leave this as an exercise to a reader.

Proposition 2.1. *Consider a mapping from the Hochschild complex for a commutative algebra A into the complex Ω_A^* with zero differential:*

$$f_0 \otimes \cdots \otimes f_k \mapsto \sum_{\sigma \in \mathfrak{S}_k} f_0 df_{\sigma_1} \wedge \cdots \wedge df_{\sigma_k} \in \Omega_A^k.$$

This mapping induces an isomorphism on homologies.

In the case of cyclic homology the description is a little bit more complicated. We need to use the mapping of shift $S\colon CC_k \to CC_{k-2}$ here. The first observation is that the above mapping $H_k(A, A) \to \Omega_A^k$ sends an element with a trivial projection on the space $CC_k(A)$ into a closed form. Therefore the same formula as above defines a mapping

$$CC_k(A) \xrightarrow{\alpha} \Omega_A^k / d\Omega_A^{k-1}.$$

We can again consider this mapping as a mapping in the complex with zero differential. Now the compositions $\alpha \circ S^m$ define a mapping of complexes

$$CC_k(A) \xrightarrow{\beta} \Omega_A^k / d\Omega_A^{k-1} \oplus \Omega_A^{k-2} / d\Omega_A^{k-3} \oplus \Omega_A^{k-4} / d\Omega_A^{k-5} \oplus \dots .$$

Consider the following subspace of the space in the right-hand side:

$$W_k = \Omega_A^k / d\Omega_A^{k-1} \oplus H_{DR}^{k-2}(A) \oplus H_{DR}^{k-4}(A) \oplus \cdots \subset \Omega_A^k / d\Omega_A^{k-1} \oplus \Omega_A^{k-2} / d\Omega_A^{k-3} \oplus \Omega_A^{k-4} / d\Omega_A^{k-5} \oplus \dots .$$

We claim that the image of a cycle in $CC_k(A)$ lies in that subspace, and

Proposition 2.2. *The corresponding to β mapping of homology is an isomorphism onto the subspace W_*.*

It is easy to understand that the corresponding to S operator on W_* is

$$\begin{array}{ccc}
\Omega_A^k / d\Omega_A^{k-1} & H_{DR}^{k-2}(A) & H_{DR}^{k-4}(A) \dots \\
\big\downarrow & \big\downarrow \scriptstyle{\mathrm{in}} & \big\downarrow \scriptstyle{\mathrm{id}} \\
\mathbf{0} & \Omega_A^k / d\Omega_A^{k-1} & H_{DR}^{k-2}(A) \dots .
\end{array}$$

Here in is the canonical inclusion.

The described above mappings from Hochschild complex and Lie complex into the cyclic complex are correspondingly taking the quotient by $d\Omega_A^{k-1}$ and taking the jet on a diagonal Δ_M in M^{k+1} (which is a k-form) and taking the same quotient.

In particular, we can see that any class of cyclic homology from $\operatorname{Ker} S$ has a representative that is a skewsymmetric chain.

3. Cocycles for the algebra of global sections

3.1. A strange pairing. Let A be an associative K-algebra with a trace $\operatorname{Tr} : A \to K$ (a trace is a linear mapping satisfying $\operatorname{Tr}[x, y] = 0$).

Definition 3.1. Consider a cyclic complex $CC_k(A) = A^{\otimes k+1}/\mathbb{Z}_{k+1}$. Consider the following pairing between $CC_k(A)$ and itself:

$$((x_0, \dots, x_k) \cdot (y_0, \dots, y_k)) = \sum_l (-1)^{kl} \operatorname{Tr} x_0 y_l x_1 y_{l+1} \dots x_k y_{k+l}$$

(here $y_{k+1+l} \stackrel{\text{def}}{=} y_l$). It is correctly defined, hence it sends the graded vector space $CC_*(A)$ into the complex $CC^*(A)$. Let us denote this mapping as i.

