THE ALEXANDER–SPANIER COHOMOLOGY AS A PART OF CYCLIC COHOMOLOGY

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ABSTRACT. Basing on a sheaf $\mathcal{O}$ with a fixed section 1 on a manifold M we introduce the notions of the de Rham, cyclic and Hochschild cohomological complexes of the Alexander–Spanier type for M with coefficients in $\mathcal{O}$. We show when these complexes are quasi-isomorphic to the usual cohomology of M and how to build cocycles for these complexes basing on cocycles for M. If $\mathcal{O}$ is a sheaf of algebras with a trace on the ring $\mathcal{A}$ of global sections, we construct mappings from these complexes to the corresponding cohomology of $\mathcal{A}$. In the case of the ring of pseudodifferential operators these mappings are isomorphisms if we consider cyclic or Hochschild complexes.

Moreover, for an arbitrary sheaf of algebras the Hochschild complex of the algebra of global sections has a natural structure of a module over the cohomological Hochschild complex of the base (with a natural product). On the level of cohomology we get an analoguous fact: algebraic Hochschild cohomology is a module over cohomological ring of the base. In the case of the sheaf of differential operators we show that this module is a free module with one generator and build this generator.

These two descriptions are compatible with known descriptions of the cohomology for corresponding algebras, however they provide also explicit constructions of cocycles. We also construct a lot of cocycles for Poisson algebras, what generalizes the Gelfand—Mathieu construction [?GelMat] to the case of an arbitrary Poisson manifold.

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0. Introduction

In the last couple of years there was a big progress in construction of cocycles for non-commutative algebras with local multiplication. In fact the first results in this direction were achieved a long time ago, when there appeared a description of cohomology of algebras of differential or pseudodifferential operators ([?BryGet, ?Wod]). However, these description were nonconstructive, so the first sign of the progress was the description of one particular Lie-algebraic cocycle of the Lie algebra of pseudodifferential operators with a use of the symbol for $\log \partial$[?KheKra].

It was a very easy task to pinpoint the topological origin of the Khesin—Kravchenko construction, and it seems now that the generalization of this construction is a common knowledge between specialists. The description of the cohomology obtained in the "ancient" papers [?BryGet, ?Wod] shows that there is a tight connection between the cohomology of the support of the algebra and the cohomology of the algebra itself. So the generalizations assign to a topological cocycle of some kind an algebraic cocycle. The best candidates for that are Čech cohomology and de Rham cohomology.

The discussion below has two targets: to give the simplest examples of the cocycles we will obtain later and to provide the reader with euristics why these cocycles are in the best cases nontrivial. We do not restrict ourselves to be *absolutely* correct with the second target, therefore the reader who needs proofs should skip all the vague arguments like "if some conditions of non-degeneracy are satisfied...". However, even in this section any construction of cocycles is still correct, hence even the most demanding reader *can* get something if he will not skip to the section 1.

0.1. **A construction of 2-cocycles.** Let us give a construction of a 2-cocycle as an example. Consider a manifold M over a field K and a sheaf $\mathcal{O}$ of associative algebras with units on M. Let $\mathcal{A}$ be $\Gamma(M, \mathcal{O})$, and suppose that there is a trace on the algebra $\mathcal{A}$, i.e., a linear functional $\operatorname{Tr} : \mathcal{A} \to K$ such that

$$\operatorname{Tr}(ab - ba) = 0$$

for any two elements $a, b \in \mathcal{A}$. The best example would be the sheaf $\mathcal{D}$ of differential operators, however, this sheaf allows only trivial trace $\operatorname{Tr} a = 0$. We will explain how to correct this deficiency later, when we use *pseudodifferential operators*.

We can consider (though approximately) a differential operator or a pseudodifferential operator as a function on a cotangent bundle. In the same way the trace on pseudodifferential operators is an analogue of integration of functions on a symplectic manifold. Therefore the reader should now imagine that there is some non-commutative deformation of the sheaf of functions on a manifold, and that the *integration* of functions deforms to a non-trivial trace on this algebra. Or, if the reader is too recalcitrant, he should consider instead *any* sheaf of algebras with a global trace. What we want to do is to construct a morphism from $H^1(M, K)$ to $H^2_{\text{Lie}}(\mathcal{A}, K)$. As we see below, in good cases this morphism is an isomorphism.

We stole the following innocent statement from [**?**KhesKra91Coc] (though it is present there only virtually): let $X \in \mathcal{A}$ and $c^1 : \mathcal{A} \to K$ given by

$$c^1 : A \mapsto \operatorname{Tr} X \cdot A \tag{0.1}$$

be a 1-cochain for $\mathcal{A}$ (here we consider, say, cochain complex for the Lie algebra that correspond to $\mathcal{A}$). Then we can rewrite a coboundary of c^1 Label equ0.3,

$$dc^1 : \mathcal{A} \otimes \mathcal{A} \to K : \ (A, B) \mapsto \operatorname{Tr} X \cdot [A, B]$$

as

$$dc^1(A, B) = \operatorname{Tr}[X, A] \cdot B. \tag{0.2}$$

Let us note that we can represent any 1-cochain on $\mathcal{A}$ in the form (0.1) if the trace on $\mathcal{A}$ is "sufficiently nondegenerate". Therefore under this condition of non-degeneracy *any* 2-*coboundary* for $\mathcal{A}$ can be written in the form (0.2). Moreover, the cochain (0.2) is remarkable by its *locality* property: let us call a 2-cochain c^2 local if there is a mapping Label equ0.6,

$$\mathcal{X} : \mathcal{O} \to \mathcal{O} : \Gamma(U, \mathcal{O}) \ni \varphi \mapsto \mathcal{X}(\varphi) \in \Gamma(U, \mathcal{O})$$

such that $c^2(A, B) = \operatorname{Tr} \mathcal{X}(A) \cdot B$.

It is clear that on local cochains the closeness is a *local property*: if we have a covering $\mathfrak{U}$ of M and a set of closed local cochains on $\mathcal{O}|_U$, $U \in \mathfrak{U}$, that are *restrictions*[1] of some global cochain, then this cochain is also closed. The following step we want to do now is to construct a local cochain that does not correspond to

[1] I.e., a local cochain coincides with the global on local sections with compact support, and such sections for different U generate the set of global sections.

any global section X. By the locality property it is closed, and since it does not correspond to any section, it cannot be a coboundary. Therefore it is a nontrivial cocycle!

Moreover, we want to do it for an arbitrary class in $H^1(M,K)$. We want here to consider a geometric realization of this cocycle as the intersection index with an (coorientable) hypersurface $H \subset M$. Consider a pair of tubular neighborhoods U_1, U of H such that $\overline{U}_1 \subset U$ and a section X_1 on U that is identically 0 near one boundary of U and identically 1 near another. Let X_2 be a 0 section on $M \smallsetminus \overline{U}_1$. The sections $X_{1,2}$ define by (0.2) two local cochains on their domains,[2] and these cochains "coincide" on the intersection of these domains. As we explained it above, that determines a cochain *on* M, and in order this cochain to be a coboundary, the section X_1 should extend to the entire M as a local constant (i.e., as a section in $K \subset \mathcal{O}$). We can write this cochain as

$$c^2(M, X_{1,2}) \colon (A,B) \mapsto \operatorname{Tr} \mathcal{X}(A) \cdot B, \quad \mathcal{X}(A) \stackrel{\text{def}}{=} [X_{1,2}, A].$$

In the definition of $\mathcal{X}$ we should take a different function X_1 or X_2 depending on the region of M we are currently in—the result $\mathcal{X}(A)$ does not depend on the choice anywhere a choice is possible.

If H divides M into two parts, then X_1 can be extended into one part as 0 and into another as 1. However, in this case H represent a trivial cohomology class. Therefore we constructed a promised mapping

$$H^1(M,K) \to H^2_{\text{Lie}}(\mathcal{A},K).$$

The cheating in this construction is the choice of the section X_1. If $\mathcal{O}$ is indeed the isomorphic as a sheaf to the sheaf of functions, and M is a C^∞-manifold, then there is no problem in providing such a section. Otherwise the notion of such a section is correctly defined (since $\mathcal{A}$ is an algebra with unity, there is a constant subsheaf $K \subset \mathcal{O}$, so there is a notion of section being locally 0 or locally 1), but to find it we need some additional "nice" properties of the sheaf $\mathcal{O}$, like $\mathcal{O}$ being soft.

We can consider $X_{1,2}$ as a (global) section of the sheaf $\mathcal{O}/K$. Then the discussion above can be rewritten in one phrase: the mapping

$$\mathcal{X} \colon \mathcal{O} \to \mathcal{O} \colon A \to [X,A]$$

is correctly defined even in the case when X is not an element of $\mathcal{A} = \Gamma(M,\mathcal{O})$, but an element of $\Gamma(M,\mathcal{O}/K)$, and the sequence

$$0 \to K = \Gamma(M,K) \to \mathcal{A} = \Gamma(M,\mathcal{O}) \to \Gamma(M,\mathcal{O}/K) \to H^1(M,K) \to H^1(M,\mathcal{O}) \to \dots$$

is exact. However, as the following generalization shows, this abstraction is too concrete to sustain useful modifications.

[2] More precise, on the rings of global sections with compact support on their domains.

0.2. **2-cocycles for pseudodifferential symbols.** As we will see later (when we give a precise definition of a pseudodifferential symbol on a circle), this ring is a ring of global section over a product of two circles: one ordinary, another infinitesimal. This manifold has 2-dimensional space H^1, therefore we can construct two 2-cocycles. However, this two 2-cocycles correspond to different geometrical objects (since the radii of the circles are so different), therefore we need two slightly different constructions.

Example 0.1. Consider the sheaf of pseudodifferential symbols on a circle S^1. We consider them as "functions" $\varphi(x,\xi)$ on the cotangent bundle T^*S^1. In fact these functions are just asymptotic expansions when $\xi\to\infty$, so they are defined on the infinitesimal neighborhood of the infinity in the cotangent bundle. There are two classes in H^1 of this manifold: one corresponds to a hypersurface $x=\text{const}$, another one to

$$\xi = a \text{ very-very big const}.$$

Consider a first one of these two classes and the corresponding function X_1. We can suppose that X_1 depends only on x, and that it has a "jump" near the point $x=0$. Now we want to expand X_1 to be as near as it is possible to a function on a circle, i.e., a function with period 1. This function (where defined) is 0 if $x<-c$, is 1 if $x>c$. Let us extend it as 0 on the interval $-1+c<x<-c$ and as 1 on the interval $c<x<1-c$. Now this function is already non-periodic, but it satisfies the relation

$$X_1(x+1) = X_1(x)+1$$

instead. Moreover, we can uniquely extend it to a function $\widetilde{X}_1$ on the entire line leaving this relation true. However, since for any function $A(x,\xi)$ with period 1 in x the expression

$$\left[\widetilde{X}_1, A\right]$$

is periodic with period 1, we can still apply the formula (0.2) and get a 2-cocycle

$$(A,B)\mapsto \operatorname{Tr}\left[\widetilde{X}_1, A\right]\cdot B.$$

(And we do not need to know the precise law of multiplication for pseudodifferential operators, the only thing we need to know is the translation-invariance of this multiplication.)

However, we can still simplify this formula a lot. Let as note that an addition of a periodical function to $\widetilde{X}_1$ results in changing this cocycle by a coboundary, as the formula (0.2) shows. Therefore we can substitute the function x instead of $\widetilde{X}_1(x)$, since $\widetilde{X}_1(x)-x$ is a periodical function. We result in the following formula for a cocycle:

$$(A(x,\xi), B(x,\xi)) \mapsto \operatorname{Tr}[x,A]\cdot B = -\operatorname{Tr}\frac{\partial A}{\partial \xi}\cdot B.$$

Example 0.2. To deal with the second case is a little bit more tricky, especially since we cannot formulate precisely what we mean by "a very-very big const". Let us proceed first as in the first example. Consider a hypersurface $\xi = \text{const}$ and a corresponding function X_1. The big problem is that the functions we consider should also have good symmetry properties. In the previous example they should have been invariant with respect to translation in x, here they should have a good decomposition with respect to the action of expansions in ξ, as the definition of a psudodifferential symbol shows.

One way to circumvent this is to consider a family of surfaces that are "approximately invariant" with respect to expansions in ξ, say

$$\xi = \text{const} \cdot \alpha^k, \quad k \in \mathbb{Z},\ \alpha > 1.$$

The corresponding function X_1 is locally constant away from these surfaces and has a "jump" 1 near any one of them. This modification is in direct analogy with the step from a locally defined function X_1 to an "almost periodical" function $\widetilde{X}_1$.

This function X_1 satisfies the property

$$X_1\left(\alpha^k \xi\right) = X_1(\xi) + k$$

of "almost-invariance" with respect to a discrete group of expansions. If we consider instead of a discrete family of hypersurfaces a "continuous family", or if we take the limit $\alpha \to 1$ with the corresponding scaling of X_1, we get a function

$$X_1(\xi) = \log \xi.$$

If the reader believes what was discussed so far, he should understand now that the formula

$$(A, B) \mapsto \operatorname{Tr}[\log \xi, A] \cdot B$$

is correct, defines a cocycle for Lie algebra of pseudodifferential operators, and that this cocycle cannot be a coboundary (since $\log \xi$ is not a pseudodifferential symbol). Moreover, it should be clear that the classes of two defined cocycles are linearly independent, since no linear combination of x and $\log \xi$ is simultaneously periodic and a sum of homogeneous in ξ functions.

Remark 0.3. The second cocycle has certain advantages comparing with the first. While the first cocycle is trivial after restriction on the ring of differential operators, the second one gives (the only nontrivial) 2-cocycle for this ring. This is a reason why the much simpler first cocycle was missed so far—and while it is discovered, the discussed in this paper theory becomes almost obvious.

We want to note also that though it is possible to *consider* the second cocycle on differential operators only, to *define* it we need pseudodifferential symbols.

0.3. **3-cocycles and 4-cocycles.** Here we want to construct a generalization of the above construction to higher codimensions. Again, we want to begin with constructions of (local) cochains and coboundaries.

Call an n-cochain c on $\mathcal{A}$ a *local cochain* if

$$c\left(A_1,\dots,A_n\right)=0 \text{ if } \bigcap_{i=1}^{n}\operatorname{Supp}A_i=\varnothing.$$

Suppose that the sheaf of algebras $\mathcal{O}$ is isomorphic to a sheaf of functions on M. In this case such a cochain is just a skew-symmetric generalized function with a support on a diagonal in M^n. Locally we can write any such function (i.e., a functional on the space of functions) as a linear combination of the terms

$$A_1\otimes\dots\otimes A_n\mapsto \operatorname{Tr}\operatorname{Alt}\mathcal{D}_1A_1\cdot\dots\cdot\mathcal{D}_nA_n,$$

and

$$A_1\otimes\dots\otimes A_n\mapsto \operatorname{Tr}\operatorname{Alt}\mathcal{D}_1A_1\cdot\dots\cdot\mathcal{D}_{n-1}A_{n-1}\cdot f_0A_n,$$

where $\mathcal{D}_i$ are differential operators without a term of degree 0, and f is a function on M. Now suppose that the product on $\mathcal{O}$ is a deformation of the commutative product on the sheaf of functions with respect to a non-degenerate Poisson structure. In this case we can write the operator $\mathcal{D}_i$ as a composition of vector fields, i.e., of Poisson brackets with functions on M. We can see that in this case we can write any local cochain as

$$A_1\otimes\dots\otimes A_n\mapsto \operatorname{Tr}\operatorname{Alt}\left[f_1^1,\left[f_1^2,\left[\dots,\left[f_1^{k_1},A_1\right]\right]\right]\right]\cdot\dots\cdot\left[f_{n-1},\left[\dots\left[f_{n-1}^{k_{n-1}},A_{n-1}\right]\right]\right]\cdot f_0A_n,$$

or as the analogous expression without f_0. Now we can write any commutator as a difference of products, therefore any such function can be written as

$$A_1\otimes\dots\otimes A_n\mapsto \operatorname{Tr}\operatorname{Alt}f_1\cdot A_1\cdot f_2\cdot A_2\cdot\dots\cdot f_n\cdot A_n.$$

Therefore we obtained a general formula for local cocycles, and we can write a general formula for local coboundaries (all under the above assumptions). If we avoid the question of a local cochain being a coboundary, but of non-local cochain only, then to construct a non-trivial cocycle we can try to find a local coboundary that is not a global coboundary. To do this we need to fix a geometrical realization of a class of cohomology on M, say a submanifold in M.

Suppose that codimension is 2. Let X_1, X_2 be two functions on M. Consider a cochain

$$c^2_{\{X_i\}}\left(A_1,A_2\right)=\operatorname{Tr}\operatorname*{Alt}_{\sigma,\tau\in\mathfrak{S}_2}X_{\sigma_1}\cdot A_{\tau_1}\cdot X_{\sigma_2}\cdot A_{\tau_2}.$$

Then we can write a coboundary of this cochain as

$$\begin{aligned}dc^2_{\{X_i\}}\left(A_1,A_2,A_{n+1}\right)=\operatorname{Tr}\operatorname*{Alt}_{\sigma\in\mathfrak{S}_2,\tau\in\mathfrak{S}_3}&\left(\frac{1}{3}\left[X_{\sigma_1},A_{\tau_1}\right]\cdot\left[X_{\sigma_2},A_{\tau_2}\right]\cdot A_{\tau_3}\right.\\&\left.+\frac{1}{12}\left[A_{\tau_1},A_{\tau_2}\right]\cdot\left[X_{\sigma_1},X_{\sigma_2}\right]\cdot A_{\tau_3}\right).\end{aligned}\tag{0.3}$$

Suppose that codimension is 3. Let X_i, $i = 1, \dots, 3$, be functions on M. Consider a cochain Label equ0.10,

$$c^3_{\{X_i\}}(A_1, A_2, A_3) = \operatorname{Tr} \operatorname*{Alt}_{\sigma,\tau\in\mathfrak{S}_3} X_{\sigma_1} \cdot A_{\tau_1} \cdot X_{\sigma_2} \cdot A_{\tau_2} \cdot X_{\sigma_3} \cdot A_{\tau_3}.$$

Then we can write a coboundary of this cochain as

$$\begin{aligned} dc^3_{\{X_i\}}(A_1, \dots, A_4) = \operatorname{Tr} \operatorname*{Alt}_{\sigma\in\mathfrak{S}_3,\tau\in\mathfrak{S}_4} \Big(\frac{1}{4} [X_{\sigma_1}, A_{\tau_1}] \cdot \dots \cdot [X_{\sigma_3}, A_{\tau_3}] \cdot A_{\tau_4} \\ + \frac{1}{16} [X_{\sigma_1}, A_{\tau_1}] \cdot [A_{\sigma_2}, A_{\sigma_3}] \cdot [X_{\tau_2}, X_{\tau_3}] \cdot A_{\tau_4} \\ + \frac{1}{16} [X_{\tau_1}, X_{\tau_2}] \cdot [A_{\sigma_1}, A_{\sigma_2}] \cdot [X_{\sigma_3}, A_{\tau_3}] \cdot A_{\tau_4} \Big). \end{aligned} \tag{0.4}$$

Now we want to show that (at least in some particular cases) we can use these two formulae for generation of cocycles, and we can hope that in reasonable cases these cocycles should be non-trivial. We see that in a formula for a local coboundary in the codimension 2 and 3 any occurence of X_i is in the form Label equ0.11,

$$[X_i, \text{something}].$$

Therefore if we know X up to a (locally defined) constant only, we can still use these formulae and we get a cocycle. If we cannot find global X_i with the specified non-constant part, then there is a big hope that this cocycle is non-trivial.

Now consider a submanifold S of codimension n in M and let us try to repeat the above construction in these conditions. One particular case is when this submanifold is a complete intersection in its neighborhood. We mean that we can construct hypersurfaces H_i, $i = 1, \dots, n$, in a neighborhood of S such that M is a transversal intersection of H_i. Now let X_i be the functions with a change 1 in a narrow neighborhood of H_i and locally constant far from it. Consider the right-hand sides of the formulae (0.3)–(0.4). They define some $(n+1)$-cochains of $\mathcal{A}$. Indeed, though X_i are defined only in a neighborhood of S, but the function under the trace sign is non-zero only in a smaller neighborhood. Therefore we can extend it everywhere as 0 and take the trace.

In the same way as above what we get is a cocycle (since locally it looks as a coboundary). If the class of S in $H^n(M, K)$ is non-trivial, there is a big hope that we get a non-trivial cochain.

Example 0.4. Let us combine the two discussed above examples of cocycles to construct a 3-cocycle for pseudodifferential symbols. We get the following formula:

$$(A, B, C) \mapsto \operatorname{Tr} \left(\frac{\partial A}{\partial \xi} \cdot [\log \xi, B] \cdot C - [\log \xi, A] \cdot \frac{\partial B}{\partial \xi} \cdot C \right).$$

This cocycle corresponds to the intersection of the plane $x = \text{const}$ with the plane $\xi = \text{const}$, i.e., to a cohomological class of a point.

0.4. **Higher dimensions.** In the case codim > 3 we do not know if we can write a differential of a local cochain in a form similar to (0.3)–(0.4). However, it is not necessary. Let X_i, $i = 1, \dots, n$, be functions on M. Consider a cochain

$$c^n_{\{X_i\}}(A_1, \dots, A_n) = \operatorname{Tr} \operatorname*{Alt}_{\sigma, \tau \in \mathfrak{S}_n} X_{\sigma_1} \cdot A_{\tau_1} \cdot \dots \cdot X_{\sigma_n} \cdot A_{\tau_n}.$$

Then we can write a coboundary of this cochain as

$$dc^n_{\{X_i\}}(A_1, \dots, A_n, A_{n+1}) = \pm \operatorname{Tr} \operatorname*{Alt}_{\sigma \in \mathfrak{S}_n, \tau \in \mathfrak{S}_{n+1}} A_{\tau_1} \cdot X_{\sigma_1} \cdot A_{\tau_2} \cdot \dots \cdot X_{\sigma_n} \cdot A_{\tau_{n+1}}. \tag{0.5}$$

Now it is very easy to see that if $X_1 = \text{const}$, then the alternation vanishes. Therefore we can substitute a section of $\mathcal{O}/K$ instead of X in this formula, therefore any argument above is still applicable. Again under some non-degeneracy conditions any cochain can be written as a linear combination of such, therefore there is a hope to write down a cocycle that is locally of the same form. What does the word "locally" mean here? We can see that if any one of X_i vanishes in a neighborhood of some point, then the expression under the trace sign vanishes there. Therefore we can consider a function $X_1 \otimes \dots \otimes X_n$ on $M \times \dots \times M$: Label equ0.12,

$$X_1 \otimes \dots \otimes X_n (m_1, \dots, m_n) = X_1(m_1) \dots X_n(m_n).$$

This function uniquely determines the corresponding cochain, moreover, the above remark on locality shows that it is sufficient to know this function in a neighborhood of the diagonal. So "locally" means exactly this consideration in a neighborhood of the diagonal.

The only problem now is what to do with the case of when S is not a local intersection. In less demanding cohomological theories we could consider a decomposition of unity. To do this in our case we should put some cut-off functions in the formula (0.5). However, there are too many places to "put a horse into", therefore it is not so easy to do this in such a way that the result will remain closed. Another problem is that we have too many degrees of freedom: we can get a mapping of cohomology groups, but this mapping is too far away from the "cohomological dream", when we have mapping of complexes themselves.

0.5. **The appearing of Alexander–Spanier theory.** One of the possible constructions is the use of Alexander–Spanier theory as a source for the initial cocycle on M. Consider the construction of a 2-cocycle basing on a section of $\mathcal{O}/K$. This section is essentially a closed 1-form on M, if $\mathcal{O}$ is the sheaf of functions. In fact we can write the basic element $[X, A]$ from (0.2) as

$$X \cdot A \cdot 1 - 1 \cdot A \cdot X.$$

In both terms A is in between, therefore we just consider the action of the element $1 \otimes X - X \otimes 1 \in \mathcal{A} \otimes \mathcal{A}$ on $A \in \mathcal{A}$ with respect to the usual left-right action. Now come two crucial observations: if we change X by a constant, the element $1 \otimes X - X \otimes 1$ does not change, and we need to know $1 \otimes X - X \otimes 1$ only on a neighborhood of a diagonal in $M \times M$ (we consider $\mathcal{A} \otimes \mathcal{A}$ as sections of $\mathcal{O} \boxtimes \mathcal{O}$ on $M \times M$). Indeed, if an element of $\mathcal{A} \otimes \mathcal{A}$ is zero in a neighborhood of the diagonal, it acts as 0 on $\mathcal{A}$. Hence this element of $\mathcal{A} \otimes \mathcal{A}$ (i.e., a section of $\mathcal{O} \boxtimes \mathcal{O}$ on $M \times M$) is correctly defined in a neighborhood of a diagonal if X is defined up to a locally constant section.