The first question is: can we describe what differential (of degree +1!) on $CC_*(A)$ "corresponds" to the differential on $CC^*(A)$ under this inclusion. *A priory* we cannot expect that such a differential exists at all.

Proposition 3.1. *The following diagram is commutative:*

$$\begin{array}{ccc} CC_k(A) & \xrightarrow{\wedge 1} & CC_{k+1}(A) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ CC^k(A) & \xrightarrow{d} & CC^{k+1}(A). \end{array}$$

Here $\wedge 1$ denotes the mapping of the shuffle product with $1 \in A$.

Remark 3.1. Due to associativity of the shuffle product it is evident that the square of the operation of the shuffle product with 1 is 0:

$$(a \wedge 1) \wedge 1 = a \wedge (1 \wedge 1) = 0.$$

Therefore we got the mapping of complexes

$$(CC_*, \wedge 1) \to CC^*.$$

The remarkable fact about this mapping is that the structure of the first (but not the second!) complex does not depend on the ring structure of A at all.

Remark 3.2. It is easy to see that in the same way we can define strange pairings between $C_*^{\mathrm{Lie}} \stackrel{\mathrm{def}}{=} \Lambda^* \operatorname{Lie}(A)$ and itself:

$$(f_1 \wedge \dots \wedge f_k, g_1 \wedge \dots \wedge g_k) = \operatorname{Tr} \sum_{\substack{\sigma,\tau \in \mathfrak{S}_k \\ \sigma_1 = 1}} f_{\sigma_1} g_{\tau_1} \dots f_{\sigma_k} g_{\tau_k},$$

and between the Hochschild complex (or the acyclic Hochschild complex) $CH_*(A) = A^{\otimes *+1}$ and itself:

$$(f_0 \otimes \dots \otimes f_k, g_0 \otimes \dots \otimes g_k) = \operatorname{Tr} f_0 g_0 \dots f_k g_k.$$

The dual to the differentials mappings (of degree 1) in these graded vector spaces are the wedge product with 1 in the case of the Lie algebra cohomology,

$$f_0 \otimes \dots \otimes f_k \stackrel{m(1)}{\mapsto} (-1)^{k+1} 1 \otimes f_0 \otimes \dots \otimes f_k - (-1)^{k+1} f_0 \otimes 1 \otimes \dots \otimes f_k + \dots \\ + f_0 \otimes \dots \otimes f_k \otimes 1,$$

and

$$f_0 \otimes \dots \otimes f_k \stackrel{m(1)}{\mapsto} -(-1)^{k+1} f_0 \otimes 1 \otimes \dots \otimes f_k + \dots + f_0 \otimes \dots \otimes f_k \otimes 1$$

in two Hochschild complexes correspondingly (up to a sign).

3.2. A mapping from the Alexander—Spanier complex. Now we want to consider a sheaf of associative algebras $\mathcal{O}$ over a topological space M with an algebra $\mathcal{A}$ of global sections. Suppose again that the algebra $\mathcal{A}$ has a trace

$$\operatorname{Tr}\colon \mathcal{A}/[\mathcal{A},\mathcal{A}] \to K.$$

We construct here a mapping from the Alexander—Spanier complex for $\mathcal{O}$ to the Lie-algebraic complex of the algebra $\mathcal{A}$ considered as a Lie algebra.

We have already constructed the mapping $\mathcal{I}$ from the complex $(\Lambda^\bullet \mathcal{A}, \wedge 1)$ to the cochain complex $(\Lambda^\bullet \mathcal{A}^*, (\wedge 1)^*)$. So the only fact we need is what this mapping can be routed via the Alexander—Spanier complex, that is a factor of $(\Lambda^\bullet \mathcal{A}, \wedge 1)$.