Therefore we come to the following construction: basing on a section $X \in \Gamma(M, \mathcal{O}/K)$ we get a section $1 \otimes X - X \otimes 1$ of $\mathcal{O} \boxtimes \mathcal{O}$ in a neighborhood of diagonal in $M \times M$. However, this section is just a representation of dX in the Alexander–Spanier complex. What remains to do is to find a more natural place for B from (0.2) and construct a generalization to the case of cocycles of higher order (this is a definition of "strange pairing").

So the topic of this article is a strange observation that while there is a big ambiguity in a construction of the mapping from the, say, Čech complex to a cyclic complex, this ambiguity is washed out if we start with an Alexander–Spanier complex. That means that, in fact, all the ambiguity is lying in the step from the Čech complex to the Alexander–Spanier one.

We remind here several useful mapping (including ambiguities) from various topological complexes to the Alexander–Spanier one and construct a *canonical* mapping from the latter complex to the cocyclic complex. (This in fact gives us also a mapping to the Hochschild complex and the Lie-algebraic one.) A remarkable property of this mapping is that it *does not depend* on the structure of the algebra, only on sheaf-theoretical structure of the corresponding sheaf.

We also show that the described set of cocycles give the entire cohomology of the corresponding algebra in cases when this cohomology is known.

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These papers together with what is written here suggest that it is interesting to try to rewrite some "standard" proof of this theorem using the Alexander–Spanier cohomology instead of the usual one.

1. Alexander–Spanier cohomology

Label h1

In this section, the sheaves we consider are going to be sheaves of K-modules for a commutative algebra K with unity. The tensor products are taken over K. Unless specified otherwise, K is going to be a field.

If you have a differential manifold M, usually there is a lot of different ways to describe the same object: the cohomology of M. You can write a lot of different complexes that are all quasi-isomorphic. In various geometrical situations you can apply the complex that you feel is more suitable for it.

However, there is one particular type of complex that appears very rare if you need a geometrical description of cohomology. I mean the *Alexander—Spanier complex*, applications of which are usually met in hard topological papers. Here I want to show that (quite unanticipated) it is very useful in descriptions of highly geometrical objects: cyclic cohomology, that are just a non-commutative analogue of the de Rham cohomology.

1.1. **Alexander–Spanier complex.** Consider a topological space M and the vector space $\mathcal{A}$ of (say, continuous) functions on M. Let

$$\underbrace{\mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes}\dots\hat{\otimes}\mathcal{A}}_{n\text{ times}}=\mathcal{A}^{\hat{\otimes}n}$$

be the space of functions[3] on M^n. We can consider the inclusion

$$\underbrace{\mathcal{A}\otimes\mathcal{A}\otimes\dots\otimes\mathcal{A}}_{n\text{ times}}=\mathcal{A}^{\otimes n}\subset\mathcal{A}^{\hat{\otimes}n}$$

of the space of functions *of finite rank* into this space. Let me remind you that a function of rank 1 is just a function of the form

$$f\left(m_1,m_2,\dots,m_n\right)=f_1\left(m_1\right)f_2\left(m_2\right)\dots f_n\left(m_n\right),$$

and a function of rank k can be represented as a linear combination of such functions. Let $\Lambda^k\mathcal{A}\subset\widehat{\Lambda}^k\mathcal{A}$ denote the spaces of skewsymmetric functions on M^n of finite rank and of any type correspondingly. This vector spaces form two complexes, if we consider the operation of exterior multiplication[4] by $1\in\mathcal{A}$

$$\wedge 1\colon f_1\wedge f_2\wedge\dots\wedge f_n\mapsto f_1\wedge f_2\wedge\dots\wedge f_n\wedge 1\colon \Lambda^k\mathcal{A}\to\Lambda^{k+1}\mathcal{A}$$

as a differential of degree 1. We can extend this operation on $\widehat{\Lambda}^k\mathcal{A}$ if we note that this operation can be written as

$$f\left(x_1,x_2,\dots,x_k\right)\mapsto df\left(x_1,x_2,x_k,\dots,x_{k+1}\right)=\sum_i\left(-1\right)^{k+1-i}f\left(x_1,x_2,\dots,\widehat{x}_i,\dots,x_{k+1}\right).$$

[3]Here the completed tensor product $\hat{\otimes}$ is *by definition* what is written above. Since we do not need this notion below, we skip the discussion of this notion.

[4]Usually, people define exterior power as a *quotient* of the tensor power; we consider instead a *subspace* of the tensor power (they cannot be identified unless the base field/ring contains $\mathbb{Q}$). On the quotient, $\wedge$ is naturally defined as the "quotient" of the operation $\otimes$; in particular, $v\wedge v'$ is the image of $v\otimes v'$ in the quotient. The proper definition of $\wedge$ on skew tensors t,t' of degrees k,k' is $\sum\operatorname{sgn}(\sigma)\,\sigma\left(t\otimes t'\right)$ with σ running over permutations of $\{1,\dots,k+k'\}$ which increase on $\{1,\dots,k\}$ and $\{k+1,\dots,k+k'\}$. (When one can identify the subspace and the quotient space flavors, this differs by a binomial coefficient from $\wedge$ on the quotient space.) In particular, $v\wedge v'$ is $v\otimes v'-v'\otimes v$, which is a skewsymmetrization of $2v\otimes v'$.

A similar dichotomy exists in definitions of exterior forms. While the original of [?KobNom] and [?SteLec] define the product "as in the quotient", [?DubNovFom] defines it "as in the subspace". To mud the water yet more, Russian translator of [?KobNom] adds a footnote recommending use of the other definition...

Remark 1.1. The geometrical realization of the bigger complex is as following: Call an n-tuple of points in M considered up to an alternation a simplex in a manifold. There is a natural operation of taking a boundary in the vector space spanned by simplices. Now we can consider a skewsymmetric function on M^n as a function on the set of simplices. It is easy to understand that the differential above is exactly the combinatorial differential on the simplicial complex.

At last, let M_Δ be diagonal subset in M^n, $\Delta\colon M_\Delta \hookrightarrow M^n$ denote the inclusion and $\Delta^*\left(\Lambda^k\mathcal{A}\right) \subset \Delta^*\left(\widehat{\Lambda}^k\mathcal{A}\right)$ denote[5] the spaces of *germs* of skewsymmetric continuous functions at a neighborhood of the diagonal (of finite rank and arbitrary correspondingly).

Definition 1.2. The Alexander–Spanier complex $\operatorname{AS}(\mathcal{A})$ consists of the vector spaces $\Delta^*\left(\widehat{\Lambda}^k\mathcal{A}\right)$ (or $\Delta^*\left(\Lambda^k\mathcal{A}\right)$). The differential in this complex is the image of the differential in the complex $\left(\widehat{\Lambda}^k\mathcal{A}, \wedge 1\right)$ (or $\left(\Lambda^k\mathcal{A}, \wedge 1\right)$).

Remark 1.3. To get a geometrical description of this complex we should call an n-tuple of nearby points on M a simplex. Then an element of a complex is a skewsymmetric function on simplices, and $\wedge 1$ is dual (up to a sign) to taking a sum of faces of a simplex.

Remark 1.4. In what follows we use primarily the smaller complex. However, it is known that in nice situations the inclusion of the smaller complex into the bigger is a quasi-isomorphism.

Remark 1.5. Consider a pushforward $\Delta_*\Delta^*\left(\Lambda^k\mathcal{A}\right)$ (or the same with $\widehat{\Lambda}$). It consists of global sections of the corresponding sheaf on M^n, and, as a vector space, is naturally isomorphic to $\Delta^*\left(\Lambda^k\mathcal{A}\right)$. If any germ near diagonal M_Δ can be extended to a global function on M^n (e.g., when $\mathcal{A}$ consists of global sections of soft sheaf on a locally compact space), then $\Delta_*\Delta^*\left(\Lambda^k\mathcal{A}\right)$ consists of functions on M^n modulo functions vanishing near the diagonal M_Δ.

In general, this shows that one can multiply elements of $\Delta^*\left(\Lambda^k\mathcal{A}\right)$ by functions on M^n of finite rank. On the other hand, this also demonstrates why $\Delta^*\left(\Lambda^k\mathcal{A}\right)$ has no natural structure of $\mathcal{O}_M$-module.

1.2. **A case with an arbitrary sheaf.** Let us consider instead of the vector space $\mathcal{A}$ of functions on M the corresponding sheaf $\mathcal{O}$ of vector spaces over M. We can easily see that the definition of the complex $\left(\Lambda^k\mathcal{A}, \wedge 1\right)$ in fact does not depend on

[5] In other words, Δ^* is taking the global sections of the pullback *in category of sheaves of sets/groups*. While most of the sheaves considered in this paper have a structure of $\mathcal{O}$-module, we do not use the inverse image in the category of $\mathcal{O}$-modules, so do not need to denote it. (Note that when people need different flavors of pushforward/pullback, they use notations like $\Delta^\bullet$, Δ^+ etc.)

anything but the sheaf structure of $\mathcal{O}$ and the global section 1 of this sheaf. So we are going to rewrite this definition using only these data.

Definition 1.6. Let $\mathcal{O}$ be a sheaf of vector spaces over M. Denote as $\mathcal{O}^{\boxtimes n}$ the *exterior tensor product* of the sheaf $\mathcal{O}$ with itself. This sheaf over M^n is defined by the *sheafification* of the following rule:

$$\Gamma\left(U_1 \times \cdots \times U_n, \mathcal{O}^{\boxtimes n}\right) = \Gamma\left(U_1, \mathcal{O}\right) \otimes \cdots \otimes \Gamma\left(U_n, \mathcal{O}\right);$$

in other words, the stalk of $\mathcal{O}^{\boxtimes n}$ over $(m_1, \dots, m_n)$ is the tensor product of stalks over $m_1, \dots, m_n$ ([**?**Bredon]). It is clear that the symmetric group $\mathfrak{S}_n$ is acting on M^n and on the sheaf $\mathcal{O}^{\boxtimes n}$. Denote as $\operatorname{Alt} \mathcal{O}^{\boxtimes n}$ the subsheaf of skewsymmetric sections (i.e., sections φ on $U \subset M^n$ such that for any $s \in \mathfrak{S}_n$ the section $s\varphi$ satisfies the relation $s\varphi|_{sU\cap U} = (-1)^s \varphi|_{sU\cap U}$).

For any fixed global section of $\mathcal{O}$ (call it 1) the sheafes $\operatorname{Alt} \mathcal{O}^{\boxtimes n}$ form a natural complex with the exterior product by 1 as a differential; more precisely, ; then $\Delta^* \operatorname{Alt} \mathcal{O}^{\boxtimes n}$ form a complex of sheaves on M.

Let us denote by $\Lambda^k\mathcal{O}$ the sheaf $\Delta^*\left(\operatorname{Alt} \mathcal{O}^{\boxtimes k}\right)$ on M_Δ; identify M and the diagonal M_Δ. Sheaves $\Lambda^k\mathcal{O}$ with differential $\wedge 1$ form a natural complex of sheaves on M for any fixed global section 1 of the sheaf $\mathcal{O}$. A section of $\Lambda^k\mathcal{O}$ over $U \subset M$ is a skewsymmetric section of $\mathcal{O}^{\boxtimes k}$ over a small neighborhood of $\Delta(U) \subset M^k$. Let $C^k_{\mathrm{AS}}(\mathcal{O}) = \Lambda^{k+1}\mathcal{O}$, $k \geq 0$.

1.3. **Realization of Alexander–Spanier cocycles.** Here we are going to give several examples of mappings from some complexes calculating the cohomology of M to the Alexander–Spanier complex. These constructions give us a possibility to provide explicit formulae for cocycles in case we need one.

Label cs1

Case 1.7. Let M be covered by open subsets U_i. Let σ_i be a unity decomposition for the covering $\{U_i\}$.

Consider a Čech cochain $c_{i_0 i_1 \dots i_n}$ for $\{U_i\}$. Let us associate to c the following Alexander–Spanier cochain:

$$f(x_0, \dots, x_n) = \sum_{i_0, \dots, i_n} \sigma_{i_0}(x_0) \dots \sigma_{i_n}(x_n)\, c_{i_0 \dots i_n}. \tag{1.1}$$

It is easy to see that this mapping from the Čech complex to the Alexander–Spanier complex is compatible with differentials. Label equ1.10,

A chain of the cosimplicial complex is a function on the set of embedded simplices. To construct a chain in the Alexander–Spanier complex we need only to assosiate with an $(n+1)$-tuple of nearby points on M an embedded simplex (or a linear combination thereof). To proceed in this way we need a further structure on M.

Label cs2

Case 1.8. M is a Riemannian manifold.

In this case given two nearby points $m_1, m_2 \in M$ we can consider a geodesic arc $\mathcal{S}^1(m_1 m_2)$ with ends in this points. Given a point $m \in M$ and a subset $V \subset M$ we can construct

$$\operatorname{Arc}(m, V) = \bigcup_{v \in V} \mathcal{S}^1(mv).$$

Let us associate (using induction) to the ordered $(n+1)$-tuple $(m_0, \dots, m_n)$ of points of M a simplex

$$\mathcal{S}^n(m_0, \dots, m_n) = \operatorname{Arc}\left(m_0, \mathcal{S}^{n-1}(m_1, \dots, m_n)\right)$$

in M. Note that the natural mapping from an affine n-dimensional simplex to $\mathcal{S}^n$ is *not* C^1-smooth! (The tangent cone at m_0 is "curvilinear".) However, the induction shows that the images of faces of the affine simplex are of the form $\mathcal{S}^k(m_{s_0}, \dots, m_{s_k})$ with $s_0 < s_1 < \dots < s_k$.

Taking the antisymmetrization of this map, we associate to the $(n+1)$-tuple $(m_0, \dots, m_n)$ a linear combination

$$\frac{1}{(n+1)!} \sum_{s \in \mathfrak{S}_{n+1}} (-1)^s \mathcal{S}(m_{s_0}, \dots, m_{s_n})$$

of imbedded simplices in M. It is easy to see that this mapping is compatible with taking a boundary.

Now given an n-form ω we can integrate it over this linear combination of simplices (possible since any $\mathcal{S}^n$ is a smooth image of a cube). It is easy to see that the resulting skew-symmetric function on M^{n+1} is closed if ω is closed (essentially, it follows from the mapping of a face of the cube being degenerate whenever the corresponding mapping into an affine simplex is degenerate).

Label cs3

Case 1.9. Let M be covered by subsets U_i with an identification of U_i with an open convex subset in an affine space. Let σ_i be a unity decomposition for the covering $\{U_i\}$. Let ω be a differential k-form on M.

In this case we can proceed as in the previous one. If ω has a support in one of subsets U_i we can define the following Alexander–Spanier cochain in U_i: to $k+1$ given points in U_i we associate the integral of ω over the oriented convex hull of this points. We can extend this function to the entire M (more precise, to the neighborhood of the entire $\operatorname{diag} M$ in M^n) to get a cochain on M. Now we can apply this construction to the forms $\sigma_i \omega$.

1.4. **The analogues for the cases of cyclic and Hochschild complexes.** We will see below that the discussed above complex is adopted to the case of cohomology of Lie algebra. Here we introduce two other complexes adopted to calculations of cyclic and Hochschild cohomology. In what follows, K is a commutative ring; unless explicitly specified otherwise, K is assumed to be a field.

Althow the definitions below are stated in terms of sheaves of vector spaces, the case of M being a point is interested as well. In this case, one deals with one vector space V instead of a sheaf $\mathcal{O}$.

Definition 1.10. Let $\mathcal{O}$ be a sheaf of vector spaces over M with a marked section 1. Consider the following differential in the graded sheaf $\bigoplus_n \Delta^* \mathcal{O}^{\boxtimes n+1}$:

$$d\left(f_0 \boxtimes \cdots \boxtimes f_n\right) = (-1)^{n+1}\, 1 \boxtimes f_0 \boxtimes \cdots \boxtimes f_n \\ + (-1)^n\, f_0 \boxtimes 1 \boxtimes \cdots \boxtimes f_n + \cdots + f_0 \boxtimes \cdots \boxtimes f_n \boxtimes 1, \qquad d^2 = 0.$$

Let $C^\bullet_{\mathrm{HAS}}(\mathcal{O}) = \left(\Gamma\left(M, \Delta^*\left(\mathcal{O}^{\boxtimes \bullet+1}\right)\right), d\right)$, $\bullet \geq 0$. Call this complex a Hochschild–Alexander–Spanier complex for $\mathcal{O}$.

Definition 1.11. Let $\mathcal{O}$ be a sheaf of vector spaces over M with a marked section 1. Consider the following differential in the graded sheaf $\bigoplus_n \mathcal{O}^{\boxtimes n+1}$:

$$d_a\left(f_0 \boxtimes \cdots \boxtimes f_n\right) = (-1)^n\, f_0 \boxtimes 1 \boxtimes \cdots \boxtimes f_n + \dots \\ - f_0 \boxtimes \cdots \boxtimes 1 \boxtimes f_n + f_0 \boxtimes \cdots \boxtimes f_n \boxtimes 1, \qquad d_a^2 = 0.$$

Let $C^\bullet_{\mathrm{aHAS}}(\mathcal{O}) = \left(\Gamma\left(M, \Delta^*\left(\mathcal{O}^{\boxtimes \bullet+1}\right)\right), \Delta^* d_a\right)$, $\bullet \geq 0$. Call this complex an “acyclic” Hochschild–Alexander–Spanier complex for $\mathcal{O}$.

Remark 1.12. In what follows we are not so rigorous and use often the notation $\otimes$ instead of $\boxtimes$.

When we want to consider a particular graded component of d or d_a, we use the notations $d^{[n+1]}$ and $d_a^{[n+1]}$ for the components written above. (So in this notation we use the grading by the valence of the tensor d or d_a acts on.)

Definition 1.13. Consider a product $V^{\otimes k} \otimes V^{\otimes l} \to V^{\otimes k+l}$ defined by the following rule: to define the image of

$$(f_1 \otimes \cdots \otimes f_k) \otimes (g_1 \otimes \cdots \otimes g_l)$$

consider all the decomposition of the set $\{1, \dots, k+l\}$ into two subsets of k and l elements. Insert the elements f_i on the places of the first subset and the element g_j on the places in the second subset in the expression

$$\underbrace{\bullet \otimes \cdots \otimes \bullet}_{k+l \text{ times}}$$

preserving the order in both sets of elements. Now sum the resulting elements with signs corresponding to the the substitution being even or odd. Call this associative product a *shuffle product*.

Definition 1.14. Consider the action $t = t_k$ of $\mathbb{Z}_k$ in $V^{\otimes k}$ (here V is a vector space) by

$$v_1 \otimes \cdots \otimes v_k \overset{t}{\mapsto} (-1)^{k+1}\, v_k \otimes v_1 \cdots \otimes v_{n-1}.$$

Call the space of invariants of this action $\left(V^{\otimes k}\right)^{\mathbb{Z}_k}$ the cyclic k-th power of V. Considering action of t on one term of shuffle product shows that the shuffle product sends cyclic powers into cyclic powers. Let $\mathbb{Z}_{n+1}$ acts in this way on C^n_{HAS}, and consider the corresponding space of invariants $(C^n_{\mathrm{HAS}})^{\mathbb{Z}_n}$. Consider a mapping of shuffle product with 1:

$$\wedge 1\colon (C^n_{\mathrm{HAS}})^{\mathbb{Z}_n} \to \left(C^{n+1}_{\mathrm{HAS}}\right)^{\mathbb{Z}_{n+1}}.$$

Since the shuffle product is associative, the square of the mapping $\wedge 1$ vanishes. Call this complex the cyclic Alexander–Spanier complex and denote it $C^\bullet_{\mathrm{cAS}}(\mathcal{O})$.

Define the operator $N = N_k = 1 + t_k + \dots + t_k^{k+1}\colon V^{\otimes k} \to V^{\otimes k}$; denote by $\mathbb{1}_L$ and $\mathbb{1}_R$ the operators of tensor multiplication by 1 on the left and right correspondingly. Then $\mathbb{1}_L = (-1)^k t_{k+1}\mathbb{1}_R = (-1)^k t_{k+1}^{-k}\mathbb{1}_R t_k^k$ on $V^{\otimes k}$; moreover, the operators d_a, d on $V^{\otimes k}$ may be written as $\sum_{l=0}^{k-\varepsilon} t_{k+1}^{-l}\mathbb{1}_R t_k^l$ on $V^{\otimes k}$ for $\varepsilon = 1, 0$ correspondingly. This immediately implies that $(1-t)\, d = d_a (1-t)$, and $Nd_a = dN$. Note that the first of these identities shows that $\operatorname{Ker}(1-t)$ is preserved by d, hence $C^\bullet_{\mathrm{cAS}}$ is indeed a subcomplex of $C^\bullet_{\mathrm{HAS}}$.

Note also that the subsheaf $C^\bullet_{\mathrm{AS}}$ consisting of skew-symmetric sections of $C^\bullet_{\mathrm{HAS}}$ (and of $C^\bullet_{\mathrm{cAS}}$) is preserved by d. Hence one gets inclusion of complexes $C^\bullet_{\mathrm{AS}} \subset C^\bullet_{\mathrm{cAS}} \subset C^\bullet_{\mathrm{HAS}}$. Similarly, consider subsheaf $C^\bullet_{\mathrm{AS}'}$ of $C^\bullet_{\mathrm{aHAS}}$ consisting of sections skew-symmetric in all indices but the first one; it is preserved by d_a. Hence one gets inclusion of complexes $C^\bullet_{\mathrm{AS}'} \subset C^\bullet_{\mathrm{aHAS}}$. (Later we construct duality between $C^\bullet_{\mathrm{AS}}$ and the Lie-algebraic complex $C_\bullet(\mathfrak{g}, k)$, $\mathfrak{g} = \Gamma(M, \mathcal{O})$; likewise $C^\bullet_{\mathrm{AS}'}$ will turn out to be dual to $C_\bullet(\mathfrak{g}, \mathfrak{g})$.)

Remark 1.15. Until this moment we considered (say) the exterior power of a vector space as a subspace in the tensor power. However, the usual definition presents this space as a quotient of the tensor power, and the difference becomes apparent if we consider not vector spaces in char = 0, but finite characteristic, or modules over a ring—to take an antisymmetrization, we should be able to divide by $n!$. The same is applicable to the cyclic case.

Investigate shortly how things change in this "politically correct" case. Denote "quotient" spaces of tensors with certain symmetry by prepending q to the lower index, as in C_{qcAS}, C_{qAS}, $C_{\mathrm{qAS}'}$. Given spaces V_G, V_H of coinvariants of action of groups $H \subset G$ in V, there are natural mappings of projection $\Pi = \Pi_{HG}\colon V_H \to V_G$, and of symmetrization $\sum g\colon V_G \to V_H$; the latter map exists if $[G : H] < \infty$, and summation is over a set of representatives of $H\backslash G$. For example, the operator N_{n+1} works as a symmetrization operator $C^n_{\mathrm{qAS}} \to C^n_{\mathrm{qAS}'}$. For what follows, denote by $N_{\mathfrak{S}}$ the operator $\sum g$ with sum going over all permutations; likewise, use $N'_{\mathfrak{S}}$ for sum over permutations stabilizing the first element; obviously, operators $N_{\mathfrak{S}}$ send vector spaces $C^\bullet_{\mathrm{qAS}}$ to $C^\bullet_{\mathrm{HAS}}$; likewise, $N'_{\mathfrak{S}}$ send $C^\bullet_{\mathrm{qAS}'}$ to $C^\bullet_{\mathrm{HAS}}$.

In our case, we consider complexes instead of vector spaces, and both d, $\wedge 1 = \mathbb{1}_R$, and d_a have the form $\sum_{l=0}^{L} \boldsymbol{d}_l$ with $\boldsymbol{d}_0 = \mathbb{1}_R$, and $\boldsymbol{d}_l = t^{-l}\boldsymbol{d}_0 t^l$. Note that any

$\boldsymbol{d}_l$ has a property that $\boldsymbol{d}_l g = g_{[l]}\boldsymbol{d}_{l^g}$; here l, g are arbitrary, and $l^g, g_{[l]}$ are choosen appropriately. (Moreover, for groups of permutation of all indices, and all indices but the first one, one can take $l^g = l$, and $g_{[l]} = \iota_l(g)$ with appropriate inclusion ι_l of permutation groups. When one wants g and $g_{[l]}$ be in cyclic permutations, one may need to take l^g distinct from l.) We want to investigate when there are mappings between complexes of coinvariants which are *intrinsic* in the sense that they exist in any such situation. For the differential to send H-coinvariants into H'-coinvariants, one needs $l \mapsto l^h$ to be a permutation of indices l appearing in differential $\sum \boldsymbol{d}_l$ for any $h \in H$, and one needs $h_{[l]}$ to be in H'. The former condition is automatically satisfied when one consider all permutations, or permutations stabilizing the first element; hence all operators $\mathbb{1}_R$, d_a, d induce mappings of spaces of coinvariants $C^\bullet_{\mathrm{qAS}}$ or $C^\bullet_{\mathrm{qAS'}}$. In fact, $\mathbb{1}_R$, d_a, d (and $\boldsymbol{d}_l$) induce proportional mappings in $C^\bullet_{\mathrm{qAS}}$; likewise, $\mathbb{1}_R$, d_a (and $\boldsymbol{d}_l$ except the last one) induce proportional mappings of $C^\bullet_{\mathrm{qAS'}}$.