We want to prove now that the mapping $\mathcal{I}$ can be direct via the space $\Gamma\left(M, \Lambda^k \mathcal{O}\right)$ (that is a factor of the space $\Lambda^k \mathcal{A} = \Gamma\left(M^k, \operatorname{Alt} \mathcal{O}^{\boxtimes k}\right)$). We need to prove that if the function $f(x_1, \dots, x_k) \in \Lambda^k \mathcal{A}$ is zero in a neighborhood of the diagonal, then $\langle \mathcal{I}(f), g \rangle$ is zero for any chain $g = (g_1, \dots, g_k) \in CC_k(\mathcal{A})$. Consider a representation of f of the form

$$f(x_1, \dots, x_k) = \sum_\alpha f_1^{(\alpha)}(x_1) \wedge \dots \wedge f_k^{(\alpha)}(x_k), \tag{3.2}$$

We have Label equu5.2,

$$\langle \mathcal{I}(f), g \rangle = \sum_\alpha \sum_{\sigma \in \mathfrak{S}_k} (-1)^\sigma \operatorname{Tr}\left(f_{\sigma_1}^{(\alpha)} g_1 f_{\sigma_2}^{(\alpha)} g_2 \dots f_{\sigma_k}^{(\alpha)} g_k\right).$$

We want to prove that in fact already

$$\sum_{\alpha}\sum_{\sigma\in\mathfrak{S}_k}(-1)^{\sigma} f_{\sigma_1}^{(\alpha)}g_1 f_{\sigma_2}^{(\alpha)}g_2\dots f_{\sigma_k}^{(\alpha)}g_k=0. \quad (5.3)$$

Indeed, consider a point $m\in M$. If U is a sufficiently small neighborhood of m, then $f|_{U\times\dots\times U}=0$, therefore in calculation of (?equ5.3?) in U we can substitute instead of representation (3.2) just $f(x_1,\dots,x_k)=0$.

This defines in fact the mappings

$$C^*_{\mathrm{AS}}(\mathcal{O})\xrightarrow{\mathcal{I}}C^*_{\mathrm{Lie}}(\Gamma(\mathcal{O}))$$

of complexes and the corresponding mapping of homologies:

$$H^*_{\mathrm{AS}}(\mathcal{O})\xrightarrow{\mathcal{I}}H^*_{\mathrm{Lie}}(\Gamma(\mathcal{O})).$$

We want to remind that the left-hand side does not depend on the multiplication law in $\mathcal{O}$! Moreover, if the sheaf $\mathcal{O}$ coinsides as a sheaf of vector spaces with the structure sheaf of M, then the left-hand side coincides with the singular cohomology of M (under mild general-topological assumptions).

A simple generalization gives the mappings

$$\begin{aligned} C^*_{\mathrm{HAS}}(\mathcal{O})&\to HC^*(\Gamma(\mathcal{O}),\Gamma(\mathcal{O})^*),\\ C^*_{\mathrm{cAS}}(\mathcal{O})&\to CC^*(\Gamma(\mathcal{O})),\\ C^*_{\mathrm{aHAS}}(\mathcal{O})&\to HC^*(\Gamma(\mathcal{O})).\end{aligned}$$

Moreover, this mappings are compatible with natural mappings between complexes in the left-hand side (described above) and mappings between the complexes in the right-hand side (described, say, in [?LodQuill84Cyc]). We should note, however, that the situation with algebraic complexes is not so simple as with topological complexes, where two complexes in question were subcomplexes in the third. In the algebraic case we have defined the following mappings only:

$$\begin{array}{ccccc} HC(A) & \longleftarrow & CC(A) & \longleftarrow & HC(A,A^*) \\ & & \big\downarrow & & \\ & & C_{\mathrm{Lie}}(\mathrm{Lie}(A)) & & \end{array},$$

and the natural mapping $C_{\mathrm{Lie}}(\mathrm{Lie}(A))\xrightarrow{\mathrm{Alt}}CC(A)$ is not compatible with differential. Other mappings in the topological case are due to the existence of the trace.

In fact we described some "topological part" of the different cohomological complexes for the ring $\Gamma(\mathcal{O})$ and can write explicit cocycles for this part.

4. Example: pseudodifferential symbols

4.1. The sheaf of pseudodifferential symbols. Here we use a synthetic approach and intertwine definitions of pseudodifferential operators and pseudodifferential symbols. The operators are only intermediate steps in definition of symbols for us.

Definition 4.1. A function $\widetilde{A}(x,\xi)$ on $T^*\mathbb{R}^n$ is a *classical pseudodifferential symbol* of order $k \in \mathbb{Z}$ if for any given N it has a decomposition

$$\widetilde{A}(x,\xi) = \sum_{j=-N}^{k} A_j(x,\xi) + A^{(N)}(x,\xi),$$

where A_j is a smooth (outside 0 section of $T^*\mathbb{R}^n$) homogeneous in ξ function of homogeneity degree j and $A^{(N)}$ is $o\left(\xi^{-N}\right)$ locally in x when $\xi \to \infty$. We say that

$$\widetilde{A}(x,\xi) \simeq \sum_{j=-\infty}^{k} A_j(x,\xi)$$

is the asymptotic expansion of $\widetilde{A}$.

We consider two symbols the same if they have the same asymptotic expansion.

Consider an operator $A\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. Consider the point $x_0 \in \mathbb{R}^n$, the δ-function δ_{x_0} in this point and the linear functional

$$A^*\delta_{x_0}\colon f \mapsto (Af)(x_0).$$

on $C^\infty(\mathbb{R}^n)$. Let us translate this generalized function on the vector $-x_0$

$$f(x) \mapsto f_{x_0}(x) = f(x+x_0) \mapsto (Af_{x_0})(x_0)$$

and denote it φ_{A,x_0}. For not to worry about the behavior at large x, fix a cut-off function $\omega(x)$ and denote $\omega\varphi_{A,x_0}$ by $\widetilde{\varphi}_{A,x_0}$.

Definition 4.2. An operator $A\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a *classical pseudodifferential operator with a symbol* $\widetilde{A}(x,\xi) = \sum_{j=-\infty}^{k} A_j(x,\xi)$ if the generalized function $\widetilde{\varphi}_{A,x_0}(x)$

$$\varphi_{A,x_0}\colon f \mapsto \langle \varphi_{A,x_0}, f\rangle = \omega(x) A(f(x+x_0))|_{x_0}$$

has Fourier transform $F\varphi_{A,x_0}(\xi)$ with the asymptotic expansion

$$F\varphi_{A,x_0}(\xi) \simeq \sum_{j=-\infty}^{k} A_j(x_0,\xi), \qquad |\xi| \to \infty.$$

Example 4.1. The operator M_α of multiplication by the function $\alpha(x)$ is pseudodifferential with symbol $A(x,\xi) = \alpha(x)$. Indeed, in this case the generalized function φ_{x_0} is just the δ-function at 0 (this is why we shift the argument of the function f in the definition) with coefficient $\alpha(x_0)$, and the Fourier transform of the δ-function is 1.

Example 4.2. The operator $\frac{\partial}{\partial x_1}$ is pseudodifferential with symbol $i\xi_1$. Moreover, any vector field corresponds to a pseudodifferential operator and the symbol is the corresponding linear function on $T^*\mathbb{R}^n$.

Proposition 4.1. *A composition of two pseudodifferential operators is again a pseudodifferential operator and its symbol has the following asymptotic expansion:*

$$\widetilde{A \circ B} = \sum_{N \geq 0} \frac{1}{N!} \frac{\partial^{|N|}}{\partial \xi^N} \widetilde{A}(x,\xi) \frac{\partial^{|N|}}{\partial x^N} \widetilde{B}(x,\xi). \tag{4.1}$$

(The terms in this sum have the order that goes to infinity, therefore to compute a component of $\widetilde{A \circ B}$ of given homogeneity degree we need to compute a sum of a finite number of summands.) Label equ6.3,