In the case of cyclic permutations of k elements, l^t must be $l+1$ if $0 \leq l < k-1$, $k^t = 1$, and there are two possible choices for $(k-1)^t$: either 0, or k. One concludes that in the cyclic case, only $d_a = \sum_0^{k-1} \boldsymbol{d}_l$ sends gives a mapping of spaces of coinvariants $C^{k-1}_{\mathrm{qcAS}} \to C^k_{\mathrm{qcAS}}$.

The next question is: given two complexes $C^\bullet_\alpha$, $C^\bullet_\beta$ of coinvariants, when a sequence of symmetrization mapping gives an intrinsic mapping of complexes? Given $H \subset H'$ and $G \subset G'$, a mappings $\sum_{l\in L} \boldsymbol{d}_l$ from H-coinvariants to G-coinvariants, likewise for $\sum_{l\in L'} \boldsymbol{d}_l$, and mappings of symmetrization $\sum_{g\in R_H} g$ and $\sum_{g\in R_G} g$ (with R_H being representatives for $H\backslash H'$, likewise for R_G), one wants to check commutative square conditions. One must compare $\sum_{l\in L'} \boldsymbol{d}_l \sum_{g\in R_H} g = \sum_{l\in L'} \sum_{g\in R} g_{[l]}\boldsymbol{d}_{l^g}$ with $\sum_{g\in R_G} g \sum_{l\in L} \boldsymbol{d}_l$; for an "intrinsic" equality, one wants the summations to be identical. This leads to a necessary condition $[H' : H]\,|L'| = [G' : G]\,|L|$.

Joining all together, one gets a diagram

$$\begin{array}{ccccc}
 & & \left(C^\bullet_{\mathrm{qAS}}, d_a(d,\mathbb{1}_R)\right) & \xrightarrow{N} & (C^\bullet_{\mathrm{AS}}, d) \\
 & & \Big\| & & \Big\uparrow \Pi \\
\left(C^\bullet_{\mathrm{qAS'}}, d_a(\mathbb{1}_R)\right) & \xrightarrow{\Pi} & \left(C^\bullet_{\mathrm{qAS}}, d_a(d,\mathbb{1}_R)\right) & & (C^\bullet_{\mathrm{HAS}}, d) \\
{[\mathbb{1}_R]}\Big\uparrow\Big\downarrow N'_{\mathfrak{S}} & & {[\mathbb{1}_R]}\Big\uparrow\Big\downarrow N'_{\mathfrak{S}} & & \Big\uparrow \iota \\
(C^\bullet_{\mathrm{aHAS}}, d_a) & \xrightarrow{\Pi} & \left(C^\bullet_{\mathrm{qcAS}}, d_a\right) & \xrightarrow{N} & (C^\bullet_{\mathrm{cAS}}, d) \\
\iota\Big\uparrow & & & & \Big\uparrow \iota \\
(C^\bullet_{\mathrm{AS'}}, d_a) & = & (C^\bullet_{\mathrm{AS'}}, d_a) & \xrightarrow{N} & (C^\bullet_{\mathrm{AS}}, d)\,.
\end{array}$$

Additionally, there is a diagonal arrow $N_{\mathfrak{S}}$ from $\left(C^\bullet_{\mathrm{qAS}}, \mathbb{1}_R\right)$ to $(C^\bullet_{\mathrm{AS}}, d)$. (One can also add the map of complexes $1-t$: $(C^\bullet_{\mathrm{HAS}}, d) \to (C^\bullet_{\mathrm{aHAS}}, d_a)$ we know already; however, it is of different nature than other maps in this diagram.)

In this diagram, Π denotes a natural projection of coinvariants, ι a natural inclusion of invariants. The notation $(C^\bullet, d_1\,(d_2, \dots d_n))$ means that $(C^\bullet, d_n)$ is a complex, and any d_k is proportional to d_n; unless indicated otherwise, we consider d_1 as the differential in $C^\bullet$. Finally, $(C^\bullet, d_1\,(d_2, \dots)) \xrightarrow{[d_k]\,f} (C'^\bullet, d)$ means that f is a mapping of complexes $(C^\bullet, d_k) \to (C'^\bullet, d)$. Note that absense of intrinsic maps between complexes in this diagram which are not compositions of arrows in the diagram may be shown using the necessary condition above; the fact that the mappings N, $N_{\mathfrak{S}}$, $N'_{\mathfrak{S}}$ in the diagram are mappings of complexes are easy to deduce.

In the case of characteristic 0 the natural projection from a subcomplex to the corresponding quotient complex is an isomorphism. Therefore the diagram above can be contracted to a smaller one.

On the other hand, note that there is a natural pairing between invariants of $\mathbb{Z}_k$ in $V^{\otimes k}$ and coinvariants of $\mathbb{Z}_k$ in $(V^*)^{\otimes k}$; likewise for any subgroup of permutations. Since our interest in Alexander–Spanier complexes is due to their pairing with corresponding (Hochschild, cyclic or Lie-algebraic) complexes in algebraic situation, one should work with invariants in geometric (Alexander–Spanier) situation if one wants to work with coinvariants in the algebraic situation.

Remark 1.16. To clarify the notion of "intrinsic" maps and "intrinsic" identities, consider *permutation objects*: assume given an additive category $\mathcal{C}$ with finite limits and colimits; then for action of a group, one can consider invariants and coinvariants. Consider a (semi)simplicial object $(C_k, \boldsymbol{d}_{kl})$, $0 \leq l \leq k$, with $C_\bullet$, $\boldsymbol{d}_\bullet$ in $\mathcal{C}$. Assume that the permutation groups $\mathfrak{S}_k$ act by automorphisms $p_k\,(g)$ of C_k, $g \in \mathfrak{S}_k$, so that $p_{k+1}\,(g)\,\boldsymbol{d}_{kl} = \boldsymbol{d}_{kl'} p_k\,(g')$; here $l' = g\,(l+1)$, and g' is $(l'\,k+1) \circ g \circ (l\,k+1)$ restricted to become a permutation of $\{1, \dots, k\}$.

Composing $\boldsymbol{d}_\bullet$ and $p_\bullet\,(g)$, one gets a functor from category with objects being sets $\{1, \dots, n\}$ and morphisms being injective mappings. (In the same way, *cyclic objects* correspond to restricting attention to injective mappings which preserve the cyclic order.) The "intrinsic" maps considered above are maps which are associated to spaces of (co)invariants of *any* permutation object.

Consider an example of how the constructions above are applicable to permutation objects. Given a morphism $\psi\colon G \to \operatorname{Aut} C$, $C \in \mathcal{C}$, consider $H \subset G$ with finite $[G : H]$, put $\Psi := \sum_{g \in R} \psi_g \in \operatorname{End} C$ with R being a particular set representatives of $H \backslash G$. Consider $H-$ and G-coinvariants of C, given by $\Pi_H : C \gg C_H$, $\Pi_G\colon C \gg C_G$ (which are coequalizers of ψ_g, $g \in H$ and $g \in G$ correspondingly). Then $\Pi_H \circ \Psi \circ g = \Pi_H \circ \Psi$ for any $g \in G$, which induces a mapping $\Phi\colon C_G \to C_H$ such that $\Pi_H \circ \Psi = \Phi \circ \Pi_G$. Moreover, $\Pi_H \circ \Psi$ does not depend on the choice of the set R. If $H = \{e\}$, so $R = G$, then Ψ can be passed through the inclusion $C^G \hookrightarrow C$ of invariants of G. Above, in the case of $N\colon C^\bullet_{\text{qAS}} \to C^\bullet_{\text{qAS}'}$, R was a subgroup of cyclic permutations, $\Psi = N$.

Denote by $\mathcal{O}_1$ the (constant) subsheaf of $\mathcal{O}$ consisting of sections locally proportional to the section 1 of $\mathcal{O}$. Consider the corresponding to $\mathcal{O}_1$ complexes (HAS, qAS etc); they are naturally subcomplexes of the corresponding complexes for $\mathcal{O}$; moreover, the differential on aHAS is acyclic, and on HAS or AS it is acyclic in degree > 0, and has cohomology K in degree 0. Note that in the case of HAS/AS this implies that the hypercohomology of the complex of sheaves corresponding to $\mathcal{O}_1$ is quasi-isomorphic to cohomology of M with coefficients in K (the quasi-isomorphism is given by inclusion of $K = \mathcal{O}_1$ into the complex). In the case of cAS the complex has cohomology $2K = \{2x \mid x \in K\}$ in positive even degrees and is quasi-isomorphic to its cohomology. (These statements hold even if $\mathcal{O}$ is not a complex of K-algebras, but a complex of rings if one replaces K by $K/\operatorname{Ann} 1$.)

Call a sheaf $\mathcal{S}$ of groups on M *space-acyclic* if $H^k(M, \mathcal{S}) = 0$ for $k > 0$.

Theorem 1.17. (1) *Assume that the sheaves $\Delta^* \mathcal{O}^{\boxtimes n}$ are space-acyclic. Then the "acyclic" Hochschild–Alexander–Spanier complex is acyclic indeed, the Hochschild–Alexander–Spanier complex is quasi-isomorphic to the complex of cohomology of M with coefficients in K. If $K \supset \mathbb{Q}$ the cyclic Alexander–Spanier complex is quasi-isomorphic to a direct sum of an infinite number of such complexes with non-negative even shifts.*

(2) *If the sheaves $\Lambda^n \mathcal{O}$ are space-acyclic, the Alexander–Spanier complex is quasi-isomorphic to the complex of cohomology of M with coefficients in K.*

(3) *If the sheaves $\mathcal{C}^\bullet_{qAS'}$ are space-acyclic, the complex $\left(C^\bullet_{qAS'}, \mathbb{1}_R\right)$ is exact.*

(4) *If the sheaves $\mathcal{C}^\bullet_{qAS}$ are space-acyclic, the the complex $\left(C^\bullet_{qAS}, \mathbb{1}_R\right)$ is quasi-isomorphic to the complex of cohomology of M.*

Proof. Fix a mapping φ from $\mathcal{A} = \Gamma(M, \mathcal{O})$ to K that sends $1 \in \mathcal{A}$ to $1 \in K$. Let us construct a *homotopy* for the complex $\left(\mathcal{A}^{\boxtimes n+1}, d_a\right)$:

$$s \cdot f_0 \otimes \cdots \otimes f_n = \varphi(f_n) f_0 \otimes \cdots \otimes f_{n-1}, \qquad s \cdot f_0 = 0.$$

It is easy to check that $sd_a + d_a s = \mathrm{id}$ indeed, therefore the complex is acyclic. Fix a point $m \in M$ and consider a local section Ψ of $\Delta^*\left(\mathcal{O}^{\boxtimes n+1}\right)$ over $U \subset M$. Lessening U we can suppose that φ corresponds to a section of $\mathcal{O}^{\boxtimes n+1}$ over U^{n+1}. Changing M to U in the discussion above we get a local homotopy. This means that for any closed local section we can find a section on a smaller subset such that the boundary of the latter section is the former. Therefore the differential d_a on the *complex of sheaves* $\mathcal{C}^\bullet_{\mathrm{aHAS}}$ is acyclic.

Now the complex of vector spaces $C^\bullet_{\mathrm{aHAS}}$ is the complex of global sections of this complex of sheaves $\mathcal{C}^\bullet_{\mathrm{aHAS}}$. Recall that the cohomology of a sheaf $\mathcal{S}$ may be calculated by taking cohomology of a complex $\boldsymbol{C}^*(M, \mathcal{S})$; here $\boldsymbol{C}^*(M, \bullet)$ is an appropriate functor from sheaves of abelian groups on M to complexes of abelian groups; moreover, one may assume that the functor $\boldsymbol{C}^*(M, \bullet)$ is exact. Applying this functor to elements and arrows in $\mathcal{C}^\bullet_{\mathrm{aHAS}}$, one gets a bicomplex

$$\boldsymbol{C}^*\left(M, \mathcal{C}^\bullet_{\mathrm{aHAS}}\right);$$

its columns (i.e., $\bullet = \text{const}$) compute the cohomology of the sheaves $\mathcal{C}^{\bullet}_{\text{aHAS}}$. By exactness, the rows of this complex are exact, therefore the row spectral sequence implies that the total complex associated with this bicomplex is also exact.

By assumption of space-acyclicity, the columns of the bicomplex are acyclic in degree ≥ 1; hence, by the column spectral sequence, the total complex of the bicomplex is quasi-isomorphic to the subcomplex of vertically-closed elements of the first row, which are

$$H^0\left(M, \mathcal{C}^{\bullet}_{\text{aHAS}}\right) = C^{\bullet}_{\text{aHAS}}.$$

Consider now the complex $C^{\bullet}_{\text{HAS}}$. The same homotopy as above satisfies

$$sd + ds = \text{id}$$

in degree ≥ 1, and if $f \in \mathcal{A}$

$$(sd + ds) f = f - \varphi(f) \cdot 1.$$

Therefore the mapping $(\mathcal{A}^{\otimes n+1}, d) \to K$ given by φ if $n = 0$ and 0 otherwise is a quasi-isomorphism. Hence the analogues inclusion $K \to (\mathcal{A}^{\otimes n+1}, d)$ is also a quasi-isomorphism; one concludes that the corresponding to $K \to \mathcal{A}\colon \alpha \mapsto \alpha \cdot 1$ inclusion of the complex $(K^{\otimes n+1}, d)$ into $(\mathcal{A}^{\otimes n+1}, d)$ is a quasi-isomorphism. Repeating this argument on the level of sheaves, one can see that the complex of sheaves $\mathcal{C}^{\bullet}_{\text{HAS}}$ is quasi-isomorphic to its constant subsheaf K, and to its subcomplex corresponding to replacing $\mathcal{O}$ by $\mathcal{O}_1$.

To get information about the complex of global sections of this complex of sheaves consider the corresponding bicomplex $\boldsymbol{C}^*\left(M, \mathcal{C}^{\bullet}_{\text{HAS}}\right)$ and its row spectral sequence. Since an exact functor applied to complexes sends quasi-isomorphisms of complexes to quasi-isomorphisms, inclusion of $\mathcal{O}_1$ into $\mathcal{O}$ gives a quasi-isomorphism of the terms E_1 of the spectral sequence; moreover, since E_1 is concentrated in the first row, the spectral convergence converges at this term. Therefore one gets a quasi-isomorphisms of total complexes of the bicomplexes between themselves, and with the cohomology of the rows, i.e., the complex $C^*(M, K)$. Again, by space-acyclicity, the total complex is quasi-isomorphic to its first row, i.e., $C^{\bullet}_{\text{HAS}}$.

Obviously, s preserves the graded subspaces $C^{\bullet}_{\text{AS}}$ and $C^{\bullet}_{\text{AS'}}$. The same way as above, one concludes that when space-acyclicity is applicable, the complex of sheaves $(C^{\bullet}_{\text{AS'}}, d_a)$ is acyclic, and the complex of sheaves $(C^{\bullet}_{\text{AS}}, d)$ is quasi-isomorphic to its constant subsheaf K in grading 0. Of course, the similar statements on relation of $\mathcal{O}$ and $\mathcal{O}_1$ hold.

Consider now the quotient complexes $\left(C^{\bullet}_{\text{qAS}}, \mathbb{1}_R\right)$ etc. Consider a homotopy

$$s_{\varepsilon} \cdot f_0 \otimes \cdots \otimes f_n := \sum_{k=\varepsilon}^{n} (-1)^{n-k} \varphi(f_k) f_0 \otimes \cdots \otimes \widehat{f_k} \otimes \cdots \otimes f_n, \quad s \cdot f_0 = 0.$$

Obviously, s_0 descends to the quotient $C^{\bullet}_{\text{qAS}}$, and s_1 descends to $C^{\bullet}_{\text{qAS'}}$. It is easy to see that $\mathbb{1}_R s_0 + s_0 \mathbb{1}_R = \text{id}$ if $n > 0$ and $(\mathbb{1}_R s_0 + s_0 \mathbb{1}_R) f_0 = f_0 - \varphi(f_0) 1$. Likewise,

$\mathbb{1}_R s_1 + s_1 \mathbb{1}_R = \mathrm{id}$. Therefore the same argument as above shows that $\left(C^{\bullet}_{\mathrm{qAS}'}, \mathbb{1}_R\right)$ is an acyclic complex of sheaves, and $\left(C^{\bullet}_{\mathrm{qAS}}, \mathbb{1}_R\right)$ is quasi-isomorphic to $C^{*}(M, K)$.

Consideration of $C^{\bullet}_{\mathrm{cAS}}$ in the case $K \supset \mathbb{Q}$ is a little bit more tricky. We use an analogue of the construction from [?LodQuill84Cyc]. Consider a bicomplex

$$\mathcal{C}^{\bullet}_{\mathrm{HAS}} \xrightarrow{1-t} \mathcal{C}^{\bullet}_{\mathrm{aHAS}} \xrightarrow{N} \mathcal{C}^{\bullet}_{\mathrm{HAS}} \xrightarrow{1-t} \mathcal{C}^{\bullet}_{\mathrm{aHAS}} \xrightarrow{N} \dots . \tag{1.2}$$

Here t is the action of $\mathbb{Z}_{n+1}$ on $\mathcal{C}^{n}_{\mathrm{HAS}} = \mathcal{C}^{n}_{\mathrm{aHAS}}$, N is equal to $1 + t + t^2 + \dots + t^n$ on $\mathcal{C}^{n}_{\mathrm{HAS}}$. It is easy to check the conditions of bicomplex for this system of mappings. Now the rows are acyclic in all the terms but the first, the homology in the first term are exactly $\mathcal{C}^{\bullet}_{\mathrm{cAS}}$. Now the row spectral sequence shows that the complex $\mathcal{C}^{\bullet}_{\mathrm{cAS}}$ is quasi-isomorphic to the total complex of this bicomplex. Label equ1.20,

On the other side, the column spectral sequence shows that the total complex is quasi-isomorphic to the complex

$$K \to 0 \to K \to 0 \to K \to \dots$$

of constant sheaves, or a direct sum of constant sheaves K in even degrees. □

Remark 1.18. If $K \supset \mathbb{Q}$, then space-acyclicity of $\mathcal{C}^{\bullet}_{\mathrm{HAS}}$ implies space-acyclicity of other sheaves mentioned in the theorem. Indeed, these sheaves are isomorphic to direct summands of $\mathcal{C}^{\bullet}_{\mathrm{HAS}}$.

Note that the sequence (1.2) is not exact if $\operatorname{char} K > 0$ (apply it to constant sheaf). Hence these arguments do not work unless $K \supset \mathbb{Q}$.

One can consider an analogue of (1.2)

$$\dots \xrightarrow{N} \mathcal{C}^{\bullet}_{\mathrm{HAS}} \xrightarrow{1-t} \mathcal{C}^{\bullet}_{\mathrm{aHAS}} \xrightarrow{N} \mathcal{C}^{\bullet}_{\mathrm{HAS}} \xrightarrow{1-t} \mathcal{C}^{\bullet}_{\mathrm{aHAS}}.$$

The rows are quasi-isomorphic to $\mathcal{C}^{\bullet}_{\mathrm{qcAS}}$, the columns to

$$\dots \to K \to 0 \to K \to 0.$$

However, this bicomplex is in a "wrong" quadrant, therefore we should not (and do not) have the isomorphisms of cohomology (as one can see comparing to cohomology of $C^{\bullet}_{\mathrm{cAS}}$—which is isomorphic provided $K \supset \mathbb{Q}$).

As duality with the theory of cyclic cohomology will show, it is "more correct" to consider a different definition of the cyclic complex: as associated complex $C^{\bullet}_{\mathrm{dcAS}}$ of the bicomplex of global sections of (1.2).

Amplification 1.19. *With $C^{\bullet}_{dcAS}$ replacing $C^{\bullet}_{cAS}$, in the preceding theorem, the restriction $K \supset \mathbb{Q}$ may be dropped.*

Proof. As above, consideration of the total complex of the bicomplex (1.2) is reduced to consideration of the corresponding complex of sheaves. Therefore, one must consider the total complex of bicomplex of stalks corresponding to (1.2).

Considering the column spectral sequence, it is enough to shwo that the inclusion of direct sum of K in the direct sum of columns $\mathcal{C}^{\bullet}_{\mathrm{HAS}}$ gives a is a quasiisomorphism.

And it since it is enough to do separately for each column, the considerations above are sufficient. ☐

Another generalization of the theorem comes from replacing the definition of $C^{\bullet}_{\mathrm{HAS}}$ from global section of $\mathcal{C}^{\bullet}_{\mathrm{HAS}}$ to global sections with compact support (or with support in arbitrary "system of supports" Φ).

Amplification 1.20. *Homology of the resulting complexes $C^{\bullet}_{HAS,\Phi}$ etc. may be computed the same way as in the preceeding theorem provided the corresponding sheaves $\mathcal{C}^{\bullet}_{HAS}$ etc are Φ-space acyclic, and one replaces cohomology of M by cohomology of M with supports in Φ.*

In fact, with Φ being consisting of compact sets, this leads to an important corollary:

Corollary 1.21. *If $\mathcal{O}$ is soft and M is locally compact, then the conclusions of the theorem hold.*

Proof. In fact, it is enough to assume that $\mathcal{O}$ is compact-soft.[6] Recall that external tensor power preserves compact-softness on locally compact spaces. (See [**?**Bre] Exer. ???.) And pullback to a closed subset preserves compact-softness on ??? Hausdorff spaces. ☐

Remark 1.22. Another case when acyclicity of $C^{\bullet}_{\mathrm{HAS}}$ follows from acyclicity of $\mathcal{O}$ is the case when $\mathcal{O}$ is a quasicoherent sheaf on an affine scheme. One can immediately see that $\Delta_*\Delta^*\mathcal{O}^{\boxtimes n}$ is quasicoherent as well (however, recall that Δ_* and Δ^* are taken in the category of sheaves of abelian groups, not sheaves of $\mathcal{O}_M$-modules), hence $\Delta_*\Delta^*\mathcal{O}^{\boxtimes n}$ is acyclic. On the other hand, cohomologies of M^n with coefficients in $\Delta_*\Delta^*\mathcal{O}^{\boxtimes n}$ coincide with cohomology of M with coefficients in $\Delta^*\mathcal{O}^{\boxtimes n}$.

Would the same hold over a Stein variety??? Yes if one replaces the tensor product by the completion. . . Note that the duality extends to completion for algebras of locally finite order (filtered so that operators of multiplication on subquotients are given by differential operators of finite order).

In the bicomplex (1.2) there is a remarkable periodicity operation S_{bi}: the translation on two columns to the right. It commutes with the differentials, therefore it results in an operation in cohomology. The remarkable fact is that for $K \supset \mathbb{Q}$ one can express this operation on the complex C_{cAS} (which is quasi-isomorphic to the total complex of the bicomplex provided $K \supset \mathbb{Q}$).

In general, assume that horizontal arrows δ in a bicomplex $C^{\bullet\bullet}$ with differentials δ, d allow a homotopy s_δ. Then in the total complex, id is homotopic to $-\sigma$, $\sigma = ds_\delta + s_\delta d$ (which has degree $(-1,1)$), hence to any power of $-\sigma$; note that σ is a morphism of bicomplexes. Hence σ sends $\operatorname{Ker}\delta$ to $\operatorname{Ker}\delta$; and, when restricted to $\operatorname{Ker}\delta$, σ coincides

[6] I.e., any section of Φ over a compact subset may be extended to a global section.

with $\sigma_0 = \delta s_\delta d s_\delta$; likewise, σ^N coincides with σ_0^N on $\operatorname{Ker}\delta$. Note that if $s_\delta^2 = 0$, then $s_\delta \delta s_\delta = s_\delta$, hence σ_0^N may be simplified to $\delta s_\delta (ds_\delta)^N$. Therefore there is a mapping homotopic to id which sends $c \in \operatorname{Ker}\delta$ to $\delta s_\delta (-ds_\delta)^N c$.