If the symbol of a pseudodifferential operator vanishes, then this operator is an operator with a smooth kernel $K(x,y)\,dy$, $x, y \in \mathbb{R}^n$:

$$f(x) \mapsto (Af)(x) = \int K(x,y) f(y)\,dy.$$

Now we want to define a notion of a pseudodifferential operator on a manifold. Consider a pseudodifferential operator P on $\mathbb{R}^n$ and a pair of cut-off functions φ and ψ defined in a neighborhood of $x \in \mathbb{R}^n$. Then $\psi P \varphi$ is the operator sending a locally defined function into a locally defined function with a compact support. It is obvious that this operator is pseudodifferential, moreover, if for any functions φ_i, φ_j from a decomposition of unity

$$\sum_i \varphi_i = 1$$

the operator $\varphi_i P \varphi_j$ is pseudodifferential then the initial operator P is also pseudodifferential. This gives a localization of the notion of a pseudodifferential operator, therefore we can define a pseudodifferential operator on a manifold. However, we want also define a notion of pseudodifferential symbol on a manifold, and this is a little bit more tricky.

We know that the operators with a smooth kernel on a manifold should form a kernel of the mapping from operators to symbols. in any local chart $M \supset U \to \mathbb{R}^n$ we can associate to the pseudodifferential operator its symbol, that is an asymptotic expansion in $T^*\mathbb{R}^n$. Consider two intersecting local charts. The symbol in one of them determines the operator up to addition of an operator with a smooth kernel, therefore it determines the symbol in the part of the other chart that corresponds to intersection of charts.

What we get is the action of "local diffeomorphisms" of $\mathbb{R}^n$ on pseudodifferential symbols. This action is difficult to describe explicitly, however, if we could do it, then we could just define the notion of a pseudodifferential symbol on a manifold without a reference to pseudodifferential operators. For convinience of the reader we want to show that this action is not a new entity, but just a corollary of the formula for the multiplication.

Indeed, consider for simplicity the differential operators on $\mathbb{R}^n$. We know how diffeomorphisms of $\mathbb{R}^n$ acts on this algebra, however, we can *deduce* this action as a corollary of the commutation law for differential operators. Indeed, we can represent a diffeomorphism as an intergral of a *flow* corresponding to some vector field. Now the change in some small time of the operator under the action of this flow is described by the commutator of the vector field and the operator. Now we can integrate these changes and get the image under this diffeomorphism. We can repeat this program literally in the case of pseudodifferential operators.

Corollary 4.1. *Consider a 1-parametric group of diffeomorphisms h_t of $\mathbb{R}^n$ corresponding to a vector field V. It can be raised to $T^*\mathbb{R}^n$, so it determines a group of diffeomorphisms h'_t of $T^*\mathbb{R}^n$ and a vector field V' on $T^*\mathbb{R}^n$. Consider a pseudodifferential symbol P_0 and the equation*

$$-\frac{d}{dt}P_t = V \circ P_t - P_t \circ V.$$

Call a solution of this equation the translation of P by the flow h_t.

The leading terms of $[V, P]$ and of the Lie derivative of the symbol P with respect to the field V' coincide, hence the leading term of P_t moves with the flow h'_t. Moreover, in the equation above we can restict our attention to any fixed number of terms in the symbol P, since the commutator with V preserves degree. Hence if $P = \sum P_j$, and

$$P_j^{(t)} = \left(h'_t\right)^* P_{j,t}$$

then the equation on $P_j^{(t)}$ is upper-triangular:

$\frac{d}{dt}P_j^{(t)} = \sum_{k>j} \alpha_k \left(P_k^{(t)}\right)$. *Here α are some differential operators. Therefore the solution always exists, it*

Consider a manifold M and an operator $A\colon C^\infty(M) \to C^\infty(M)$. We call this operator a *pseudodifferential operator* on M if it is locally of such type, i.e., if for a local chart $h\colon M \supset U \to \mathbb{R}^n$ it acts on functions with compact support in U as some psuedodifferential operator in $\mathbb{R}^n$. This means that for a cut-off function σ with support on U the corresponding operator

$$h^{-1*} \circ M_\sigma \circ A \circ M_\sigma \circ h^* \colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

is pseudodifferential. It is easy to see that we can consider the symbol of this operator in this coordinate frame and that the highest order term of this symbol is correctly defined function on T^*M. We can consider a complete symbol of A as an asymptotic expansion of a function on T^*M with a "twisted" transformation law under chart changes on M: only the highest term is just transferred by the flow, to the lower terms some additional terms (depending on the higher order terms) are added.