We would apply this with $N = 2$ to an element of the form $c = S_{\mathrm{bi}} c_0$ with $c_0 \in \operatorname{Ker} d$. Note that $S = \delta s_\delta (ds_\delta)^2 S_{\mathrm{bi}}$ has grading (0,2), so acts inside one column of the bicomplex (1.2). Taking an even column, $\operatorname{Ker}\delta$ consists of cyclically symmetric chains, and d coincides with differential in $C^\bullet_{\mathrm{HAS}}$. Since S would preserve $\operatorname{Ker}\delta$, it would give a mapping in the cyclic complex commuting with differential.

To construct homotopy for horizontal arrows in (1.2) it is enough to find operators τ and ν commuting with t such that $\tau(1-t)+\nu N = \mathrm{id}$; to ensure $s_\delta^2 = 0$, it is enough to take ν proportional to N, and $\tau N = 0$. Since N is proportional to a projection, one can take $\nu = \nu_{n+1} = N_{n+1}/(n+1)^2$; here n is is the degree in $C^\bullet_{\mathrm{HAS}}$, so $n+1$ is the valence of the tensors. Now one can solve for $(1-t)\tau + \nu N = \mathrm{id}$; requiring that t vanishes on the $\operatorname{Im} N$, one concludes that $\tau = \tau_{n+1} = \widehat{\tau}_{n+1}/(2(n+1))$ with $\widehat{\tau}_{n+1} = \sum_{l=0}^{n} (n-2l)\, t_{n+1}^l$. Hence on must assume $K \supset \mathbb{Q}$.

One concludes that the operator $S = N\nu d\tau d_a \nu$ of degree 2 in $C^\bullet_{\mathrm{HAS}}$ (and $\mathcal{C}^\bullet_{\mathrm{HAS}}$) preserves the subcomplex $C^\bullet_{\mathrm{cAS}}$ and commutes with differential on $C^\bullet_{\mathrm{cAS}}$. Note that $N\nu d\tau d_a\nu \colon C^n_{\mathrm{HAS}} \to C^{n+2}_{\mathrm{HAS}}$ is a shortcut for $N_{n+3}\nu_{n+3} d\tau_{n+2} d_a \nu_{n+1} = \frac{1}{2(n+3)(n+2)(n+1)^2} N_{n+3} d\widehat{\tau}_{n+2} d_a N_{n+1}$ (since $N_k \nu_k = N_k/k$).

One can make further simplifications: if one replaces d_a by d in this formula, and notes that τ is divisible by $1-t$ (say, $\tau = (1-t)\upsilon$), one can note that $\tau d\nu = \upsilon d_a (1-t)\nu = 0$; hence the result is 0. Therefore, one can replace d_a by $d_a - d = \pm \mathbb{1}_L$ without changing the result. Moreover, replacing d in the formula by d_a gives 0, since $N d_a \tau = dN\tau = 0$; thus one can replace d by $d - d_a = \pm\mathbb{1}_L$; the signs match, hence one can replace $Nd\tau d_a\nu$ by $N\mathbb{1}_L \tau \mathbb{1}_L \nu$.

When $t_{n+3}^\alpha \mathbb{1}_L t_{n+2}^\beta \mathbb{1}_L t_{n+1}^\gamma$ is applied to $f_0 \otimes \cdots \otimes f_n$, $1s$ appear at positions $(\alpha)_{n+3} + 1$ (with the leftmost position indexed as 1) and $\left(\alpha + (\beta)_{n+2} + 1\right)_{n+3} + 1$ in the tensor product; here $(k)_l$ is the minimal nonnegative representative of $k \mod l$. The index of the first $f_\bullet$ "after" (in the cyclic sense) these $1s$ are $f_{\left(-(\beta)_{n+2}-\gamma\right)_{n+1}}$ and $f_{(-\gamma)_{n+1}}$. Hence up to permutation of these $1s$, t_{n+3}^α, t_{n+2}^β, t_{n+1}^γ are uniquely determined by the result (provided 1 and f_k are independent). One gets a dependency between $t_{n+3}^\alpha \mathbb{1}_L t_{n+2}^\beta \mathbb{1}_L t_{n+1}^\gamma$ and $t_{n+3}^{\alpha+(\beta)_{n+2}+1} \mathbb{1}_L t_{n+2}^{-1-\beta} \mathbb{1}_L t_{n+1}^{(\beta)_{n+2}+\gamma}$; indeed, $\mathbb{1}_L t_{n+2}^\beta \mathbb{1}_L$ and $t_{n+3}^{\beta+1} \mathbb{1}_L t_{n+2}^{n+1-\beta} \mathbb{1}_L t_{n+1}^\beta$ result in proportional monomials for $0 \leq \beta \leq n+1$. Compare the sign of resulting monomials: $\mathbb{1}_L t_{n+2}^\beta \mathbb{1}_L$ involves $(n+3)\beta$, and $t_{n+3}^{\beta+1} \mathbb{1}_L t_{n+2}^{n+1-\beta} \mathbb{1}_L t_{n+1}^\beta$ involves $(n+4)(\beta+1) + (n+3)(n+1-\beta) + (n+2)\beta \equiv_2 1 + (n+1)\beta$. Hence the signs are opposite.

On the other hand, the terms with $\beta = \beta_0$ and $\beta = n+1-\beta_0$ already have opposite coefficients in the formula for $\widehat{\tau}_{n+2}$. Therefore, in the formula $\sum_{\alpha\beta\gamma} \left(n+1-2(\beta)_{n+2}\right) t^\alpha \mathbb{1}_L t^\beta \mathbb{1}_L t^\gamma$ for $N_{n+3} \mathbb{1}_L \widehat{\tau}_{n+2} \mathbb{1}_L N_{n+1}$ one can restrict summation so that only one of two terms above is included if one multiplies the result by 2. Hence

$$S_n = \frac{1}{(n+3)(n+2)(n+1)^2} \sum_{\substack{0\leq\alpha\leq\beta\leq n+1\\ 0\leq\gamma\leq n}} (n+1+2\alpha-2\beta)\, t_{n+3}^{\alpha} \mathbb{1}_L t_{n+2}^{\beta-\alpha} \mathbb{1}_L t_{n+1}^{\gamma} \colon C^n_{\mathrm{HAS}} \to C^n_{\mathrm{HAS}},$$

and between these terms, there is no cancellation. Note that we are going to restrict this operator to subspace C^n_{cAS} of cyclically symmetric tensors, hence the last factor t^{γ}_{n+1} acts as 1. Choose one particular value of γ; it is convenient to take $\gamma = -\beta$, so that the order of initial tensor factors is preserved. One arrives at

$$\frac{1}{(n+3)(n+2)(n+1)} \sum_{1\leq\alpha<\beta\leq n+3} (n+3+2\alpha-2\beta)\, t_{n+3}^{\alpha-1} \mathbb{1}_L t_{n+2}^{\beta-\alpha-1} \mathbb{1}_L t_{n+1}^{2-\beta}$$

(here we shifted α by 1, β by 2, so they denote positions of 1s) which coincides with S_n on C^n_{cAS}. The sign of a term is associated with $(n+4)(\alpha-1)+(n+3)(\beta-\alpha-1)+(n+2)(2-\beta) \equiv_2 n\alpha+(n+1)(\alpha+\beta)+n\beta+1 = \alpha+\beta+1$. Therefore, one is lead to consider

$$U = U^{[n+1]} = \sum_{1\leq\alpha<\beta\leq n+3} (-1)^{\beta-\alpha-1} (n+3+2\alpha-2\beta)\, \mathbb{1}^{[n+1]}_{\alpha\beta} \colon V^{\otimes n+1} \to V^{\otimes n+3};$$

here $\mathbb{1}^{[k]}_{\alpha\beta}$ sends $v_1\otimes\cdots\otimes v_k$ to $v_1\otimes\cdots\otimes f_{\alpha-1}\otimes 1\otimes f_\alpha\otimes\cdots\otimes f_{\beta-2}\otimes 1\otimes f_{\beta-1}\otimes\cdots\otimes f_k$ (so $1s$ are at positions α,β). (Define $\mathbb{1}^{[k]}_{\alpha_1\dots\alpha_l} \colon V^{\otimes k} \to V^{\otimes k+l}$ likewise.)

One can immediately see that $d^{[n+3]}U^{[n+1]} = (n+1)\, T^{[n+1]}$ with $T^{[n+1]} = \sum_{\alpha<\beta<\gamma} (-1)^{\alpha+\beta+\gamma} \mathbb{1}^{[n+1]}_{\alpha\beta\gamma}$, and $U^{[n+2]}d^{[n+1]} = (n+4)\, T^{[n+1]}$. (This is why one *needs* denominator $(n+3)(n+2)(n+1)$ in the definition of operator S_n which commutes with d.) Define $U^{[n+1]}_a$ by the same formula as $U^{[n+1]}$, but with summation over $1<\alpha<\beta\leq n+3$. Then a calculation immediately shows that $U_a(1-t) = (1-t)\, U$ and $UN = NU_a$.

Define $\widehat{d} = \widehat{d}^{[k]}$, $\widehat{d}'_a = \widehat{d}'_a{}^{[k]}$ as $\sum_{\alpha=\varepsilon}^{k+1} (-1)^{k+1-\alpha} (k+1-\alpha)\, \mathbb{1}^{[k]}_{\alpha}$ with $\varepsilon = 0,1$ correspondingly. One can immediately see that $t\widehat{d} = \left(\widehat{d}'_a + d_a\right) t$, and that

$$\widehat{d}d = \sum_{\alpha<\beta} (-1)^{\beta-\alpha} (\beta-\alpha)\, \mathbb{1}_{\alpha\beta}, \qquad d\widehat{d} = -\sum_{\alpha<\beta} (-1)^{\beta-\alpha} (\beta-\alpha-1)\, \mathbb{1}_{\alpha\beta}$$

(similar formulae hold for $\widehat{d}'_a d_a$ and $d'_a\widehat{d}_a$ with $\alpha=1$ removed from the summation). Hence $k\widehat{d}^{[k+1]}d^{[k]} + d^{[k+1]}(k+2)\,\widehat{d}^{[k]} = U^{[k]}$, likewise $k\widehat{d}'_a{}^{[k+1]}d^{[k]}_a + d^{[k+1]}_a (k+2)\,\widehat{d}'_a{}^{[k]} = U^{[k]}_a$; this implies that the anticommutator $\left[d, \frac{\widehat{d}^{[k]}}{k(k+1)}\right]_+$ is $\frac{U^{[k]}}{k(k+1)(k+2)}$, likewise for a-flavors. Note that in these formulae one can replace $\widehat{d}$ by $\widehat{d}+\varepsilon d$ with arbitrary ε; likewise for $\widehat{d}'_a$; the formula for $t\widehat{d}$ shows it is more convenient to consider $\widehat{d}_a = \widehat{d}'_a + d_a$.

Basing on these formulae, define operator $\widehat{D}_{k+1} \colon H^k_{\mathrm{HAS}} \to H^{k+1}_{\mathrm{HAS}}$ as $D_{k+1} = \frac{\widehat{d}_{k+1}}{(k+1)(k+2)}$; define $\widehat{D}_{a,k+1}$ likewise.

Given a pair of operators Y, Y_a acting in $C^{\bullet}_{\mathrm{HAS}}$ and $C^{\bullet}_{\mathrm{aHAS}}$ correspondingly, denote by Y_v an operator in the cyclic bicomplex acting as Y in HAS-columns, and as $-Y_a$ in aHAS-columns. Define the operator $\boldsymbol{k}$ as acting by multiplication by k in tensors of valence k. Then $\left[\widehat{d}_v \frac{1}{\boldsymbol{k}(\boldsymbol{k}+1)}, d_v\right]_+ = S_v$; note that d_v coincides with the vertical component of the differential in bicomplex.

Moreover, $\left[\widehat{d}_v, 1-t\right]_+$ acts as $\widehat{d} - \widehat{d}'_a - d_a$ on even columns. Note that $\widehat{d}^{[k]} - \widehat{d}'_a{}^{[k]} = k\left(d^{[k]} - d_a^{[k]}\right)$; hence $\left[\widehat{d}_v, 1-t\right]_+ = nd^{[k]} - d_a^{[k]}(k+1)$. In other words, $\left[\widehat{d}_v \frac{1}{k(k+1)}, 1-t\right]_+ = \frac{1}{k+1} d^{[k]} - d_a^{[k]} \frac{1}{k}$. Note that the RHS can be written as $\left[\frac{1}{\boldsymbol{k}}, d_v\right]_+$.

The next step is to find how $\widehat{d}_v$ commutes with the operator $N \colon C^{\bullet}_{\mathrm{HAS}} \to C^{\bullet}_{\mathrm{aHAS}}$. One can write $\widehat{d}N - N\widehat{d}_a$ as

$$\sum (-1)^{k+1-\alpha} (k+1-\alpha)\, \mathbb{1}_{\alpha}^{[k]} t^{\beta} - \left(\sum_{\beta<\alpha-1} (-1)^{k+1-\alpha} (k+1-\alpha+\beta)\, \mathbb{1}_{\alpha}^{[k]} t^{\beta} + \sum_{\beta\geq\alpha-1} (-1)^{k+1-\alpha} (k+1-\alpha')\, \mathbb{1}_{\alpha}^{[k]} t^{\beta}\right);$$

here $\alpha' = k + \alpha - \beta$; in the last term we used $t^{\beta'} \mathbb{1}_{\alpha'}^{[k]} = (-1)^{\beta'-1+k}\, \mathbb{1}_{\alpha}^{[k]} t^{\beta'-1}$ for $\alpha = \alpha' + \beta' - k - 1$ if $\alpha' + \beta' > k+1$, $1 \leq \beta' \leq k$. Hence

$$\widehat{d}N - N\widehat{d}_a = -\sum (-1)^{k+1-\alpha} \beta \mathbb{1}_{\alpha}^{[k]} t^{\beta} + k \sum_{\beta\geq\alpha-1} (-1)^{k+1-\alpha}\, \mathbb{1}_{\alpha}^{[k]} t^{\beta}.$$

The first term is $-d\widetilde{\tau}$, with $\widetilde{\tau} = \sum_{\beta=1}^{k-1} \beta t_k^{\beta}$. Note that, as above, $\widetilde{\tau} d_a = \sum (-1)^{k+1-\alpha'} \beta' t^{\beta'} \mathbb{1}_{\alpha'}^{[k]} = \sum_{\beta<\alpha-1} (-1)^{k+1-\alpha} \beta \mathbb{1}_{\alpha}^{[k]} t^{\beta} + \sum_{\beta\geq\alpha-1} (-1)^{n+1-\alpha} (\beta+1)\, \mathbb{1}_{\alpha}^{[k]} t^{\beta}$. Therefore $d\widetilde{\tau} - \widetilde{\tau} d_a = -\sum_{\beta\geq\alpha-1} (-1)^{k+1-\alpha}\, \mathbb{1}_{\alpha}^{[k]} t^{\beta}$.

One concludes that $\widehat{d}N - N\widehat{d}_a = -(k+1)\, d\widetilde{\tau} + k\widetilde{\tau} d_a$; in other words, $\left[\widehat{d}_v \frac{1}{\boldsymbol{k}(\boldsymbol{k}+1)}, N\right]_+ = -\left[\frac{\widetilde{\tau}}{\boldsymbol{k}}, d_v\right]_+$. Basing on these commutators of $\widehat{d}_v$ with the components $1-t$ and N of the horizontal part d_h of differential in the cyclic bicomplex, define $\widehat{d}_h$ as an operator in bicomplex of degree (1,0) acting as $-\operatorname{id}$ from $C^{\bullet}_{\mathrm{HAS}}$ to $C^{\bullet}_{\mathrm{aHAS}}$, and as $\widetilde{\tau}$ from $C^{\bullet}_{\mathrm{aHAS}}$ to $C^{\bullet}_{\mathrm{HAS}}$. One can write the found commutators in a compact form

$$\left[\widehat{d}_v \frac{1}{\boldsymbol{k}(\boldsymbol{k}+1)}, d_h\right]_+ + \left[\frac{\widehat{d}_h}{\boldsymbol{k}}, d_v\right]_+ = 0.$$

In other words, $\left[\widehat{d}_v \frac{1}{\boldsymbol{k}(\boldsymbol{k}+1)} + \frac{\widehat{d}_h}{\boldsymbol{k}}, d_h + d_v\right]_+$ contains only terms of degree (2,0) and (0,2). An immediate calculation shows that the term of degree (2,0) is $-S_h$; i.e., up to the sign it is a shift to the right by 2 units. Therefore S_h is homotopic to the term of degree (0,2), which is $\left[\widehat{d}_v \frac{1}{\boldsymbol{k}(\boldsymbol{k}+1)}, d_v\right]_+ = U_v \frac{1}{\boldsymbol{k}(\boldsymbol{k}+1)(\boldsymbol{k}+2)}$.

On the other hand, coefficients in U_{n+1} are even if n is odd; hence it is more natural to define $S_n^{\mathbb{Z}}$ as $U_{n+1}/\gcd(n+1,2)$. Now $d_{n+2}S_n^{\mathbb{Z}} = (n+1)/\!/2\,T_n$, $S_{n+1}^{\mathbb{Z}}d_n = (n+4)/\!/2\,T_n$; here $m/\!/l := m/\gcd(m,l)$. Hence a natural operator in the complex to consider is $S_n^{\mathbb{Q}} := \frac{6}{(n+3)/\!/2\,(n+2)/\!/2\,(n+1)/\!/2}S_n^{\mathbb{Z}} = \frac{12\gcd(n+1,2)}{(n+3)(n+2)(n+1)}S_n^{\mathbb{Z}}$. Note that in characteristic p, $p \geq 5$, it has no advantages over S_n (it is defined for the same values of n as S_n); however, S_n is not defined in characteristics 2,3, but $S_n^{\mathbb{Q}}$ is defined in characteristic 3 unless $n \equiv_9 -1,-2,-3$, and in characteristic 2 unless $n \equiv_8 -3,-2,-1$; hence there are many values of n for which both $S_n^{\mathbb{Q}}$ and $S_{n+1}^{\mathbb{Q}}$ are defined, and $d_{n+2}S_n^{\mathbb{Q}} = S_{n+1}^{\mathbb{Q}}d_n$.

Two of operators $S_n^{\mathbb{Q}}$ have integer coefficients: $S_0^{\mathbb{Q}} = 2S_0^{\mathbb{Z}}$, $S_1^{\mathbb{Q}} = S_1^{\mathbb{Z}}$. Note also that $S_n^{\mathbb{Q}} = 0$ in characteristic 2 if $8|n$. In slightly different vane, one can consider $S_n^{[p]} = \frac{1}{(n+3)/\!/p^\infty\,(n+2)/\!/p^\infty\,(n+1)/\!/p^\infty}S_n^{\mathbb{Z}}$; this operator is defined for any n in characteristic p, and satisfies $d_{n+2}S_n^{[p]} = S_{n+1}^{[p]}d_n$ unless $p=2$ or $n \equiv_{p^k} -1,-4$ with $k=2$ for $p=3$, $k=1$ otherwise.

Remark 1.23. Note that we defined the operator U_{n+1} via compatibility with S_n in $\operatorname{Ker}(1-t)$. This gives no reason for $S_n^{\mathbb{Q}}$ to commute with d on the whole space C^n_{HAS}. However, this is what a calculation shows. We do not know any "deeper" explanation of this phenomenon.

Note that in characteristic 2 the operators $S_n^{\mathbb{Q}}$ are defined for $n \not\equiv_8 -3,-2,-1$ and vanish for $n \equiv_8 0,4$; extend them to $n \equiv_8 -3,-2,-1$ as 0. One can immediately see that these 0 values "guard" the undefined values, so the resulting sequence $S_n^{\{8\}}$ still commutes with d. In fact, take any sequence a_n of integers; then $a_{\lfloor \frac{n}{8} \rfloor}S_n^{\{8\}}$ commutes with d.

In fact, one can do the same modulo 4, or even modulo p^e with $e=2$ for $p \leq 3$, and $e=1$ otherwise. Indeed, take $n_0 = mp^{e'}$, $e' \geq e$, $p \nmid m$; then $n_0/12\,S_n^{\mathbb{Q}}$ is has p-integral coefficients for $n_0 - 4 \leq n \leq n_0$ and vanishes for $n = n_0-4, n_0$. So define $S_n^{\{p\}}$ as $n_0/12\,S_n^{\mathbb{Q}}$ if n may be written as $n_0 - n'$, $p^e|n_0$, $1 \leq n' \leq 3$, and as 0 otherwise. Then the operators with graded components $a_{\lfloor \frac{n}{p^e} \rfloor}S_n^{\{p\}}$ are well-defined in characteristic p (provided $a_\bullet$ takes integer values), and commute with d in $C^\bullet_{\mathrm{HAS}}$. One can check that if $p^e|n_0$, then $S_{n_0-1}^{\{p\}} = cS_{n_0-1}^{\mathbb{Z}}$, $S_{n_0-2}^{\{p\}} = -S_{n_0-2}^{\mathbb{Z}}$, $S_{n_0-3}^{\{p\}} = cS_{n_0-3}^{\mathbb{Z}}$ with $c=1$ for even n_0, and $c=1/2$ for odd n_0.

Note a significant difference of $S^{\{2\}}$ compared with $S^{\{p\}}$, $p>2$: for the latter, $\left(S^{\{p\}}\right)^3 = 0$. However, $S^{\{2\}}$ is not nilpotent.

Sum up the properties of operator $S_v^{\mathbb{Z}}$. Taking into account that $S_v^{\mathbb{Q}}$ commutes with d_v, $1-t$ and N of the cyclic bicomplex (if $K \supset \mathbb{Q}$), one concludes that (if $K \supset \mathbb{Q}$):

$$\widehat{d}_v d_v \boldsymbol{k}/\!/2 + \boldsymbol{k}/\!/2\,d_v\widehat{d}_v = S_v^{\mathbb{Z}}, \qquad \boldsymbol{k}/\!/2\,d_vS_v^{\mathbb{Z}} = S_v^{\mathbb{Z}}d_v\,\boldsymbol{k}/\!/2;$$

$$(1-t)\,S^{\mathbb{Z}} = S_a^{\mathbb{Z}}\,(1-t)\,, \qquad NS_a^{\mathbb{Z}} = S^{\mathbb{Z}}N.$$

In particular, these properties hold if V is a vector space over $\mathbb{Q}$; since all mentioned operators have integer coefficients, this holds if V is a free module over $K = \mathbb{Z}$ too. Therefore, these identities hold in arbitrary module V without assuming K to be a field; likewise, these hold for arbitrary sheaf of K-modules.

In particular, the operator $S^{\mathbb{Z}}$ acts in $C^{\bullet}_{\mathrm{cAS}}$ (but is not commuting with d). It acts on cyclic cocycles the same as $\boldsymbol{k}/\!\!/2\, d\widehat{d}$; hence it sends them to cyclic cocycles (moreover, it sends them into dC_{HAS}). Moreover, if $n/\!\!/2$ is invertible in K, then $S^{\mathbb{Z}}d$ is proportional to $dS^{\mathbb{Z}}$ on C^{n-1}_{HAS}; hence $S^{\mathbb{Z}}$ sends a coboundary in C^{n}_{cAS} to a coboundary in C^{n}_{cAS} provided $n/\!\!/2$ is invertible in K. Under this assumption, $S^{\mathbb{Z}}$ defines an operator in cohomology $H^{n}_{\mathrm{cAS}} \to H^{n+2}_{\mathrm{cAS}}$. More specifically, $\sigma = S^{\mathbb{Z}}\,(\boldsymbol{k}-1)/\!\!/2$ satisfies $\sigma d = \boldsymbol{k}/\!\!/2\, dS^{\mathbb{Z}}\,\boldsymbol{k}/\!\!/2$, hence sends a coboundary to a coboundary; therefore $\sigma = \boldsymbol{k}/\!\!/2\, d\widehat{d}\,(\boldsymbol{k}-1)/\!\!/2 = d\widehat{d}\,(\boldsymbol{k}-1)\,(\boldsymbol{k}+2)/2$. induces an operator in cyclic cohomology $H^{\bullet}_{\mathrm{cAS}}$.

On the other hand, since $d\widehat{d}dN = d\widehat{d}Nd_a = dN\widehat{d}_a d_a + d\,(c_1 d\widetilde{\tau} + c_2\widetilde{\tau}d_a)\,d_a = dN\widehat{d}_a d_a$, the operator $d\widehat{d}$ sends $d\,\mathrm{Im}\,N$ to $d\,\mathrm{Im}\,N$. However, $\mathrm{Im}\,N$ in C^{n-1}_{HAS} coincides with C^{n-1}_{cAS} unless $\mathrm{char}\,K \mid n$ (moreover, $nC^{n-1}_{\mathrm{cAS}} \subset \mathrm{Im}\,N$ for any K); hence $d\widehat{d}\,(\boldsymbol{k}-1)$ send $d\,\mathrm{Ker}\,(1-t)$ to $d\,\mathrm{Im}\,N$; hence it defines an operator in cyclic cohomology. Therefore $d\widehat{d}\,(\boldsymbol{k}-1)/\!\!/2$ defines an operator in cyclic cohomology (since $\gcd\,(k, k\,(k+3)/2) = k/\!\!/2$).

Note, however, that $d\widehat{d}$ does not preserve the subcomplex $d\,\mathrm{Ker}\,(1-t)$ of cyclic coboundaries even if M is a point. Indeed, take $v \in V = \mathcal{O}$ which is not proportional to 1; then $v^{\otimes k} \in C^{k-1}_{\mathrm{cAS}}$ (and, if $\mathrm{char}\,K|k$, $v^{\otimes k} \notin \mathrm{Im}\,N$). We claim that $d\widehat{d}d\left(v^{\otimes k}\right) \notin d\,\mathrm{Ker}\,(1-t)$ if $\mathrm{char}\,K|k$ (while $d\left(v^{\otimes k}\right) \in d\,\mathrm{Ker}\,(1-t)$) (at least if $\mathrm{char}\,K \neq 2$). Indeed, if $c \in V^{\otimes k}$, then $d\widehat{d}dc = -c \vee (1 \otimes 1 \otimes 1)$; here $\vee$ denotes the shuffle product. Note that $\mathrm{char}\,K \nmid k+2$, hence $\mathrm{Ker}\,(1-t)$ coincides with $\mathrm{Im}\,N$ in $V^{\otimes k+2}$. If $d\widehat{d}d\left(v^{\otimes k}\right) \in \mathrm{Im}\,dN$, then $d\widehat{d}d\left(v^{\otimes k}\right) = N\mathbb{1}_L d\sum_\alpha f_\alpha \mathbb{1}^{[k]}_\alpha v^{\otimes k}$ for some coefficients f_α, $\alpha = 1, \ldots, k+1$. A calculation shows that the condition that $N\mathbb{1}_L d\sum_\alpha f_\alpha \mathbb{1}^{[k]}_\alpha v^{\otimes k}$ and $v^{\otimes k} \vee (1 \otimes 1 \otimes 1)$ are proportional may be written as $(-1)^{(k+1)\alpha}\, h_{\beta-\alpha} + (-1)^{(k+1)(\beta+1)}\, h_{\gamma-\beta} + (-1)^{(k+1)(\gamma+1)}\, h_{k+3+\alpha-\gamma}$ does not depend on a choice of $1 \leq \alpha < \beta < \gamma \leq k+3$; here $h_m = (-1)^m \left(f_m + (-1)^{(k+1)m}\, f_{k+2-m}\right)$. If $2|k$, this immediately implies $h_m = 0$; if k is odd, then this can be reduced to $h_n + h_m - h_{n+m-1}$ not depending on $n, m \geq 1$ with $n + m \leq k+2$; plugging in $n = 2$ shows that h is a linear function; this, together with $h_{k+2-m} = -h_m$, implies that h_m is proportional to $k + 2 - 2m$.

However, for this choice of h_n, $h_n + h_m - h_{n+m-1}$ vanishes in K. This implies $N\mathbb{1}_L d\sum_\alpha f_\alpha \mathbb{1}^{[k]}_\alpha v^{\otimes k} = 0$. Therefore $d\widehat{d}d\left(v^{\otimes k}\right) \notin d\,\mathrm{Ker}\,(1-t)$.

Definition 1.24. Let the shift S send the class of $a_0 \otimes \cdots \otimes a_n$ in C^n_{qcAS} into the class of

$$\sum_{0 \le k \le l \le n} (2\,(l-k) - n - 1)\, a_0 \otimes \cdots \otimes a_k \otimes 1 \otimes a_{k+1} \otimes \cdots \otimes a_l \otimes 1 \otimes a_{l+1} \otimes \cdots \otimes a_n$$

in C^{n+2}_{qcAS}.

Proposition 1.25. *The operation of shift is correctly defined and commutes with differential. If $k \supset \mathbb{Q}$, then S is quasi-isomorphic to the operation of translation on two columns to the right in* (1.2)*. The natural inclusion of C_{cAS} into the first column of* (1.2) *is a quasi-isomorphism to the quotient by the image of the shift operator. The image* $\operatorname{Im} S$ *is therefore quasi-isomorphic to the kernel of the cyclization* Cycl*, moreover, the corresponding sequence of cohomology*

$$\dots \xrightarrow{\mathrm{Cycl}} H^{n+1}_{HAS} \xrightarrow{B} H^{n}_{qcAS} \xrightarrow{S} H^{n+2}_{qcAS} \xrightarrow{\mathrm{Cycl}} H^{n+2}_{HAS} \xrightarrow{B} H^{n+1}_{qcAS} \to \dots$$

is exact.

Note also that when $C^\bullet_{\mathrm{aHAS}}$ is acyclic, it has a homotopy $h^{\mathrm{aHAS}}_\bullet$ (as any acyclic complex of vector spaces). However, it must not have a direct relationship to the the local homotopy s for $\mathcal{C}^\bullet_{\mathrm{aHAS}}$; in nontrivial cases, this operator is "global" in nature. Indeed, by inspection, $h^{\mathrm{aHAS}}_\bullet$ would allow one to reconstruct $f - f(x_0)$ given a germ of $f(x) - f(y)$ near diagonal $x = y \in M$ which is morally equivalent to reconstruction of $f - f(x_0)$ given df. Here x_0 is a fixed in advance point of M.

The homotopy $h^{\mathrm{aHAS}}_\bullet$ leads to the mapping $B\colon C^\bullet_{\mathrm{HAS}} \to C^\bullet_{\mathrm{cAS}}[-1]$, $B = -N \circ h^{\mathrm{aHAS}}_\bullet \circ (1-t)$. It is easy to see that this mapping is compatible with differentials. We use it below in the exact sequence relating cyclic and Hochschild–Alexander–Spanier cohomology.

Remark 1.26. In general, the complex $C^\bullet_{\mathrm{aHAS}}$ of vector spaces does not allow an explicit homotopy h^{aHAS}. Above, we did not construct a homotopy, but only used local homotopies for complex $\mathcal{C}^\bullet_{\mathrm{aHAS}}$ of sheaves of vector spaces to deduce that a global homotopy *exists*. However, there is a situation in which such an explicit homotopy may be constructed: assume that the vector space of global sections of $\mathcal{O}$ (with a distinguished element 1) allows a (not necessarily associative) multiplication $\odot$ for which 1 is an identity (in fact, "a right unit" is enough for our purposes). Then

$$h^{\mathrm{aHAS}}_n\colon f_0 \otimes \cdots \otimes f_n \mapsto (-1)^n (f_0 \odot f_1) \otimes f_2 \cdots \otimes f_n$$

is a (global) homotopy for $C^\bullet_{\mathrm{aHAS}}$. Note that for this case, we do not need the assumption of space-acyclicity.

Note that existence of the product $\odot$ may be broken into two parts: first, that the mapping $f \mapsto f \otimes 1\colon \mathcal{O} \to \mathcal{O} \otimes \mathcal{O}$ is an injection, and, second, that the image of this mapping is a direct summand of $\mathcal{O} \otimes \mathcal{O}$. The first condition[7] is equivalent to Supp 1

[7]For the second condition, it is not clear whether assumption of space-acyclicity would allow to simplify it.

coinciding with $\operatorname{Supp}\mathcal{O}$ (or, if K is not a field, equivalent to $\operatorname{Ann} 1 \subset \operatorname{Ann} f$ holding locally for any local section f).

Denote by $\mathcal{C}^{\bullet\bullet}_{\mathrm{HAS2}}$ the part of bicomplex (1.2) consisting of the first two columns; use $\mathcal{C}^{\bullet}_{\mathrm{HAS2}}$ for the total complex; likewise for $C^{\bullet\bullet}_{\mathrm{HAS2}}$, $C^{\bullet}_{\mathrm{HAS2}}$. The exact sequence $0 \to \mathcal{C}^{\bullet\bullet}_{\mathrm{dcAS}} \xrightarrow{S_{\mathrm{bi}}} \mathcal{C}^{\bullet\bullet}_{\mathrm{dcAS}} \xrightarrow{\pi} \mathcal{C}^{\bullet\bullet}_{\mathrm{HAS2}} \to 0$ leads to the corresponding long exact sequence of cohomology of total complexes. Note that the column spectral sequence for $C^{\bullet\bullet}_{\mathrm{HAS2}}$ (or, what is the same, the long exact sequence for the inclusion $C^{\bullet}_{\mathrm{aHAS}}[1] \hookrightarrow C^{\bullet}_{\mathrm{HAS2}}$) shows that the natural projection $C^{\bullet}_{\mathrm{HAS2}} \to C^{\bullet}_{\mathrm{HAS}}$ is a quasiisomorphism provided the complex $C^{\bullet}_{\mathrm{aHAS}}$ is acyclic. This implies that

$$\dots \xrightarrow{\pi} H^{n+1}_{\mathrm{HAS}} \xrightarrow{B_d} H^{n}_{\mathrm{dcAS}} \xrightarrow{S_{\mathrm{bi}}} H^{n+2}_{\mathrm{dcAS}} \xrightarrow{\pi} H^{n+2}_{\mathrm{HAS}} \xrightarrow{B_d} H^{n+1}_{\mathrm{dcAS}} \to \dots$$

is exact; here B_d is the operator B described above composed with inclusion of $C^{\bullet}_{\mathrm{cAS}}$ into $C^{\bullet}_{\mathrm{dcAS}}$ as the first column.

Indeed, using acyclicity of $C^{\bullet}_{\mathrm{aHAS}}$ via homotopy $h^{\bullet}_{\mathrm{aHAS}}$, one can lift $C^{\bullet}_{\mathrm{HAS}}$ into $C^{\bullet}_{\mathrm{HAS2}}$ as $\iota\colon c_n \mapsto c_n \oplus \left(-h^{n}_{\mathrm{aHAS}}\,(1-t)\,c_n\right)$ (compatibility with differential may be checked immediately). When $H^{\bullet}_{\mathrm{HAS}}$ is replaced with $H^{\bullet}_{\mathrm{HAS2}}$ above, one gets a long exact sequence of inclusion S_{bi}; denote its connecting mapping by ∂. So all one needs to check is that $B = \partial\iota$ which is immediate.

One concludes that if $k \supset \mathbb{Q}$ (so all versions of cyclic complex are quasiisomorphic) and $C^{\bullet}_{\mathrm{aHAS}}$ is acyclic, then

$$\dots \xrightarrow{j} H^{n+1}_{\mathrm{HAS}} \xrightarrow{B} H^{n}_{\mathrm{cAS}} \xrightarrow{S^{\mathbb{Q}}} H^{n+2}_{\mathrm{cAS}} \xrightarrow{j} H^{n+2}_{\mathrm{HAS}} \xrightarrow{B} H^{n+1}_{\mathrm{cAS}} \to \dots$$

is exact; here j is the mapping induced by inclusion $C^{\bullet}_{\mathrm{cAS}}$ into $C^{\bullet}_{\mathrm{HAS}}$. Note also that the operator $B\colon C^{\bullet}_{\mathrm{HAS}} \to C^{\bullet-1}_{\mathrm{cAS}}$ depends on the choice of homotopy h^{aHAS}, but the induced operator $H^{\bullet}_{\mathrm{HAS}} \xrightarrow{B} H^{\bullet-1}_{\mathrm{cAS}}$ does not. Moreover, one can replace the operator $S^{\mathbb{Q}}$ by any proportional operator, e.g., $S^{\mathbb{Z}}$. In fact, since $S^{\mathbb{Z}}_k = U^{[k]}/\gcd(2,k) = k /\!\!/ 2\,\widehat{d}^{[k+1]}d^{[k]} + (k+2)/\!\!/2\,d\widehat{d}^{[k+1][k]}$, when $S^{\mathbb{Z}}$ is applied to cocycles, it coincides with $(k+2)/\!\!/2\,d\widehat{d}^{[k+1][k]}$. Therefore, one can replace $S^{\mathbb{Q}}$ in the exact sequence by $d\widehat{d}$.

Now, analyse what changes if one drops the assumption $k \supset \mathbb{Q}$. The operator $S^{\mathbb{Q}}$ does not make sense in general, but proportional operators $S^{\mathbb{Z}}$ and $d\widehat{d}$ do. Obviously, the "smallest" choice of integer multiple of $S^{\mathbb{Q}}$ when acting on cycles in $C^{\bullet}_{\mathrm{cAS}}$ is $d\widehat{d}$. However, already showing that $d\widehat{d}$ actually induces an operator in $H^{\bullet}_{\mathrm{cAS}}$ is not completely straightforward.

One needs to show that $d\widehat{d}dC^{\bullet}_{\mathrm{cAS}} \subset dC^{\bullet}_{\mathrm{cAS}}$. One can see that $d\widehat{d}dc = 1^{\otimes 3} \vee c$; here $\vee$ stands for shuffle product. Given coefficients $b_\bullet$, define $\mathcal{D}_b$ as $\sum_{m=0}^{k} b_m \mathbb{1}^{[k]}_{1\,m+2}$; we want to find b so that $dN\mathcal{D}_b c = d\,\hat{\mathrm{d}}\mathrm{dc}$ if $c = tc$.

Note that $t^a \mathbb{1}^{[k]}_{1\,m} t^{-a}$ is $\mathbb{1}^{[k]}_{1+a\,m+a}$ if $a \geq 0$, $m+a \leq k+2$, and $t^a \mathbb{1}^{[k]}_{1\,m} t^{-a+1}$ is $(-1)^{k+1}\,\mathbb{1}^{[k]}_{m+a-k-2\,1+a}$ if $m+a > k+2$, $a \leq k+1$. Hence $N\mathcal{D}_b = \sum_{l<m} \mathbb{1}^{[k]}_{lm} \Big(b_{m-l-1}t^{1-l} + (-1)^{k+1}\,b_{k-m+l+1}$

So denote $\widetilde{b}_m = (-1)^m \left(b_m + (-1)^{k+1} b_{k-m}\right)$; obviously, $\widetilde{b}_m = -\widetilde{b}_{k-m}$, and this is the only restriction on values of $\widetilde{b}_m$. One can see that $dN\mathcal{D}_b c = (-1)^{l+m+n+1} \sum_{l<m<n} \Big(\widetilde{b}_{n-m-1} - \widetilde{b}_{n-l-2} + \widetilde{b}_{n-}$ if $tc = c$. Therefore we want to solve equations $\widetilde{b}_r + \widetilde{b}_s - \widetilde{b}_{r+s} = -1$, $\widetilde{b}_r = -\widetilde{b}_{k-r}$. The first equation implies that $b_r = ar - 1$; the second one implies that there is no solution if $\operatorname{char} K | k$.

Taking $c = v^{\otimes k}$, one can immediately see that $tc = c$, $c \notin \operatorname{Im} N$ when $\operatorname{char} K | k$, $2 \nmid k$. If v is not proportional to 1, then the argument above implies that $d\widehat{d}dc \notin \operatorname{Im} dN$ if $\operatorname{char} K | k$, $2 \nmid k$ (even if one considers c locally). Since these conditions imply that $\operatorname{char} K \nmid k+2$, hence $\operatorname{Im} N_{k+2} = \operatorname{Ker}(1 - t_{k+2})$, one gets that $d\widehat{d}$ does not induce a mapping $H^k_{\text{cAS}} \to H^{k+2}_{\text{cAS}}$.

If $2 \neq \operatorname{char} K | k$, $k = 2k'$, then take two non-proportional vectors v', v''; one can see that if $c = (v' \otimes v'')^{\otimes k'} - (v'' \otimes v')^{\otimes k'}$, then $c \in \operatorname{Ker}(1 - t)$, $c \notin \operatorname{Im} N$. Assume that v', v'' are not proportional to 1; extending $\{1, v', v''\}$ to a basis in V gives a grading in tensor power of V by the number of occurrence of 1, of v' and of v''. To check whether $1^{\otimes 3} \vee c = Ndc'$ for some c', consider elements of gradings $(3, k_1, k_1)$ (for $1^{\otimes 3} \vee c$) and $(2, k_1, k_1)$ (for c'). In the former subspace, consider span of tensor monomials starting with v', where v' may be followed only by v'', and v'' and 1 may be followed only by 1 or v'. One may assume that monomials appearing in c' start with 1 and satisfy the same "follow" rules. Now one can immediately see that the matrix coefficients of Nd between these two subspaces are exactly the same as those considered in the case $c = v^{\otimes k}$ (only with k replaced by k'); the correspondence consists of replacing $v' \otimes v''$ by v. Therefore $1^{\otimes 3} \vee c \notin \operatorname{Im} Nd$, hence $d\widehat{d}$ does not induce a mapping $H^k_{\text{cAS}} \to H^{k+2}_{\text{cAS}}$.

However, $c \vee 1^{\otimes 3} = c \vee (1^{\otimes 2} \vee 1) = d(c \vee 1^{\otimes 2})$. Therefore, it is enough to show that $(1 - t)(c \vee 1^{\otimes 2}) = 0$ if $c = tc$. Recall that for $a \in V^{\otimes \alpha}$, $b \in V^{\otimes \beta}$, $a \vee b$ is sum of $(-1)^{\sigma_P} \sigma_P(a \otimes b)$; here σ_P is a permutation corresponding to a partition P of $\{1, \dots, \alpha + \beta\}$ into two subsets $\sigma_P\{1, \dots, \alpha\}$, $\sigma_P\{\alpha + 1, \dots, \beta\}$. Denote by $\boldsymbol{t}$ the cyclic permutation (so that $t = (-1)^{\boldsymbol{t}} \boldsymbol{t}$). Now $\boldsymbol{t}$ acts on the set of partitions so that $\boldsymbol{t}_{\alpha+\beta}\sigma_P = \sigma_{\boldsymbol{t}P}(\boldsymbol{t}_\alpha \times \operatorname{id}_\beta)$ or $\boldsymbol{t}_{\alpha+\beta}\sigma_P = \sigma_{\boldsymbol{t}P}(\operatorname{id}_\alpha \times \boldsymbol{t}_\beta)$ depending on whether $\sigma_P|_{\alpha+\beta} = \alpha + \beta$; here $\times$ denotes a mapping of permutation groups $\mathfrak{S}_\alpha \times \mathfrak{S}_\beta \to \mathfrak{S}_{\alpha+\beta}$.

Unfortunately, $d\widehat{d}$ does not send[8] cyclic coboundaries to cyclic coboundaries, so does not induce an operator $H^n_{\text{cAS}} \to H^{n+2}_{\text{cAS}}$ if $\operatorname{char} K \mid n+2$, $\operatorname{char} K \neq 2$ (so $d\widehat{d}$ is not proportional to $S^{\mathbb{Z}}$ when acting on cocycles). (If $\operatorname{char} K = 2$, then $(1 - t)\widehat{d}d = \widehat{d}_a d_a (1 - t) - \boldsymbol{k} d_a d$ vanishes on C^n_{cAS} if $2 | n+3$, and then $d\widehat{d}$ sends dC^n_{cAS} to dC^n_{cAS}; on the other hand, if $2 \nmid n+1$, then $C^n_{\text{cAS}} = NC^n_{\text{HAS}}$, and $d\widehat{d}dN = dN\widehat{d}_a d_a$.)

To circumvent this problem, define $\mathring{H}^n_{\text{cAS}}$ as $Z^n / \mathring{B}^n$; here $Z^n = \operatorname{Ker} d : C^n_{\text{cAS}} \to C^{n+1}_{\text{cAS}}$, and $\mathring{B}^n = dNC^{n-1}_{\text{HAS}} \subset dC^{n-1}_{\text{cAS}}$. Now $d\widehat{d}$ sends Z^n to Z^{n+2} since $(1 - t)\, d\widehat{d} = d_a\widehat{d}(1 - t) + d_a d\boldsymbol{k}$; moreover, it sends $\mathring{B}^n$ to $\mathring{B}^{n+2}$ since $d\widehat{d}dN = 0 = dN\widehat{d}_a d_a$. Hence

[8]**Demonstrate this??? Same for "smallest choice"???**

one gets an operator $d\widehat{d}\colon \mathring{H}^{n}_{\text{cAS}} \to \mathring{H}^{n+2}_{\text{cAS}}$. If $K \supset \mathbb{Q}$, or $\operatorname{char} K \nmid n+1/\!/2$, then $\mathring{H}^{n}_{\text{cAS}} = H^{n}_{\text{cAS}}$; in general, there is a surjection $\mathring{H}^{n}_{\text{cAS}} \to H^{n}_{\text{cAS}}$. Since $Bd = -dB$, $B \operatorname{Im} d \subset d \operatorname{Im} N$ and $B \operatorname{Ker} d \subset \operatorname{Ker} d \cap \operatorname{Im} N$; hence B defines an operator $H^{n}_{\text{HAS}} \to \mathring{H}^{n-1}_{\text{cAS}}$. The kernel of $\mathring{H}^{n}_{\text{cAS}} \to H^{n}_{\text{cAS}}$ is $dC^{n-1}_{\text{cAS}} / \operatorname{Im} dNC^{n-1}_{\text{HAS}}$, or $d \operatorname{Ker}(1-t)/d \operatorname{Im} N$; this vanishes unless $\operatorname{char} K | n$.

With a change of $H^{\bullet}_{\text{cAS}}$ to $\mathring{H}^{\bullet}_{\text{cAS}}$, a simple calculation shows that the diagram remains a complex even with the "smallest" choice $d\widehat{d}$ of the replacement for $S^{\mathbb{Q}}$. Indeed, the only nontrivial statement is that $d\widehat{d}B$ on cocycles gives something in $\operatorname{Im} dN$. However, $\widehat{d}N = N\widehat{d}_a + \widetilde{\tau} d_a \boldsymbol{k} - \boldsymbol{k} d\widetilde{\tau}$; the first term contributes an element of $\operatorname{Im} dN$, the last term contributs 0; so what remains is to consider $d\widetilde{\tau} d_a h\,(1-t)\,c$ with $dc = 0$. Obviously, $d_a h\,(1-t)\,c = (1-t)\,c$, and $\widetilde{\tau}\,(1-t) = N - \boldsymbol{k} \cdot \mathrm{id}$; now N contributes an element of $\operatorname{Im} dN$, and id contributes $dc = 0$.

Remark 1.27. Since $-d\widehat{d} - \widehat{d}d = \wedge 1^{\otimes 2} = 1^{\otimes 2}\wedge$ (the operators of shuffle product on the right and on the left) on $\operatorname{Im} d$, $d\widehat{d}$ coincides with $-\wedge 1^{\otimes 2}$. Therefore, the operators $\mathring{H}^{n}_{\text{cAS}} \to \mathring{H}^{n+2}_{\text{cAS}}$ induced by $d\widehat{d}$ and $-\wedge 1^{\otimes 2}$ coincide. So in this arrow of the complex, one can use any one of 3 operators S, $d\widehat{d}$ and $-\wedge 1^{\otimes 2}$; the last two coincide, and the first one is proportional to them.

Since one gets a complex

$$\dots \xrightarrow{j} H^{n+1}_{\text{HAS}} \to \mathring{H}^{B}_{\text{cAS}}{}^{n} \to \mathring{H}^{d\widehat{d}}_{\text{cAS}}{}^{n+2} \xrightarrow{j} H^{n+2}_{\text{HAS}} \to \mathring{H}^{B}_{\text{cAS}}{}^{n+1} \to \dots, \tag{1.3}$$

the next step is to inspect its cohomology. Start with the only term $B \circ j$ which does not involve S; one can replace $\mathring{H}$ by H. Note that $\operatorname{Ker} B$ consists of d-cocycles $c \in C^{k}_{\text{HAS}}$ with $Bc = dc''$ and $(1-t)\,c'' = 0$. Now, start walking over the bicomplex; $dc = 0$ immediately implies $d_a h\,(1-t)\,c = (1-t)\,c$, so putting $c' = h\,(1-t)\,c$ one gets relations $dc = 0$, $(1-t)\,c = d_a c'$, $Nc' = dc''$, $(1-t)\,c'' = 0$, $c' \in C^{k-1}_{\text{aHAS}}$, $c'' \in C^{k-2}_{\text{HAS}}$; moreover, c' is defined uniquely up to addition of $d_a c_1$ (this adds Nc_1 to c''). If the row of theh bicomplex containing c'' is exact, one can kill c'', hence one may assume $Nc' = 0$. If, additionally, the row of the bicomplex containing c' is exact, one can write $c' = (1-t)\,c_2$; then replacing c by $c - dc_2$ allows one to replace c' by 0; therefore $c - dc_2$ is a cyclic cocycle, which implies exactness of the complex (1.3) at terms $H^{\bullet}_{\text{HAS}}$. Label equ1.55,

$\boldsymbol{BS}$ starts here:

Note that if $\operatorname{char} K = p > 0$, then only rows $C^{l}_{(a)\text{HAS}}$ with $p|l$ are not exact; hence at most one of rows $k-1$, $k-2$ containing c' and c'' is not exact. Assume that the row containing c'' is exact; then the equations on c, c', c'' may be reduced to $dc = 0$, $(1-t)\,c = d_a c'$, $Nc' = 0$ with c' defined up to addition of $d_a\,(1-t)\,c_2 = (1-t)\,dc_2$. Changing c to $c + d\bar{c}$ changes c' to $c' + (1-t)\,\bar{c}$, changing c by a cyclic cocycle does not change c'. This boils down to an injection $\operatorname{Ker} B/\operatorname{Im} j \hookrightarrow \operatorname{Ker} N/(\operatorname{Im}(1-t) + Z)$ from cohomology of (1.3) to a quotient of row-cohomology of the bicomplex; here

$Z = \operatorname{Ker} N \cap \operatorname{Ker} d_a$. The image consists of classes of c' such that $d_a c' \in (1-t) \operatorname{Ker} d$. Note that given c' with $Nc' = 0$, one can assume that $d_a c' = (1-t)\,\widetilde{c}$ (one may assume that cohomology of the row containing c' is non-trivial; then cohomology of the row containing $d_a c'$ is trivial), and $\widetilde{c}$ is defined up to addition of Nc_2. Hence $d\widetilde{c}$ is defined up to $\operatorname{Im} dN$. If $c' = (1-t)\,c_3$, then $d\widetilde{c}$ may be assumed to be 0. One obtains a mapping $\Xi\colon c' \mapsto d\widetilde{c}\colon \operatorname{Ker} N_{k-1}/\operatorname{Im}(1-t_{k-1}) \to \operatorname{Im} d/\operatorname{Im} dN_k$, and cohomology of (1.3) at term $H^{\bullet}_{\text{HAS}}$ coincides with kernel of this mapping (assuming the row containing c'' is acyclic).