However, we can see that if in one chart the symbol of the operator A is 0 when ξ goes to infinity inside a given open conic subset of T^*M, then this condition is satisfied in any other coordinate chart. The composition law (4.1) shows that a product of such operator with any other operator is again of this type. This means that the restriction of the symbol of the product to an open conic subset is uniquely determined by values of the symbols of factors on the given subset.

Therefore we can consider the set $\Psi DS(M)$ of pseudodifferential symbols on M, define the multiplication law of such symbols and transformation laws under diffeomorphisms. It easy to see that this ring has a natural structure of a sheaf of rings over the "infinity in the cotangent bundle".

So consider a projective (or better, spherical) completion $\mathcal{P}T^*M$ and the infinity PT^*M in this completion. We can consider a symbol on M as a function on the "punctured infinitesimal neighborhood of PT^*M in $\mathcal{P}T^*M$". We call this (formal) manifold DT^*M. It is fibered over PT^*M with a "punctured disk of infinitesimally small radius" as a fiber. The fiber has two connected components, corresponding to the positive part of the disk and the negative one.

Here we want to show that the cyclic cohomogy of this ring are exhausted by the "topological type" cocycles defined above. To do this we use the description of the cyclic cohomology obtained in the papers [**?BryGet**], [**?Wod**] and compare this description with the image of the mapping $\mathcal{I}$.

4.2. Cohomology of psedodifferential symbols. Here we want to show that cohomology of the ring of pseudodifferential symbols are isomorphic to the cohomology of the manifold over which such symbols form a sheaf. As we have shown above this manifold is a product of a spherization of a cotangent bundle with a Spec $k((\alpha))$, i.e., with the infinitezimal punctured disk. In the discussion below we follow [**?BryGet**].

Consider the action of $k^* = k \smallsetminus \{0\}$ on T^*M by dilatations. Denote the dilatation in h times D_h. Consider a new product $\cdot_h$ on the sheaf ΨDS that is the image of the old product under the action of D_h:

$$f \cdot_h g = D^*_{h^{-1}}\left(D^*_h f \cdot D^*_h f\right).$$

Since the old product is isomorphic to the old one, the cohomologies are also isomorphic. It is easy to see that $f \cdot_h g$ considered as a function of h has a limit when $h \to 0$, and is a smooth function of h at $h = 0$. The limit is just the ordinary commutative multiplication in the sheaf of functions.

Consider now a new ring $\Psi DS \otimes k\left((h)\right)$ with multiplication $\cdot_h$:

$$\left(fh^k\right) * \left(gh^m\right) = \left(f \cdot_h g\right) h^{k+m}.$$

Since this multiplication is isomorphic to the $\cdot$-product, the cohomology are "the same": they are obtained by the tensor product with $k\left((h)\right)$. Now we can note that the product leaves $\Psi DS \otimes k\left[[h]\right]$ invariant, therefore we can consider the cohomology of this ring.[4]

At last, consider the decreasing filtration by images in multiplication by h^k on the cohomological complex of the last ring.

Theorem 4.1 ([**?BryGet**]). *The corresponding to this filtration spectral sequence degenerates at the term E_2 and converges to the cohomology of the initial complex.*

The proof of this theorem in [**?BryGet**] uses the above action of k^*. The homogeneity properties of the differential insure the fact that the higher differentials in the spectral sequence should be zero.

Until now we did not specify which cohomology theory we consider now. To follow [**?BryGet**] as near as possible let us consider the Hochschild theory. In this case the term E_0 of the spectral sequence is the Hochshild complex for the corresponding commutative algebra with multiplication $\cdot_0$ (with added variable h and additional grading in degree of h), therefore ([**?HochKosRos62Dif**], [**?Con84Non**]) E_1 coincides with the graded vector space of differential forms, and there is a compatible with the differential $d_0 E_0 \to E_1$:

$$a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto \sum_{\sigma \in \mathfrak{S}_n} a_0 da_{\sigma_1} \wedge \cdots \wedge da_{\sigma_n}.$$

The differential d_1 is given by the following formula of Koszul:

$$d_1 \colon a_0 da_1 \wedge \cdots \wedge da_n \mapsto \sum (-1)^i \left\{a_0, a_i\right\} da_1 \wedge \cdots \wedge d\widehat{a}_i \wedge \cdots \wedge da_n + \sum (-1)^{i+j} a_0 d\left\{a_i, a_j\right\} \wedge da_1 \wedge \cdots$$

here $\{,\}$ is the Poisson bracket on functions, i.e.,

$$\{f, g\} = \lim_{h \to 0} \frac{1}{h} \left(f \cdot_h g - g \cdot_h f\right).$$

Therefore all we need to compute is the cohomology of $\Omega^* DT^* M$ with respect to this differential. What we want to prove is that this operator differs from the operator of exterior derivative by a linear change. Indeed, we can consider $*$-duality in a

[4] We could not get *yo* this ring in one step, since the isomorphism of the $\cdot$-product and $*$-product does not preserve this ring.

symplectic vector space (V^{2n}, ω), that identifies $\Lambda^k V$ with $\Lambda^{2n-k} V$. The standard calculation shows that

Proposition 4.2. *Consider a $*$-operator in the exterior algebra of the cotangent bundle of a symplectic manifolds. This linear mapping transforms the above differential into the exterior derivative.*

We see that the term E^2 of the spectral sequence is given by

$$E^2_{pq} = H^{2n-p}(DT^*M) \cdot h^q.$$

To prove that the spectral sequence degenerates at this term we can note that the exterior multiplication on α supplies a homotopy for the operator of (twisted) Lie derivative with respect to the Eiler field

$$\mathcal{L}_v + n - k.$$

Here α is the canonical 1-form $p\,dq$ on T^*M, k is the degree of forms we consider. Therefore the subcomplex of E^1 formed by forms of homogeneity degree $k-n$ has the same homology, therefore the Eiler field acts on E^2_{pq} as multiplication by h^p. Now the computation of the homogeneity degree of higher differentials shows that they should vanish.

This finishes our sketch of a proof of coincidence of the Hochschild cohomology of the algebra of pseudodifferential operators and the cohomology of DT^*M. Now we want to show that constructed above mapping from the Alexander—Spanier—Hochschild complex into the Hochschild complex induces an isomorphism on cohomology.

However, we do not know how to prove this directly, so we need in fact to repeat the above arguments with the cyclic homology instead of Hochschild one. In fact, we would be much more pleased to work directly with Lie algebra homology, however, the structure of this homology of Lie algebra of differential operators is not clear now, and is conjecturally *very* complicated. Therefore we will work with cyclic homology instead, and use the fact that there is a natural mapping from Lie algebra homology to cyclic homology. We will also use some mapping in another direction, that is *not* canonically defined, and therefore will complicate the further discussion a lot.

Here we stop following the discusion in [**?BryGet**], since they used a different definition of cyclic homology than the definition used here. In this paper only the stable part of cyclic cohomology is computed, and we need the complete answer. However, the discussion remains absolutely parallel to the discussed in this paper.