On the other hand, $\operatorname{Ker} N/\operatorname{Im}(1-t)$ in $V^{\otimes pl}$, here V is a vector space over K, $\operatorname{char} K = p$, can be easily described: it consists of tensors $b^{\otimes p}$, $b \in V^{\otimes l}$, and b is defined modulo $\operatorname{Im}(1-t_l)$. Note that $b \mapsto b^{\otimes p}$ gives an inclusion $F_p\colon V^{\otimes l} \hookrightarrow V^{\otimes pl}$ compatible with action of t and identifying $\operatorname{Coker}(1-t_l)$ with $\operatorname{Ker} N_{pl}/\operatorname{Im}(1-t_{pl})$; while F_p is non-linear, the image in $\operatorname{Ker} N_{pl}/\operatorname{Im}(1-t_{pl})$ is $\mathbb{Z}_p$-linear (and K-semilinear). A similar statement holds for spaces $C^{\bullet}_{\text{HAS}}$ provided the sheaves $\mathcal{C}^{\bullet}_{\text{HAS}}$ (and a few related sheaves) are space-acyclic. Indeed, the observation about F_p is applicable on the level of sheaves. On the other hand, if $\varphi\colon \mathcal{S}^{\bullet} \to \mathcal{T}^{\bullet}$ is a quasiisomorphism of complexes of sheaves, and sheaves in $\mathcal{S}^{\bullet}$, $\mathcal{T}^{\bullet}$ are space-acyclic, then $\Gamma(\mathcal{S}^{\bullet}) \to \Gamma(\mathcal{T}^{\bullet})$ is a quasiisomorphism (apply the double-complex trick with $\boldsymbol{C}^{*}(\mathcal{S}^{\bullet})$ to the cone of φ). Taking $\mathcal{T}^{\bullet}$ to be the row of cyclic double complex, one identifies $\operatorname{Ker} N_{pl}/\operatorname{Im}(1-t_{pl})$ with $\Gamma(\operatorname{Coker}(1-t_l))$; here t_l acts in $\mathcal{C}^{l}_{\text{aHAS}}$ (provided $\operatorname{Coker}(1-t_l)$ is space-acyclic—which follows from space-acyclicity of $\mathcal{C}^{l}_{\text{aHAS}}$ if $p \nmid l$ since then Coker is a direct summand). Likewise, if both $\operatorname{Coker}(1-t_l)$ and $\operatorname{Im}(1-t_l)$ are space-acyclic, then $\Gamma(\operatorname{Coker}(1-t_l))$ coincides[9] with $\operatorname{Coker}(1-t_l)$ acting in C^{l}_{aHAS}.

Calculate the image $\widetilde{Z}$ of Z in $\operatorname{Ker} N_k/\operatorname{Im}(1-t_k)$; we claim that it is spanned by $(\operatorname{Ker} d_a)^{\otimes p}$ and $\widehat{Z}$; here $\widehat{Z} = 0$ if $2 \nmid k = lp$, and $\widehat{Z} = \left\{\psi 1^{\otimes k}\right\}$ otherwise; here ψ runs over locally constant functions. Let $\widetilde{s}$ be the local homotopy s defined above on $\mathcal{C}^{>0}_{\text{aHAS}}$, and is defined as φ on $\mathcal{C}^{0}_{\text{aHAS}}$. If $d_a(c^{\otimes p}) = 0$, then $c^{\otimes p} = d_a(s \cdot c^{\otimes p}) = d_a(c^{\otimes p-1} \otimes \widetilde{s} \cdot c) = c^{\otimes p-1} \otimes d_a(\widetilde{s} \cdot c) \pm d_a(c^{\otimes p-1}) \otimes \widetilde{s} \cdot c$. If $l > 1$, then $\widetilde{s} \cdot c = s \cdot c$, hence $d_a(\widetilde{s} \cdot c) = c - \widetilde{s} \cdot d_a c$, hence $c^{\otimes p-1} \otimes \widetilde{s} \cdot d_a c = \pm d_a(c^{\otimes p-1}) \otimes \widetilde{s} \cdot c$. Assume $\widetilde{s} \cdot d_a c \neq 0$; decomposing tensors as in $V^{\otimes l(p-1)} \otimes V \otimes V^{p-1}$, one concludes that locally there exists a section f of $\mathcal{O}$ such that $d_a(c^{\otimes p-1}) = c^{\otimes p-1} \otimes f$, and $d_a(\widetilde{s} \cdot c) = \pm f \otimes \widetilde{s} \cdot c$; the former formula (together with $d_a c^{\otimes p} = 0$) implies that $d_a c = \pm f \otimes c$ and $\widetilde{s} \cdot d_a c = \pm f \otimes \widetilde{s} \cdot c$. Together with the latter formula, this shows that $c = \operatorname{const} \cdot f \otimes \widetilde{s} \cdot c$; plugging the LHS into the RHS repeatedly implies that $c = \operatorname{const} \cdot f^{\otimes l}$. Since $d_a c$ is proportional to c, f must be proportional to 1, and c to $1^{\otimes l}$.

In the case $\widetilde{s} \cdot d_a c = 0$, one may assume $\widetilde{s} \cdot c \neq 0$; hence $d_a(c^{\otimes p-1}) = 0$. Therefore $d_a(c^{\otimes p}) = 0$ implies that $c^{\otimes p-1} \otimes d_a c = 0$, hence $d_a c = 0$.

[9] One should be careful in these considerations; e.g., if K contains l-th roots of unity, and allows l-th roots, then $\operatorname{Ker}(1-t_l)$ coincides locally with $f^{\otimes l}$, f being a section of $\mathcal{O}$. However, globally this identification does not hold even if $K \supset \mathbb{Q}$: e.g., a global section-up-to-monodromy f may lead to an honest global section $f^{\otimes l}$ if the monodromy is an l-th root of unity.

If $l=1$, then $\widetilde{s}\cdot c$ is a scalar and the formula above simplifies to $c^{\otimes p}=(\widetilde{s}\cdot c)\,d_a\left(c^{\otimes p-1}\right)=\pm(\widetilde{s}\cdot c)\left(d_a c^{\otimes p-2}\right)\otimes c+(\widetilde{s}\cdot c)\,c^{p-2}\otimes d_a c$. Again, this implies $d_a c=f\otimes c$; since $d_a c=c\otimes 1$, c must be proportional to 1. To finish description of $\widetilde{Z}$, it is enough to note that $d_a 1^{\otimes k}=0$ iff $2|k$.

Conjecture: so far I could not find any element of the kernel of the mapping Ξ from $(\operatorname{Ker} N_k/\operatorname{Im}(1-t_k))/\widetilde{Z}$. I expect that the kernel is 0, hence c' does not contribute to cohomology!

Choose a basis $\{v_s\}_{s\in S}$ in V containing 1. Identify $V^{\otimes l}$ with words in alphabet S of length l; consider a $\mathbb{Z}$-grading $\mathcal{G}_{s_0,\mathcal{L}}$ on $V^{\otimes l}$ given by number of occurences of v_{s_0}, $s_0\in S$, in positions in $\mathcal{L}\subset\{1,\dots,l\}$. We will use $\mathbb{Z}^m$-gradings components of which are $\mathcal{G}_{s_0,\mathcal{L}}$. The basis in $(\operatorname{Ker} N_k/\operatorname{Im}(1-t_k))$ is given by monomials $\mu^{\otimes l}$, here $\operatorname{char} K=p|l|k$, and μ is a word of length k/l without cyclic symmetries. Here μ should be chosen as a representative of "circular words", i.e., classes of words up to cyclic rotations). To choose such a representative, one can take the minimal (in lexicographical order) representative in a class; here one needs an ordering in S; assume that $s_1=1$, and $s\leq 1$ for any s in S.

For a word ν, denote by $\bar{\nu}$ its *reduction:* the word obtained from ν by removing symbols 1. Consider the grading of the tensor power of V by the circular word of $\bar{\nu}$, here a monomial in $V^{\otimes n}$ is considered as a word ν; note that t, d_a, d, $\widehat{d}$, $\widehat{d}_a$ preserve this grading. So one can focus the attention on one of these gradings.

Decompose $\bar{\mu}=\lambda^{l'}$ as a power; here λ has no circular symmetries; to do this, one must assume that μ contains a symbol distinct from 1 (if $\mu=1^{\otimes k}$, this requires a separate consideration—but it is trivial). Let λ' be the minimal representative of the circular class of λ (one may have $\lambda'\neq\lambda$; e.g., take $\mu=aba1a1$, $\lambda'=aaab$). Our grading is determined by (λ',l'). Apply Ξ to the monomial $\mu^{\otimes l}$; consider the component of the result corresponding to words not starting with 1, and the reduced word being $\lambda'^{ll'}$. Note that $(1-t_m)\,\widetilde{\tau}^{[m]}=N_m-m$. If $Nc'=0$, then $Nd_a c'=0$, hence $(k+1)\,c+\widetilde{\tau}d_a c'$ is killed by $(1-t)$; note that $k+1$ is invertible in K. Hence to calculate $\operatorname{Ker}\Xi$, one can replace Ξ by $\widehat{\Xi}=d\widetilde{\tau}d_a$. Let R consist of $0\leq r<k/l$ such that $t^r\mu$ does not start with 1, and $\overline{t^r\mu}=\lambda'^{l'}$ (hence $|R|=l'$). Then the component of $\widehat{\Xi}\left(\mu^{\otimes l}\right)$ of required grading coincides with the component of $\sum_{r\in R}\sum_{r'=0}^{l}\sum_{\varepsilon=0}^{1}\left(r+r'k/l+\varepsilon\right)dt^{r+r'k/l+\varepsilon}d_{a,r+r'k/l,\varepsilon}\left(\mu^{\otimes l}\right)$; here $d_{a,r,0}=d_a-d_{a,r,1}$, $d_{a,r,1}=\sum_{s=0}^{r-1}\boldsymbol{d}_s$. (This decomposition of d_a corresponds to $d_a\left(w_1\otimes w_2\right)=(-1)^{|w_2|}\left(d_a w_1\right)\otimes w_2+w_1\otimes d_a w_2$ with $|w_2|=r$.)

Note that $t^{r+1}d_{a,r,1}^{[k]}=d_{a,k-r,0}^{[k]}t^r$, $t^r d_{a,r,0}^{[k]}=d_{a,k-r,1}^{[k]}t^r$; plugging into the preceeding formula, one gets that the componet coincides with

$$\sum_{r\in R}\sum_{r'=0}^{l}\sum_{\varepsilon=0}^{1}\left(r+r'k/l+\varepsilon\right)dd_{a,k-r-r'k/l,1-\varepsilon}t^{r+r'k/l}\left(\mu^{\otimes l}\right)=\sum_{r\in R}\sum_{r'=0}^{l}\zeta^{r'}d\left(\left(r+r'k/l\right)d_a+d_{a,k-r-r'k/l,0}\right)t^r\left(\right.$$

here $\zeta=(-1)^{(k+1)k/l}$. Since $\sum_{r'=0}^{l}\zeta^{r'}(r+r'k/l)=0$, one concludes that the component is $\sum_{r\in R}\sum_{r'=0}^{l}\zeta^{r'}dd_{a,k-r-r'k/l,0}t^{r}\left(\mu^{\otimes l}\right)=\sum_{r\in R}f_{\alpha\beta rl}\mathbb{1}_{\alpha\beta}^{[k]}t^{r}\left(\mu^{\otimes l}\right)$; here[10] if $\zeta=1$, then $f_{\alpha\beta rl}$ is the number of integers r' with $(2-\beta+r)\,l/k\leq r'<(2-\alpha+r)\,l/k$; otherwise $f_{\alpha\beta rl}$ is the number of such integers which are even minus number of odd ones.

Consider now the secondary gradings; note that λ' starts with a basis element which is not 1. Fix P such that $\operatorname{char}K=p|P$ and $P|\,(ll')$; let $L=k/P$; consider words of length $k+2$ such that the symbol at positions $1+nL$ is not 1, and in the interval $1+(n-1)\,L\dots nL$ there are exactly $l'|\lambda'|$ non-1s for $n=1,\dots,P-2$. Clearly, these words form a graded component in one of secondary gradings and that the component of $\widehat{\widehat{\Xi}}\left(\mu^{\otimes l}\right)$ in this grading corresponds to $\alpha,\beta>(P-2)\,L$. In particular, if $P=p\geq 3$, then the first L symbols in the word appearing in this component of $\widehat{\widehat{\Xi}}\left(\mu^{\otimes l}\right)$ form one of the words $t^{r}\left(\mu^{\otimes L/l}\right)$, $r\in R$.

What is important to us is that this word determines l; therefore, the words with the same $(\lambda',l'l)$ but different l do not "mix", and one can consider words with the same (λ',l') separately. Taking this into account, put $L=l$, $P=k/l$; now the first l symbols determine $t^{r}\mu$, therefore r. In particular, a certain graded component of $\widehat{\widehat{\Xi}}c$ depends only on the graded component of c of type (λ',l'), and consists of terms with $r=0$, $\alpha,\beta>k-2l$.

Choose the minimal μ_1 of words $t^{r}\mu$, $r\in R$; since μ is chosen to be a certain representative of its circular class, and so far we did not care which of the representatives it is, we may assume that $\mu=\mu_1$. One concludes that the graded component may be written as $\mu^{\otimes l-2}\otimes\varphi(\mu)$ plus terms $\nu_q\otimes\kappa_q$ with $|\nu_q|=|\mu|\,(l-2)$ and ν_1 later than $\mu^{\otimes l-2}$, here $\varphi(\mu)$ is a word of length $2k/l+2$. Therefore, to show that Ξ has no kernel on graded component of type (λ',l'), it is enough to show that $\varphi(\mu)\neq 0$ for any word μ with $\bar{\mu}=\lambda^{l'}$; here $\varphi(\mu)=\sum_{\alpha\beta}f_{\alpha+|\mu|(l-2)\,\beta+|\mu|(l-2)\,0l}\mathbb{1}_{\alpha\beta}^{[k]}\mu^{\otimes 2}=\pm\sum_{\alpha=2}^{|\mu|+1}\sum_{\beta=|\mu|+2}^{2|\mu|+2}\mathbb{1}_{\alpha\beta}^{[k]}\mu^{\otimes 2}=\pm\left(d_a\mu\right)^{\otimes 2}$. One can see that for c in the complement to $1^{\otimes k}$, $\widehat{\widehat{\Xi}}c=0$ iff $d_ac=0$ (provided $p>2$).

The case of $\operatorname{char}K=p=2$ is much more delicate. The argument above allows only "the first step of refinement", one with $P=p$; it does not exclude a "mixing" of different l, μ and r, as well as mixing different μ with the same λ'. Therefore one should consider $\widehat{\widehat{\Xi}}c$ with c being a linear combination of words w^{p} with w of type $(\lambda',k/(p|\lambda'|))$; in turn, w may be of the form $\mu^{l/p}$ with μ having no circular symmetries. What one *can* do using secondary gradings is to ensure that the symbol at position $k/p+2$ is non-1, and in positions $1\dots k/p+1$ there are exactly $|\lambda'|l'k/(pl)$ non-1s; this corresponds to $\alpha\leq k/p+1$, $\beta\geq k/p+3$.

One should be more careful: l is a running parameter, and one should show that Ker does not contain linear combinations with different l!!! In fact, since $\mu_1^{k/pl}$ uniquely determines l, this is not an obstacle—if $l>2$.

[10]**Signs???**

All this argument is $\boldsymbol{BS}$: first, it works only when the base is a point, when exactness in the term H^k_{HAS} allows a simple handling; second, it ignores taking quotient by $\operatorname{Im} dN$.

Check exactness in the term $\mathring{H}^{n+2}_{\mathrm{cAS}}$; a cocycle in $\operatorname{Ker} j$ can be written as dc with $(1-t)\,dc = 0$, $c \in C^{n+1}_{\mathrm{HAS}}$, and c is defined up to HAS-cocycle. The $C^{\bullet}_{\mathrm{cAS}}$-cohomology class of dc is determined by c up to addition of $\bar{c}$ with $(1-t)\,\bar{c} = 0$; hence this class is determined by $c_1 = (1-t)\,c$ which must satisfy $d_a c_1 = 0$, and is defined up to addition of $(1-t)\operatorname{Ker} d$. If d_a is acyclic, then $(1-t)\,c = d_a c'$; now c' satisfies $d_a c' \in \operatorname{Im}(1-t)$ and is defined up to addition of "negligible" elements $\widetilde{c}'$, i.e., such that $d_a \widetilde{c}' \in (1-t)\operatorname{Ker} d$.

On the other hand, the stricter condition $d_a \widetilde{c}' \in (1-t)\operatorname{Im} d = \operatorname{Im}(d_a(1-t))$ shows that any $\widetilde{c}' \in \operatorname{Im}(1-t)$ is negligible. If the row containing c' is acyclic, the class of $C^{\bullet}_{\mathrm{cAS}}$-cohomology of dc is determined by $c'_1 = Nc'$; note that it satisfies $(1-t)\,c'_1 = 0$, $dc'_1 = 0$. Moreover, up to negligible elements, c' can be replaced by $\frac{1}{\boldsymbol{k}} c'_1$. Then $d_a c' = \pm \mathbb{1}_L c'$ and $tc' = c'$.

Now recall the construction of the homotopy S when applied to elements $\bar{c}'$ with $d\bar{c}' = 0$, $(1-t)\,\bar{c}' = 0$: we solve $N\bar{c}'_{-1} = \bar{c}'$, then solve $(1-t)\,\bar{c} = d_a \bar{c}'_{-1}$; now $S\bar{c}' = d\bar{c}$. In fact, we take solutions to these equations given by homotopies for the rows of the cyclic bicomplex which contain $\bar{c}'$, $\bar{c}$; however, if these rows are acyclic, it is clear that the arbitrariness in the choice of $\bar{c}'_{-1}$ does not contribute into $d\bar{c}$, and the arbitrariness in the choice of $\bar{c}$ contributes an element in $\operatorname{Im} dN$, hence does not contribute into the class of $d\bar{c}$ in $\mathring{H}^{\bullet}_{\mathrm{cAS}}$. We saw that modulo $\operatorname{Im} dN$, $d\bar{c}$ coincides with an element proportional to $S\bar{c}'^{\mathbb{Z}}$ and proportional to $d\widehat{d}\bar{c}'$ (under the assumptions, the coefficients are invertible).

Apply this observation to $\bar{c}' = c'_1$; one concludes that the class of dc in $\mathring{H}^{n+2}_{\mathrm{cAS}}$ is proportional to the class of $d\widehat{d}c'_1$. Hence $\operatorname{Ker} j \subset \operatorname{Im} d\widehat{d}$ in the term $\mathring{H}^{n+2}_{\mathrm{cAS}}$ if $\operatorname{char} K \nmid n+1, n+2$.

If the row containing c_1 is exact, then the condition on c' may be rewritten as $Nd_a c' = 0$. If d is exact at the cell of c, then c_1 is determined up to addition of $\operatorname{Im}((1-t)\,d) = \operatorname{Im}(d_a(1-t))$, hence c' is defined up addition of $\operatorname{Ker} d_a$ and $\operatorname{Im}(1-t)$. If, additionally, the row containing c' is exact, then c' is defined up to addition of $\operatorname{Ker} d_a$ and $\operatorname{Ker} N$; hence the cAS-class of dc is determined by $c'_1 = Nc'$ up to addition of $N\operatorname{Ker} d_a = \operatorname{Im} Nd_a = \operatorname{Im} dN$. Under these assumptions, c'_1 must satisfy $dc'_1 = 0$, $(1-t)\,c'_1 = 0$. (If the row *below* c' is exact, then c'_1 becomes a uniquely determined class of cyclic cohomology.)

Now inspect the term $\mathring{H}^{n}_{\mathrm{cAS}}$; let c'_1 be a representative of an element of $\mathring{H}^{n}_{\mathrm{cAS}}$. As above, an element c'_1 in $\operatorname{Ker}(1-t) \cap \operatorname{Ker} d$ may be extended to solutions of $Nc' = c'_1$, $(1-t)\,c = d_a c'$ if $\operatorname{char} K \nmid n+1, n+2$. Moreover, dc differs from an element proportional to $d\widehat{d}c'_1$ by something in $\operatorname{Im} dN$, which is a contribution of the arbitrariness of the choice of c. If the class of $d\widehat{d}c'_1$ in $\mathring{H}^{n+2}_{\mathrm{cAS}}$ vanishes, one can choose c so that $dc =$

0; hence c represents a class of H^{n+1}_{HAS}. Obviously, B maps this class to the class of c'_1 in $\mathring{H}^n_{\mathrm{cAS}}$. One obtains exactness in the term $\mathring{H}^n_{\mathrm{cAS}}$ provided $\operatorname{char} K \nmid n+1, n+2$.

If one replaces $S^{\mathbb{Q}}$ by $S^{\mathbb{Z}}$ (or $S^{\{\operatorname{char} K\}}$), the complex makes a perfect sense also if $\operatorname{char} k > 0$, but there is no hope that it is exact.

Remark 1.28. We see that if $k \supset \mathbb{Q}$ and $\mathcal{O}$ is soft, $C^*_{\mathrm{cAS}}(\mathcal{O})$ is quasi-isomorphic to $C^*(M, k[S])$ as $k[S]$-module. This mapping is given by the inclusion of the constant sheaf $k[S]$ into $\mathcal{C}^\bullet_{\mathrm{qcAS}}$:

$$1 \mapsto 1 \in \mathcal{O} = \mathcal{C}^0_{\mathrm{qcAS}}, \qquad S^k \mapsto S^k \cdot 1 = \mathrm{const} \cdot \underbrace{1 \otimes \cdots \otimes 1}_{2k+1 \text{ times}} \in \mathcal{C}^{2k}_{\mathrm{qcAS}}.$$

Remark 1.29. We have seen that the differential sends a skewsymmetric element of $C^\bullet_{\mathrm{qcAS}}(\mathcal{O})$ to a skewsymmetric element, therefore the Alexander–Spanier complex is a subcomplex of a cyclic Alexander–Spanier complex. Moreover, a differential sends a cyclically symmetric element of $C^\bullet_{\mathrm{HAS}}(\mathcal{O})$ to a cyclically symmetric element, therefore the cyclic complex is in turn a subcomplex of the Hochschild complex. Therefore the above constructions of Alexander–Spanier cocycles gives in fact cyclic and Hochschild Alexander–Spanier cocycles. The application of the mapping S allows to construct in this way any class of the cocycle in the case of soft $\mathcal{O}$ and $k \supset \mathbb{Q}$.

2. Complexes in algebraic situation

2.1. Definitions of complexes.

Let K be a commutative ring over $\mathbb{Q}$. We use here several complexes associated with an associative algebra A over K.

Definition 2.1. The Hochschild homological complex consists of vector spaces $CH_k(A) = A^{\otimes k+1}$ with the differential

$$d\colon f_0 \otimes \cdots \otimes f_k \mapsto \sum_l (-1)^l f_0 \otimes \cdots \otimes (f_l \cdot f_{l+1}) \otimes \cdots \otimes f_k + (-1)^k (f_k \cdot f_0) \otimes f_1 \otimes \cdots \otimes f_k.$$

The acyclic Hochschild complex differs from this one only by the absence of the last term in differential. The cyclic complex CC_* consists of coinvariant "in" the Hochschild complex with respect to the following action of $\mathbb{Z}_{k+1}$ on $A^{\otimes k+1}$:

$$t\colon f_0 \otimes \cdots \otimes f_k \mapsto (-1)^k f_1 \otimes \cdots \otimes f_k \otimes f_0.$$

(It is easy to see that the above differential sends indeed coinvariants $\left(A^{\otimes k+1}\right)_{\mathbb{Z}_{k+1}}$ into coinvariants $\left(A^{\otimes k}\right)_{\mathbb{Z}_k}$.)

In the same way we can consider the corresponding dual cohomological complexes.

We can also consider the corresponding to A Lie algebra $\mathrm{Lie}(A)$ (this algebra coincides with A as a vector space and has commutator as a Lie operation) and homological and cohomological complexes $C^{\mathrm{Lie}}_*(\mathrm{Lie}(A))$ and $C^*_{\mathrm{Lie}}(\mathrm{Lie}(A))$.

This definition has a big resemplance with the definitions of corresponding objects in the topological situation. As then, we have some maps between these complexes, however not any map extends to the topological situation.

Definition 2.2. The mapping of shift S sends the class of $f_0 \otimes \cdots \otimes f_k$ in CC_k into the class of

$$\sum_l (3-k)\, f_0 \otimes \cdots \otimes (f_l \cdot f_{l+1} \cdot f_{l+2}) \otimes \cdots \otimes f_k$$
$$+ \sum_{l+1<m} (2\,(m-l)-k+1)\, f_0 \otimes \cdots \otimes (f_l \cdot f_{l+1}) \otimes \cdots \otimes (f_m \cdot f_{m+1}) \otimes \cdots \otimes f_k.$$

in CC_{k-2}. The mapping B sends the class of $f_0 \otimes \cdots \otimes f_k$ in CC_k into the element

$$\sum_i (-1)^{ik}\, 1 \otimes f_i \otimes \cdots \otimes (f_k \cdot f_0) \otimes \cdots \otimes f_{i-1} + \sum_i (-1)^{(i+1)m}\, f_i \otimes \cdots \otimes f_{i-1} \otimes 1$$

of CH_{k+1}.

The mappings S and B commute with differentials, therefore define an exact sequence of cohomologies

$$\cdots \to HH_{k+1} \to HC_k \to HC_{k-2} \to HH_{k-1} \to \ldots.$$

2.2. **The Lie algebra complex and the cyclic complex.** We can consider any given associative algebra A as a Lie algebra $\mathrm{Lie}\,(A)$ with the commutator operation. Consider the inclusion of the homological Lie-algebraic complex for $\mathrm{Lie}\,(A)$ to the homological cyclic complex for A that sends $X_1 \wedge \cdots \wedge X_n \in \Lambda^n \mathfrak{g}$ to the corresponding element of $\mathfrak{g}^{\otimes n}/\mathbb{Z}_n$. It is easy to see that differential of these two complexes are compatible (up to a factor 2), hence there is a corresponding mapping of homologies:

$$H_*^{\mathrm{Lie}}\,(\mathrm{Lie}\,(A)) \to HC_*\,(A)$$

and of cohomologies

$$HC^*\,(A) \to H^*_{\mathrm{Lie}}\,(\mathrm{Lie}\,(A))\,.$$

2.3. **The Hochschild complex and the cyclic complex.** In the same way as above we can consider a projection from the Hochschild complex to the cyclic complex, that is (by definition) compatible with differentials. Together with the mapping from the previous section we get a diagram

$$\begin{array}{ccc} C_*^{\mathrm{Lie}}\,(A) & \longrightarrow & CC_*\,(A) \\ & & \| \\ CH_*\,(A) & \longrightarrow & CC_*\,(A)\,. \end{array}$$

We defined above three pairings of these complexes with complexes $\left(C_*^{\mathrm{Lie}}\,(A)\,, \wedge 1\right)$, $(CC_*\,(A)\,, \wedge 1)$ and $(CH_*\,(A)\,, m\,(1))$. It is easy to see that there exists a dual diagram to the previous diagram:

$$\begin{array}{ccc} \left(C_*^{\mathrm{Lie}}(A), \wedge 1\right) & \xleftarrow{\alpha} & \left(CC_*(A), \wedge 1\right) \\ & & \| \\ \left(CH_*(A), m(1)\right) & \xleftarrow{\beta} & \left(CC_*(A), \wedge 1\right). \end{array}$$

The mappings α and β are projection and symmetrization correspondingly.

2.4. **A case with a commutative ring.** Suppose that the ring A in the above situation is commutative. In this case it is possible to compute the cohomology explicitly at least in the case when A is smooth in the algebraic-geometrical case.

The simplest possible answer is in the situation of Lie algebra homologies. The differential in the homological complex vanishes, therefore

$$H_*^{\mathrm{Lie}}(A) = \Lambda^* A.$$

The situation with Hochschild homology is also very simple. If A is a space of functions on the manifold M, define Ω_A^* as the space of differential forms on M. It is possible to define this space in terms of A itself, but we do not need such complications, therefore leave this as an exercise to a reader.

Proposition 2.3. *Consider a mapping from the Hochschild complex for a commutative algebra A into the complex Ω_A^* with zero differential:*

$$f_0 \otimes \cdots \otimes f_k \mapsto \sum_{\sigma \in \mathfrak{S}_k} f_0 df_{\sigma_1} \wedge \cdots \wedge df_{\sigma_k} \in \Omega_A^k.$$

This mapping induces an isomorphism on homologies.

In the case of cyclic homology the description is a little bit more complicated. We need to use the mapping of shift $S\colon CC_k \to CC_{k-2}$ here. The first observation is that the above mapping $H_k(A, A) \to \Omega_A^k$ sends an element with a trivial projection on the space $CC_k(A)$ into a closed form. Therefore the same formula as above defines a mapping

$$CC_k(A) \xrightarrow{\alpha} \Omega_A^k / d\Omega_A^{k-1}.$$

We can again consider this mapping as a mapping in the complex with zero differential. Now the compositions $\alpha \circ S^m$ define a mapping of complexes

$$CC_k(A) \xrightarrow{\beta} \Omega_A^k / d\Omega_A^{k-1} \oplus \Omega_A^{k-2} / d\Omega_A^{k-3} \oplus \Omega_A^{k-4} / d\Omega_A^{k-5} \oplus \dots.$$

Consider the following subspace of the space in the right-hand side:

$$W_k = \Omega_A^k / d\Omega_A^{k-1} \oplus H_{DR}^{k-2}(A) \oplus H_{DR}^{k-4}(A) \oplus \cdots \subset \Omega_A^k / d\Omega_A^{k-1} \oplus \Omega_A^{k-2} / d\Omega_A^{k-3} \oplus \Omega_A^{k-4} / d\Omega_A^{k-5} \oplus \dots.$$

We claim that the image of a cycle in $CC_k(A)$ lies in that subspace, and

Proposition 2.4. *The corresponding to β mapping of homology is an isomorphism onto the subspace W_*.*

It is easy to understand that the corresponding to S operator on W_* is

$$\begin{array}{ccc} \Omega^k_A/d\Omega^{k-1}_A & H^{k-2}_{DR}(A) & H^{k-4}_{DR}(A) \ldots \\ \downarrow & \downarrow \text{in} & \downarrow \text{id} \\ \mathbf{0} & \Omega^k_A/d\Omega^{k-1}_A & H^{k-2}_{DR}(A) \ldots . \end{array}$$

Here in is the canonical inclusion.

The described above mappings from Hochschild complex and Lie complex into the cyclic complex are correspondingly taking the quotient by $d\Omega^{k-1}_A$ and taking the jet on a diagonal Δ_M in M^{k+1} (which is a k-form) and taking the same quotient.

In particular, we can see that any class of cyclic homology from $\operatorname{Ker} S$ has a representative that is a skewsymmetric chain. Moreover, in the commutative case there are natural mappings

$$C^{\text{Lie}}_*(A) \to CH_{*-1}(A, A),$$
$$CC_{*-1}(A) \to C^{\text{Lie}}_*(A).$$

3. Cocycles for the algebra of global sections

3.1. **A strange pairing.** Let A be an associative K-algebra with a trace $\operatorname{Tr} : A \to K$ (a trace is a linear mapping satisfying $\operatorname{Tr}[x, y] = 0$).

Definition 3.1. Consider a cyclic complex $CC_k(A) = A^{\otimes k+1}/\mathbb{Z}_{k+1}$. Consider the following pairing between $CC_k(A)$ and itself:

$$((x_0, \ldots, x_k) \cdot (y_0, \ldots, y_k)) = \sum_l (-1)^{kl} \operatorname{Tr} x_0 y_l x_1 y_{l+1} \ldots x_k y_{k+l}$$

(here $y_{k+1+l} \stackrel{\text{def}}{=} y_l$). It is correctly defined, hence it sends the graded vector space $CC_*(A)$ into the complex $CC^*(A)$. Let us denote this mapping as i.

The first question is: can we describe what differential (of degree +1!) on $CC_*(A)$ "corresponds" to the differential on $CC^*(A)$ under this inclusion. *A priory* we cannot expect that such a differential exists at all.

Proposition 3.2. *The following diagram is commutative:*

$$\begin{array}{ccc} CC_k(A) & \xrightarrow{\wedge 1} & CC_{k+1}(A) \\ i\downarrow & & i\downarrow \\ CC^k(A) & \xrightarrow{d} & CC^{k+1}(A). \end{array}$$

Here $\wedge 1$ *denotes the mapping of the shuffle product with* $1 \in A$.

Remark 3.3. Due to associativity of the shuffle product it is evident that the square of the operation of the shuffle product with 1 is 0:

$$(a \wedge 1) \wedge 1 = a \wedge (1 \wedge 1) = 0.$$

Therefore we got the mapping of complexes

$$(CC_*, \wedge 1) \to CC^*.$$

The remarkable fact about this mapping is that the structure of the first (but not the second!) complex does not depend on the ring structure of A at all.

Remark 3.4. It is easy to see that in the same way we can define strange pairings between $C_*^{\mathrm{Lie}} \stackrel{\mathrm{def}}{=} \Lambda^* \operatorname{Lie}(A)$ and itself:

$$(f_1 \wedge \dots \wedge f_k, g_1 \wedge \dots \wedge g_k) = \operatorname{Tr} \sum_{\substack{\sigma,\tau \in \mathfrak{S}_k \\ \sigma_1 = 1}} f_{\sigma_1} g_{\tau_1} \dots f_{\sigma_k} g_{\tau_k},$$

and between the Hochschild complex (or the acyclic Hochschild complex) $CH_*(A) = A^{\otimes * + 1}$ and itself:

$$(f_0 \otimes \dots \otimes f_k, g_0 \otimes \dots \otimes g_k) = \operatorname{Tr} f_0 g_0 \dots f_k g_k.$$

The dual to the differentials mappings (of degree 1) in these graded vector spaces are the wedge product with 1 in the case of the Lie algebra cohomology,

$$f_0 \otimes \dots \otimes f_k \stackrel{m(1)}{\mapsto} (-1)^{k+1} 1 \otimes f_0 \otimes \dots \otimes f_k - (-1)^{k+1} f_0 \otimes 1 \otimes \dots \otimes f_k + \dots \\ + f_0 \otimes \dots \otimes f_k \otimes 1,$$

and

$$f_0 \otimes \dots \otimes f_k \stackrel{m(1)}{\mapsto} -(-1)^{k+1} f_0 \otimes 1 \otimes \dots \otimes f_k + \dots + f_0 \otimes \dots \otimes f_k \otimes 1$$

in two Hochschild complexes correspondingly (up to a sign).

3.2. **A mapping from the Alexander–Spanier complex.** Now we want to consider a sheaf of associative algebras $\mathcal{O}$ over a topological space M with an algebra $\mathcal{A}$ of global sections. Suppose again that the algebra $\mathcal{A}$ has a trace

$$\operatorname{Tr}\colon \mathcal{A}/[\mathcal{A}, \mathcal{A}] \to K.$$

We construct here a mapping from the Alexander–Spanier complex for $\mathcal{O}$ to the Lie-algebraic complex of the algebra $\mathcal{A}$ considered as a Lie algebra.

We have already constructed the mapping $\mathcal{I}$ from the complex $(\Lambda^\bullet \mathcal{A}, \wedge 1)$ to the cochain complex $(\Lambda^\bullet \mathcal{A}^*, (\wedge 1)^*)$. So the only fact we need is what this mapping can be routed via the Alexander–Spanier complex, that is a factor of $(\Lambda^\bullet \mathcal{A}, \wedge 1)$.

We want to prove now that the mapping $\mathcal{I}$ can be direct via the space $\Gamma\left(M, \Lambda^k \mathcal{O}\right)$ (that is a factor of the space $\Lambda^k \mathcal{A} = \Gamma\left(M^k, \operatorname{Alt} \mathcal{O}^{\boxtimes k}\right)$). We need to prove that if the function $f(x_1, \dots, x_k) \in \Lambda^k \mathcal{A}$ is zero in a neighborhood of the diagonal, then $\langle \mathcal{I}(f), g \rangle$ is zero for any chain $g = (g_1, \dots, g_k) \in CC_k(\mathcal{A})$. Consider a representation of f of the form

$$f(x_1, \dots, x_k) = \sum_\alpha f_1^{(\alpha)}(x_1) \wedge \dots \wedge f_k^{(\alpha)}(x_k), \tag{3.1}$$

We have Label equu5.2,

$$\langle \mathcal{I}(f), g\rangle = \sum_{\alpha} \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\sigma} \operatorname{Tr}\left(f^{(\alpha)}_{\sigma_1} g_1 f^{(\alpha)}_{\sigma_2} g_2 \dots f^{(\alpha)}_{\sigma_k} g_k \right).$$

We want to prove that in fact already

$$\sum_{\alpha} \sum_{\sigma \in \mathfrak{S}_k} (-1)^{\sigma} f^{(\alpha)}_{\sigma_1} g_1 f^{(\alpha)}_{\sigma_2} g_2 \dots f^{(\alpha)}_{\sigma_k} g_k = 0. \ (5.3)$$

Indeed, consider a point $m \in M$. If U is a sufficiently small neighborhood of m, then $f|_{U \times \dots \times U} = 0$, therefore in calculation of (?equ5.3?) in U we can substitute instead of representation (3.1) just $f(x_1, \dots, x_k) = 0$.

This defines in fact the mappings

$$C^*_{\mathrm{AS}}(\mathcal{O}) \xrightarrow{\mathcal{I}} C^*_{\mathrm{Lie}}(\Gamma(\mathcal{O}))$$

of complexes and the corresponding mapping of homologies:

$$H^*_{\mathrm{AS}}(\mathcal{O}) \xrightarrow{\mathcal{I}} H^*_{\mathrm{Lie}}(\Gamma(\mathcal{O})).$$

We want to remind that the left-hand side does not depend on the multiplication law in $\mathcal{O}$! Moreover, if the sheaf $\mathcal{O}$ coinsides as a sheaf of vector spaces with the structure sheaf of M, then the left-hand side coincides with the singular cohomology of M (under mild general-topological assumptions).

A simple generalization gives the

Theorem 3.5. *The strange pairing defines the following mappings of complexes that are compatible with differentials, with natural inclusions and projections and the mappings B and S:*

$$C^*_{HAS}(\mathcal{O}) \to HC^*(\Gamma(\mathcal{O}), \Gamma(\mathcal{O})^*),$$
$$C^*_{AS}(\mathcal{O}) \to C^*_{Lie}(\mathrm{Lie}(\Gamma(\mathcal{O}))),$$
$$C^*_{cAS}(\mathcal{O}) \to CC^*(\Gamma(\mathcal{O})),$$
$$C^*_{aHAS}(\mathcal{O}) \to HC^*(\Gamma(\mathcal{O})).$$

We claim that these mappings are compatible with natural mappings between complexes in the left-hand side (described above) and mappings between the complexes in the right-hand side (described, say, in [?LodQuill84Cyc]). We should note, however, that the situation with algebraic complexes is not so simple as with topological complexes, where two complexes in question were subcomplexes in the third. In the algebraic case we have defined the following mappings only:

$$\begin{array}{ccccc} HC(A) & \longleftarrow & CC(A) & \longleftarrow & HC(A, A^*) \\ & & \downarrow & & \\ & & C_{\mathrm{Lie}}(\mathrm{Lie}(A)) & & \end{array},$$

and the natural mapping $C_{\text{Lie}}(\text{Lie}(A)) \xrightarrow{\text{Alt}} CC(A)$ is not compatible with differential. The existance of other mappings in the topological case cannot suggest the existance in the algebraic situation since there is an additional hypothesis of existance of the trace.

Example 3.6. Let us show that the natural mapping of skewsymmetrization $CC_2(\mathcal{A}) \to C_3^{\text{Lie}}(\text{Lie}(\mathcal{A}))$ is not compatible with differentials. Indeed, the differential of (c_0, c_1, c_2) in CC contains only the products in the order c_0c_1, c_1c_2, c_2c_3, therefore in the non-commutative case its skewsymmetrization should not coinside with the differential of the skewsymmetrization, that contains also the product c_1c_0.

In fact we described some "topological part" of the different cohomological complexes for the ring $\Gamma(\mathcal{O})$ and can write explicit cocycles for this part.

3.3. **A case of a commutative algebra.** It is clear that in the case of the commutative algebra a lot of the discussion above becomes degenerate.

Proposition 3.7. *Let $\mathcal{O}$ be a sheaf of commutative K-algebras over X, $\mathcal{A} = \Gamma(X, \mathcal{O})$. Consider a linear functional* $\text{Tr}\colon \mathcal{A} \to K$. *Then the mapping*

$$C_{AS}^k(X, \mathcal{O}) \to C_{Lie}^k(\mathcal{A})$$

vanishes for $k > 0$, the mapping

$$C_{cAS}^k(X, \mathcal{O}) \to CC^k(\mathcal{A})$$

vanishes for odd k and coincides with the mapping

$$(f_0, f_1, \dots, f_k) \mapsto f_0 f_1 \dots f_k$$

for even k. Here we consider $\mathcal{A}$ as included in $CC^k(\mathcal{A})$ by the rule

$$g \mapsto ((c_0, \dots, c_k) \mapsto \text{Tr}\, g c_0 \dots c_k).$$

3.4. **A case of an almost commutative algebra.** Here we investigate the cohomology of an algebra that is approximately commutative. Let $\mathcal{A}$ be a K-algebra, and $*$ be an associative product on $\mathcal{A} \otimes_K K[[h]]$ such that $\mathcal{A}[[h]]$ with this product is a $K[[h]]$-algebra. We can "fix an infinitesimally small" h and consider the corresponding associative product $\cdot_h$ on $\mathcal{A}$. In this way we get a family of associative products on $\mathcal{A}$ parametrised by infinitely small parameter h. Suppose that $\cdot_0 \stackrel{\text{def}}{=} \cdot$ is commutative. We can write this condition in terms of the product $*$:

$$fg - gf = O(h).$$

In this case we can consider the speed of change of the product $\cdot_h$ with respect to h, more precise, how quick this product becomes non-commutative:

$$\{f, g\} \stackrel{\text{def}}{=} \lim_{h \to 0} \frac{f \cdot_h g - g \cdot_h f}{h}.$$

It is clear that this bracket satisfies the Leibniz identity with respect to the commutative product $\cdot$ and the Jacobi identity. The product $\cdot$ and the bracket $\{,\}$ form so called "first approximation" to the product $*$. The formalization of this situation is the following

Definition 3.8. A Poisson algebra $\mathcal{A}$ is a vector space with a commutative product $\cdot$ and a skewsymmetric bracket $\{,\} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ that satisfy the Leibniz and the Jacobi identities.

Consider a Poisson algebra $\mathcal{A}$. Then we have a Poisson bracket on $X = \operatorname{Spec} \mathcal{A}$. If X is smooth, we have a bivector field η (i.e., a section of $\Lambda^2 TX$) on X defined by the rule

$$\langle \eta|_x, df \wedge dg|_x \rangle = \{f, g\}|_x.$$

Indeed, the right-hand side depends only on $df|_x$, $dg|_x$ because of the Leibniz identity, therefore can be written as the left-hand side with an appropriate η.

In any case the Poisson bracket is local, therefore we get a sheaf of Lie algebras $\mathcal{O}$ with the bracket $\{,\}$ on $\operatorname{Spec} \mathcal{A}$. From the other side, for any h we get a sheaf of Lie algebras $\mathcal{O}$ with the bracket $[,]_h$,

$$[f, g]_h = f \cdot_h g - g \cdot_h f.$$

It is easy to see that the bracket $\{,\}$ is the scaled limit of the brackets $[,]_h$:

$$\{,\} = \lim_{h \to 0} \frac{[,]_h}{h}.$$

Consider what is an analogue of trace in the Poisson situation. It should be a mapping $\operatorname{Tr} \colon \Gamma(\mathcal{O}) \to K$ satisfying the relation $\operatorname{Tr}\{f, g\} = 0$. If $\operatorname{Spec} \mathcal{A}$ is smooth and compact (or proper), then this defines a measure on $\operatorname{Spec} \mathcal{A}$, that is invariant with respect to the *Hamiltonian flow* of any function on $\operatorname{Spec} \mathcal{A}$. We suppose that there is a fixed Tr on $\mathcal{A}$.

Consider a class in $H^*_{\mathrm{AS}}(X)$ and the images of this class in $H^*_{\mathrm{Lie}}(\operatorname{Lie}(\mathcal{A}, [,]_h))$. Below we show that these classes have a scaled limit when h goes to 0. Therefore we get a mapping

$$H^*_{\mathrm{AS}}(X) \to H^*_{\mathrm{Lie}}(\operatorname{Lie}(\mathcal{A}, \{,\}))$$

(moreover, the corresponding mapping of complexes). We show below that this mapping can be written using only the data $\cdot$ and $\{,\}$.

Theorem 3.9. *Let $\mathcal{A}$ be a Poisson algebra corresponding to the family of products $\cdot_h$, and a linear function* Tr *on $\mathcal{A}$ that is a trace with respect to any product $\cdot_h$. Consider an arbitrary element $c^n \in C^n_{\mathrm{AS}}(\operatorname{Spec} \mathcal{A}) = \Lambda^{n+1}\mathcal{A}$. Consider the corresponding element $\widetilde{c}^n_h \in C^{n+1}_{\mathrm{Lie}}(\operatorname{Lie}(\mathcal{A}, \cdot_h)) = \Lambda^{n+1}\mathcal{A}^*$. Then*

$$\widetilde{c}^n_h = \widehat{c}^n h^n + O\left(h^{n+1}\right)$$

for some $\widehat{c}^n = \sum_k \binom{n-k-1}{k} \widehat{c}^n_{(k)} \in \Lambda^{n+1}\mathcal{A}^*$, *and if* $c^n = f_0 \wedge \cdots \wedge f_n$*, then the value of* $\widehat{c}^n_{(k)}$ *on* $g_0 \wedge \cdots \wedge g_n \in \Lambda^{n+1}\mathcal{A}$ *can be written as*

$$\begin{aligned}\widehat{c}^n_{(k)}\left(g_0 \wedge \cdots \wedge g_n\right) = \operatorname{Tr} \operatorname*{Alt}_{\sigma,\tau \in \mathfrak{S}_{n+1}} &\left\{f_{\sigma_0}, f_{\sigma_1}\right\} \cdot \left\{f_{\sigma_2}, f_{\sigma_3}\right\} \cdot \cdots \cdot \left\{f_{\sigma_{2k-2}}, f_{\sigma_{2k-1}}\right\} \\ &\cdot \left\{g_{\tau_0}, g_{\tau_1}\right\} \cdot \left\{g_{\tau_2}, g_{\tau_3}\right\} \cdot \cdots \cdot \left\{g_{\tau_{2k-2}}, g_{\tau_{2k-1}}\right\} \\ &\cdot \left\{f_{\sigma_{2k}}, g_{\tau_{2k}}\right\} \cdot \left\{f_{\sigma_{2k+1}}, g_{\tau_{2k+1}}\right\} \cdot \cdots \cdot \left\{f_{\sigma_{n-1}}, g_{\tau_{n-1}}\right\} \cdot f_{\sigma_n} \cdot g_{\tau_n}.\end{aligned}$$

Moreover, for any Poisson algebra $\mathcal{A}$ *with trace* Tr *the above formula determines (by additivity) a mapping* $\widehat{c}^*$ *from the complex* $C^*_{AS}(\operatorname{Spec}\mathcal{A})$ *into the complex* $C^*_{\mathrm{Lie}}(\operatorname{Lie}(\mathcal{A}))$*, and this mapping is compatible with differentials.*

Proof. We should compute

$$\operatorname*{Alt}_{\sigma,\tau \in \mathfrak{S}_{n+1}} f_{\sigma_0} \cdot_h g_{\tau_0} \cdot_h f_{\sigma_1} \cdot_h g_{\tau_1} \cdot_h \cdots \cdot_h f_{\sigma_{n-1}} \cdot_h g_{\tau_{n-1}} \cdot_h f_{\sigma_n} \cdot_h g_{\tau_n}.$$

up to terms of order $n+1$ in h. Let us consider one summand in this formula and write an expression that contains a lot of commutators and gives the same result after alternation. First, let us move all f_{σ_i} with $i \geq 1$ to the left of g_{τ_0} one by one beginning from f_{σ_1} using the identity

$$af = fa + [a, f].$$

The resulting expression can be written as a sum of the expressions of the following form: it begins with a product of the terms f_i, and the remaining factors are of the form

$$[\dots[[g_j, f_{i_1}], f_{i_2}], \dots, f_{i_l}], \quad l \geq 0.$$

Any such product has a coefficient 0 or 1 in this expression.