So let us consider the same operations as above, i.e., addition of the new variable h and taking the filtration by degree of h in cyclic homologies. The term E_1 is again the cyclic homology of the corresponding commutative subalgebra, therefore we can use a known description of these spaces. The differential d_1 determines a mapping

$$\Omega^k_A / d\Omega^{k-1}_A \oplus H^{k-2}_{DR} \oplus \dots \to \Omega^{k-1}_A / d\Omega^{k-2}_A \oplus H^{k-3}_{DR} \oplus \dots.$$

Taking into account the compatibility of spectral sequences with mapping of complexes, we can conclude first, that on $\Omega_A^{k-1}/d\Omega_A^{k-2}$ this mapping is given by the same formula as above

$$d_1\colon a_0 da_1\wedge\cdots\wedge da_n\mapsto\sum(-1)^i\{a_0,a_i\}\,da_1\wedge\cdots\wedge d\widehat{a}_i\wedge\cdots\wedge da_n+\sum(-1)^{i+j}a_0 d\{a_i,a_j\}\wedge da_1\wedge\cdots$$

(from the mapping from the Hochschild complex) and that on H_{DR}^{k-2m} it is zero (from the compatibility with the mapping S). Indeed, we can write d_1 in the Hochschild complex as

$$d_1\alpha=\left(i(\eta)\,d_{DR}+d_{DR}i(\eta)\right)\alpha.$$

Here d_{DR} is de Rham differential, i is the inner product and η is some bivector field on T^*M. We can in fact define η by the following relation:

$$\langle\eta,df\wedge dg\rangle=\{f,g\}\,.$$

Here $\langle,\rangle$ is the usual pairing between bivectors and 2-forms with values in functions.

This formula for d_1 shows that d_1 sends a closed form into an exact form, therefore determines a mapping $\Omega_A^k/d\Omega_A^{k-1}\to\Omega_A^{k-1}/d\Omega_A^{k-2}$ indeed. Moreover, the corresponding mapping $d_1\colon H_{DR}^k\to H_{DR}^{k-1}$ vanishes. The same argument as in [?**BryGet**] shows that the spectral sequence continues to degenerate at the term E^2. (????!!!)

We see that we obtained the following theorem: (at least at E^2?????)

Theorem 4.2. *Consider the ring of pseudodifferential symbols $\mathcal{A}$ on the n-dimensional C^∞-manifold M. Then the Hochschild homology of this ring is isomorphic to cohomology of the manifold DT^*M:*

$$HH_k(\mathcal{A},\mathcal{A})\simeq H_{DR}^{2n-k}(DT^*M)\,,$$

the cyclic homology is isomorphic to the following direct sum

$$HC_k(\mathcal{A})\simeq H_k\left(\Omega_{d_1}^*/\mathcal{B}^*\right)\oplus H_{DR}^{k-2}(DT^*M)\oplus H_{DR}^{k-4}(DT^*M)\oplus\dots.$$

Here $\mathcal{B}^*$ denotes the subcomplex of exact forms on DT^*M in the complex of all forms Ω_{d_1} with the differential $d_1=d\circ i(\eta)+i(\eta)\circ d$.

The shift mapping $S\colon HC_k(\mathcal{A})\to HC_{k-2}(\mathcal{A})$ is identical on $H_{DR}^{k-2m}(DT^*M)$, $m\geq 2$, vanishes on $H_k(\Omega^*/\mathcal{B}^*)$ and is the natural mapping $H_{DR}^{k-2}(DT^*M)\to H_{k-2}(\Omega^*/\mathcal{B}^*)$ on the remaining summand. The natural mapping $HH_k(\mathcal{A},\mathcal{A})\to HC_k(\mathcal{A})$ is the composition

$$HH_k(\mathcal{A},\mathcal{A})\simeq H_{DR}^{2n-k}(DT^*M)=H_k(\Omega_{d_1})\to H_k\left(\Omega_{d_1}^*/\mathcal{B}^*\right).$$

If we consider the ring of pseudodifferential symbols with complact support, we should change the de Rham complex above to the complex of differential forms with a compact support (along M).

Now we want to investigate the question when a class of cyclic homology has a skewsymmetric representative. We have seen that in the commutative case such a class is a class from the kernel of the shift operator S.

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