Some terms g_j occurs without a commutator in this expression. Let us move such a term to the right using the formula

$$ga = ag + [g, a].$$

As a result we get a sum of products that begin with some number of f_i, end with some number of g_i and contain terms of the form

$$[g_{j_1}, [g_{j_2}, \dots [g_{j_m}, [\dots[[g_j, f_{i_1}], f_{i_2}], \dots, f_{i_l}]] \dots]], \quad l \geq 1,\, m \geq 0.$$

in between. It is clear that the number of commutators in this term is no less than half the number of letters in this term, and the equality can occur only in the case $l+1$, $m = 0$. On the other hand, consider the beginning of such a product $f_{i_1} \cdot_h \cdots \cdot_h f_{i_k}$. We can write the alternation of this expression in $i_1, \dots, i_k$ as the alternation of

$$2^{-k/2}\left[f_{i_1}, f_{i_2}\right] \cdot_h \cdots \cdot_h \left[f_{i_{k-1}}, f_{i_k}\right]$$

if k is even, and of

$$2^{-(k-1)/2}\left[f_{i_1}, f_{i_2}\right] \cdot_h \cdots \cdot_h \left[f_{i_{k-2}}, f_{i_{k-1}}\right] f_{i_k}.$$

if k is odd. Therefore the alternation contains commutators in quantity no less than half the number of letters in this product minus $\frac{1}{2}$. The same is true for the product of g's that finishes the term we consider.

That means that we can change the expression under the alternation sign in the theorem to a sum of expressions with no less than n commutators, and any expression with exactly n commutators is of the form

$$\begin{aligned}2^{-2k}&\left[f_{i_1},f_{i_2}\right]\cdot_h\dots\cdot_h\left[f_{i_{2k-1}},f_{i_{2k}}\right]\cdot_h f_{i_{2k+1}}\\&\cdot_h\left[g_{j_1},f_{l_1}\right]\cdot_h\dots\left[g_{j_{n-2k}},f_{l_{n-2k}}\right]\\&\cdot_h\left[g_{t_1},g_{t_2}\right]\cdot_h\dots\cdot_h\left[g_{t_{2k-1}},g_{t_{2k}}\right]\cdot_h g_{t_{2k+1}}.\end{aligned}$$

Any term with more than n commutators is $O\left(h^{n+1}\right)$, and in the terms with n commutators we can change $\cdot_h$ to the commutative product $\cdot$, and $[,]$ to $h\{,\}$, with an error of order $O\left(h^{n+1}\right)$. Moreover, any term appears with a coefficient 0 or 1. Therefore the only thing we need to compute is which terms appear indeed in the resulting sum.

The indices i_α and t_β are uniquely determined by the set of indices j_γ and l_δ. It is clear that j_γ are in the same order as τ_i. Suppose that the substitutions $\sigma=\tau=id$. Then the sequence j_γ increases, and the sequence l_δ is bigger than j_γ: $l_\gamma>j_\gamma$ and contains no repeating terms. It is easy to see that any such pair of sequences appears in the sum. From the other side, suppose that for some $\gamma<\delta$ we have $l_\gamma>j_\delta$. Then we can exchange l_γ and l_δ and get another term of this expression. However, it is clear that the sum of two such terms vanishes after alternation, therefore we can consider only terms with $j_\gamma<l_\gamma\le j_{\gamma+1}$. In particular, the sequence l_γ increases.

On the other hand, if $j_\gamma<l_\gamma<j_{\gamma+1}$, then this sequence and the sequence with l_γ increased by 1 give opposite terms after alternation. Therefore we can consider only sequences with $l_\gamma=j_\gamma+1$, and odd $j_{\gamma+1}-j_\gamma$ and $n-j_{n-2k}$. All such sequences give the same contribution into the alternation, therefore it is sufficient to consider one of them (with the biggest possible j_*) and compute the number of such sequences. However, this number is the number of decompositions of k into $n-2k$ summands, i.e., $\binom{n-k-1}{k}$.

Now let us prove the claim of the theorem about Poisson algebras that may not allow deformation to an associative algebra over $K[[h]]$. Consider the difference of strange pairings between $(a_0,\dots,a_{n-1})$ and $\partial\left(x_0,\dots,x_n\right)$, and between $d\left(a_0,\dots,a_{n-1}\right)$ and $(x_0,\dots,x_n)$. Here we consider cyclic complexes, ∂ and d are differentials in the cyclic complex and the cyclic Alexander–Spanier complex correspondingly. We know that these two quantities are equal, therefore the difference is 0, however, we want to do it in a more invariant way. Therefore remember that the strange pairing is a value of Tr on some expression, and compute the difference of these expressions instead. It is easy to see that this difference is

$$\sum_k\left[a_kx_0a_{k+1}x_1\dots a_{k-1}x_{n-1},x_n\right].$$

(The trace of this expression vanishes since it is manifestly a sum of commutators.)

Now we can note if we take pairings between skewsymmetric tensors, we can apply the same procedure as above to the skewsymmetrization of the term $a_k x_0 a_{k+1} x_1 \dots a_{k-1} x_{n-1}$. As a result we present it as an expression containing $n-1$ commutator in any term. That means that *we have represented the incompatibility of the mapping from the theorem with differentials as a trace of a sum of commutators*. Moreover, the expressions in these commutators have a proper scaled limit when h goes to 0, and these limits can be expressed in terms of the commutative product and Poisson bracket only.

Therefore we have specific formula expressing the difference between the expressions in the strange pairings, and this formula is written in terms of commutative product and the Poisson bracket only. However, we have proved this formula only in the case when the Poisson algebra structure is obtained basing on the associative product over $K[[h]]$. However, we can use the structure theorem for Poisson manifold, which says that an open subset of a Poisson manifold allows deformation to an associative algebra. This means that the difference coincides with the sum of commutators on an open subset, therefore everywhere. Hence the trace of the difference vanishes, and the mapping of complexes is compatible with differentials.

Several words about the structure theorem. In the usual formulation it says that in points of an open subset we can find $m \in \mathbb{N}$ and a coordinate system $(x_1, \dots, x_{2k+m})$ such that the Poisson bracket can be written as

$$\{f, g\} = \sum_{l=1}^{k} \left(\frac{\partial f}{\partial x_l} \frac{\partial g}{\partial x_{l+k}} - \frac{\partial g}{\partial x_l} \frac{\partial f}{\partial x_{l+k}} \right).$$

Now we can write the deformation as

$$f \cdot_h g = \sum_{n \geq 0} \sum_{l=1}^{k} \frac{h^n}{n!} \frac{\partial^n f}{\partial x_l^n} \frac{\partial^n g}{\partial x_{l+k}^n}.$$

$\square$

3.5. **The case of a Poisson algebra.** Consider a Poisson algebra $\mathcal{A}$. We defined a mapping

$$C^*_{\mathrm{AS}}(\operatorname{Spec} \mathcal{A}) \to C^*_{\mathrm{Lie}}(\operatorname{Lie}(\mathcal{A}))$$

that is compatible with differentials. Now we want to show that this mapping can be routed via much more coarse complexes. Indeed, there is a natural mapping (of taking the minimal possible jet) from the Alexander–Spanier complex into the de Rham complex

$$f_0 \wedge \dots \wedge f_n \overset{J}{\mapsto} \sum_{l} (-1)^l f_l df_0 \wedge \dots \wedge d\widehat{f_l} \wedge \dots \wedge df_n,$$

and there is another mapping from the complex of differential forms with the Koszul differential into the Lie-algebraic complex for the Lie algebra of functions with Poisson bracket. We are going to show that the mapping $\mathcal{I}$ can be written as a mapping from the de Rham complex into the Koszul complex.

Consider a chain $g_0 \wedge \cdots \wedge g_n \in \Lambda^{n+1} \operatorname{Lie}(\mathcal{A})$. Let us associate a differential form

$$\sum_l (-1)^l g_l dg_0 \wedge \cdots \wedge d\widehat{g_l} \wedge \cdots \wedge dg_n$$

on $\operatorname{Spec} \mathcal{A}$ to this chain. We will denote this mapping by the same letter J. It is very simple to compute the operation on differential forms that corresponds to a differential in a Lie-algebraic complex. It is

$$\begin{aligned} g_0 dg_1 \wedge \cdots \wedge dg_n \overset{\delta}{\mapsto} & \sum_l (-1)^l \{g_0, g_l\} dg_1 \wedge \cdots \wedge d\widehat{g_l} \wedge \cdots \wedge dg_n \\ & + \sum_{l<m} (-1)^{l+m} g_0 d\{g_l, g_m\} \wedge dg_0 \wedge \cdots \wedge d\widehat{g_l} \wedge \cdots \wedge d\widehat{g_m} \wedge \cdots \wedge dg_n. \end{aligned}$$

(This differential was considered by Koszul.) We can write the operation δ as

$$\delta = d \circ i(\eta) + i(\eta) \circ d,$$

where η is the defined above bivector field associated to the Poisson bracket on $\operatorname{Spec} \mathcal{A}$. Indeed, $\{f, g\} = i(\eta)\, df \wedge dg$.

Now we can easily see that the defined above pairing between $F = f_0 \wedge \cdots \wedge f_n \in C^n_{\mathrm{AS}}(\operatorname{Spec} \mathcal{A})$ and $G = g_0 \wedge \cdots \wedge g_n \in \Lambda^{n+1} \operatorname{Lie}(\mathcal{A})$ can be written as

$$\operatorname{Tr} \sum_l \alpha_{n,l} i(\eta)^{n-l} \left(\left(i(\eta)^l J(F) \right) \wedge \left(i(\eta)^l J(G) \right) \right)$$

with appropriate constants $\alpha_{n,l}$. In particular,

Corollary 3.10. *The above formula defines the mapping from the de Rham complex for the Poisson manifold M with a trace to the cohomological Lie-algebraic complex for the Lie algebra of functions on M with respect to the Poisson bracket.*

Remark 3.11. The above analysis is applicable in the case of a Poisson manifold with a trace on functions. However, in a lot of important cases Poisson manifolds carry only a trace on the set of functions with compact support, and this trace satisfies the relation $\operatorname{Tr}\{f, g\} = 0$ if one of the functions f or g has a compact support. We can easily see that in this case the above mapping is well-defined as a mapping from the de Rham complex with compact support or *Alexander—Spanier complex with compact support* (that is obviously defined).

Remark 3.12. In the above theorem we have shown that the pairing is of order $O(h^n)$. The above remarks shows that this pairing is not of a smaller order. The following example will show that this pairing can be nontrivial even on the level of homology. Moreover, this example is a simplified version of the more elaborate example of pseudodifferential symbols which we use as a main component of the proof of non-degeneracy theorem.

Example 3.13. Let us consider the Poisson algebra $\mathcal{P}$ of germs of functions on a symplectic manifold M. Darboux theorem says that we can choose a coordinate system such that this manifold is equipped with the *standard Poisson structure*

$$\{f,g\} = \sum_{l=1}^{k} \left(\frac{\partial f}{\partial x_l} \frac{\partial g}{\partial x_{l+k}} - \frac{\partial g}{\partial x_l} \frac{\partial f}{\partial x_{l+k}} \right).$$

This manifold carries no trace, however, we can define a trace on functions with compact support as

$$\operatorname{Tr} f \stackrel{\text{def}}{=} \int f(x)\, dx_1 \dots dx_{2n}.$$

Therefore we get a mapping from the de Rham complex with compact support to the Lie-algebraic complex for the Poisson algebra of functions. We want to show here that this mapping induces *inclusion on cohomology*.

To show this it is sufficient to provide one Alexander–Spanier cocycle with compact support and one Lie-algebraic cycle with nontrivial pairing between them (since the Alexander–Spanier cohomology with compact support is one-dimensional). Consider a step function $s(x)$ in one variable, i.e., a smooth function such that $s' = 0$ outside a small neighborhood of $x = 0$ and $s(-\infty) = 0$, $s(\infty) = 1$. A simple calculation shows that

$$1 \wedge s(x_1) \wedge s(x_2) \wedge \dots \wedge s(x_{2n}) \in C^{2n}_{\text{AS}}$$

has a compact support. Moreover, this function is manifestly a cocycle, since it contains 1 as a factor.

On the other hand, consider a Lie-algebraic chain

$$1 \wedge x_1 \wedge x_2 \wedge \dots \wedge x_{2n} \in \Lambda^{2n+1}\mathcal{P}.$$

This chain is obviously a cycle, and obviously has a nontrivial pairing with the above Alexander–Spanier cocycle. Therefore both the cycle and the cocycle are non-trivial, and the pairing is nontrivial.

3.6. **The S-operations.** Consider an Alexander–Spanier cochain $c \in C^{k+1}_{\text{AS}}(X, \mathcal{O})$. We described the image $\mathcal{I}c$ of c in the Lie-algebraic complex of $\mathcal{A} = \Gamma(X, \mathcal{O})$. On the other hand, we can consider c as an element of $C^{k+1}_{\text{cAS}}(X, \mathcal{O})$ via the mapping

$$C^{k+1}_{\text{AS}}(X, \mathcal{O}) \to C^{k+1}_{\text{cAS}}(X, \mathcal{O}),$$

and the image of c in the cohomological cyclic complex of $\mathcal{A}$. In this representation we can consider also the action of S-operation on c and the cyclic cochains $S^k(\mathcal{I}c) = \mathcal{I}S^k(c)$. However, though in the algebraic situation we have no mapping that associates to a Lie-algebraic cochain a cyclic cochain, *there is* a mapping in the opposite direction. This means that we can consider $S^k(\mathcal{I}c)$ as a Lie-algebraic cochain.

Hence we constructed a mapping from $C^*_{\mathrm{AS}}[1]\otimes_K K[S]$ into $C_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{A}))$. Moreover, the latter complex is a differential graded algebra (DGA), therefore we can consider the mapping from the free DGA $\mathrm{FreeDGA}(C^*_{\mathrm{AS}}[1]\otimes_K K[S])$ generated by $C^*_{\mathrm{AS}}[1]\otimes_K K[S]$ into $C_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{A}))$. Let us remind that the free DGA is just a symmetric power in the case of vector superspaces.

This construction is defined so while only in the case when $\mathcal{O}$ is a sheaf of associative algebras. However, we know already that if we forget about S-operations the mapping above can be correctly defined also in the case of sheaves of Poisson algebras. Below we show that a similar approximation is true also in the case of S-operations: we can compute a main term in h of the image of $S^k(\mathcal{I}c)$ in the Lie-algebraic cochain complex. However, this main term coincides with an image of some element of higher degree **(????)**, therefore the difference of these two elements has a higher order in h, and the above calculations *do not* give the main term of this difference. Moreover, it is possible to show that this main term *is not* determined by the Poisson algebra structure and it depends on the higher order terms in the product. We discuss this situation below.

Definition 3.14. Consider a family of products $\cdot_h$ in $\mathcal{A}$ and the corresponding mapping $\mathcal{I}$ from $C^\bullet_{\mathrm{AS}}[1]\otimes_K K[S]$ into $\Lambda^\bullet\mathcal{A}^*[[h]] = C^\bullet_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{A}[[h]],\cdot_h))$. Define a filtration on $C^\bullet_{\mathrm{AS}}[1]\otimes_K K[S]$ as $F^k = \left\{c \mid \mathcal{I}c = O\left(h^k\right)\right\}$. Define a mapping $\mathrm{Gr}\,\mathcal{I}$ from the corresponding graded quotient $\mathrm{Gr}\,F^\bullet$ into $\Lambda^\bullet\mathcal{A}^*$ as

$$F^k/F^{k+1} = \mathrm{Gr}^k F \ni c \mapsto \lim_{h\to 0}\frac{\mathcal{I}c}{h^k}.$$

The following fact is obvious:

Lemma 3.15. *Consider a Lie algebra $\mathcal{P}$ associated with the family of priducts $\cdot_h$. There are natural differentials in $\mathrm{Gr}\,F^\bullet$ and in $C^\bullet_{\mathrm{Lie}}(\mathcal{P}) = \Lambda^\bullet\mathcal{A}^*$, and the mapping $\mathrm{Gr}\,\mathcal{I}$ is compatible with differentials.*

In their paper [**?**GelMat92Coh] I. Gelfand and O. Mathieu consider the Poisson algebra $\mathcal{P} = \mathcal{P}(\mathbb{T}^{2n})$ of functions on a symplectic torus. They have constructed an (ad hoc) DGA (that is quasi-isomophic to the above one) with a mapping from it into $C^\bullet_{\mathrm{Lie}}(\mathcal{P})$. They also stated a conjecture that is equivalent to the positive answer to the following question in the case of $X = \mathbb{T}^{2n}$

Question . *Consider a symplectic manifold X and the Lie algebra $\mathcal{P}(X)$ of functions on X with respect to the Poisson bracket. Suppose that $\cdot_h$ is the deformation of the commutative product on X that corresponds to the Poisson bracket on X. Is the above mapping from the symmetric power of the Alexander–Spanier complex with a compact support*

$$\mathrm{FreeDGA}\left(\mathrm{Gr}\left(C^\bullet_{\mathrm{AS}_c}[1]\otimes_K K[S]\right)\right) \to C^\bullet_{Lie}(\mathcal{P})$$

a quasi-isomorphism?

Though there are some indications that the Gelfand—Mathieu conjecture can be valid in the toric case, there can be additional complications in the case of an arbitrary manifold even in the compact case. The structure of the above mapping is nearly related to the failure of the Lefschetz theory in the symplectic case, therefore in the case of (say) twisted torus of Witten [**?**Wit] the structure of this mapping can be yet more complicated.

However, we want to describe the image of the element $\mathcal{I}\left(S^{r}c\right)$ in the Lie-algebraic cohomology of the Poisson algebra $\mathcal{A}$.

Theorem 3.16. *Let $\mathcal{A}$ be a Poisson algebra corresponding to the family of products $\cdot_{h}$, and a linear function* Tr *on $\mathcal{A}$ be a trace with respect to any product $\cdot_{h}$. Consider an arbitrary element $c^{n}\in C_{AS}^{n}\left(\operatorname{Spec}\mathcal{A}\right)=\Lambda^{n+1}\mathcal{A}$. Since $C_{AS}^{n}\left(\operatorname{Spec}\mathcal{A}\right)\subset C_{cAS}^{n}\left(\operatorname{Spec}\mathcal{A}\right)$, we can consider $S^{r}\left(c^{n}\right)\in C_{cAS}^{n+2r}\left(\operatorname{Spec}\mathcal{A}\right)$. Consider the corresponding element $\widehat{c}_{h}^{n,r}\in CC^{n+1+2r}\left(\mathcal{A},\cdot_{h}\right)=\mathcal{A}^{*\otimes n+1+2r}/\mathbb{Z}_{n+1+2r}$, and restrict this cochain to skewsymmetric chains, that gives as a cochain $\widehat{c}_{h}^{n,r}\in\Lambda^{n+2r+1}\mathcal{A}^{*}$ for the Lie algebra* $\operatorname{Lie}\left(\mathcal{A},\cdot_{h}\right)$*. Then*

$$\widehat{c}_{h}^{n,r}=\widehat{c}^{n,r}h^{n+2r}+O\left(h^{n+1+2r}\right)$$

*for some $\widehat{c}^{n,r}=\sum_{k}\binom{n-k-1}{k}(????)\,\widehat{c}_{(k)}^{n,r}\in\Lambda^{n+1+2r}\mathcal{A}^{*}$, (**I should compute it yet**)*

and if $c^{n}=f_{0}\wedge\cdots\wedge f_{n}$, then the value of $\widehat{c}_{(k)}^{n,r}$ on $g_{0}\wedge\cdots\wedge g_{n+2r}\in\Lambda^{n+1+2r}\mathcal{A}$ can be written as

$$\begin{aligned}\widehat{c}_{(k)}^{n,r}\left(g_{0}\wedge\cdots\wedge g_{n+2r}\right)=\operatorname{Tr}\underset{\sigma\in\mathfrak{S}_{n+1},\tau\in\mathfrak{S}_{n+1+2r}}{\operatorname{Alt}}&\left\{f_{\sigma_{0}},f_{\sigma_{1}}\right\}\cdot\left\{f_{\sigma_{2}},f_{\sigma_{3}}\right\}\cdot\dots\cdot\left\{f_{\sigma_{2k-2}},f_{\sigma_{2k-1}}\right\}\\&\cdot\left\{g_{\tau_{0}},g_{\tau_{1}}\right\}\cdot\left\{g_{\tau_{2}},g_{\tau_{3}}\right\}\cdot\dots\cdot\left\{g_{\tau_{2k-2+2r}},g_{\tau_{2k-1+2r}}\right\}\\&\cdot\left\{f_{\sigma_{2k}},g_{\tau_{2k+2r}}\right\}\cdot\left\{f_{\sigma_{2k+1}},g_{\tau_{2k+1+2r}}\right\}\cdot\dots\cdot\left\{f_{\sigma_{n-1}},g_{\tau_{n-1+2r}}\right\}\cdot f_{\sigma_{n}}\cdot g_{\tau_{n+2r}}.\end{aligned}$$

Moreover, for any Poisson algebra $\mathcal{A}$ with trace Tr *the above formula determines (by additivity) a mapping $\widehat{c}^{*,r}$ from the complex $C_{AS}^{*}\left(\operatorname{Spec}\mathcal{A}\right)$ into the complex $C_{\mathrm{Lie}}^{*}\left(\operatorname{Lie}\left(\mathcal{A}\right)\right)$, and this mapping is compatible with differentials.*

Proof. We can proceed in the same way as with the proof of the theorem .... The operation S^{r} inserts $2r$ ones in the given word in all possible places (with some coefficients). Let us consider one particular ordering of the letters f_{α} and g_{β} and one particular insertion of ones in the word $f_{0}f_{1}\dots f_{n}$. Let us call the resulting word $\widetilde{f}_{0}\widetilde{f}_{1}\dots\widetilde{f}_{n+2r}$, any $\widetilde{f}_{\gamma}$ being f_{α} or 1. The strange pairing gives as a word $\widetilde{f}_{0}g_{0}\widetilde{f}_{1}g_{1}\dots\widetilde{f}_{n+2r}g_{n+2r}$. Call two noncommutative polynomials congruent if they become the same after alternation in indices α and β. Now we can make the same transformations as before with the polynomial $\widetilde{f}_{0}g_{0}\widetilde{f}_{1}g_{1}\dots\widetilde{f}_{n+2r}g_{n+2r}$,

until we write this expression as a sum of terms of the form

$$\widetilde{f}_{i_{1}}\cdot_{h}\dots\cdot_{h}\widetilde{f}_{i_{2k+1}}\cdot_{h}\left[g_{j_{1}},f_{l_{1}}\right]\cdot_{h}\dots\left[g_{j_{n-2k}},f_{l_{n-2k}}\right]\cdot_{h}g_{t_{1}}\cdot_{h}\dots\cdot_{h}g_{t_{2k+1+2r}}$$

and of a remainder of order $O\left(h^{n+2r}\right)$.

Here $\widetilde{f_\bullet}$ denotes either some f_i or 1. We can suppose again that $j_1 < l_1 \le j_2 < l_2 \le \cdots \le j_{n-2k} < l_{n-2k}$. Moreover, if $j_i < l_i < l_i' \le j_i$, both $\widetilde{f_{l_i}}$ and $\widetilde{f_{l_i'}}$ are some f_α, and any $\widetilde{f_\gamma}$ is 1 for $l_i < \gamma < l_i'$, then the exchange of l_i and l_i' results in the change of the sign of the alternation. Therefore we can suppose that in the set $\left\{\widetilde{f}_{j_i+1}, \ldots, \widetilde{f}_{j_{i+1}}\right\}$ there is odd numbers of f_γ. Now it is easy to check that if we choose l_i to be the maximal possible index $l_i \le j_{l+1}$ such that $\widetilde{f_{l_i}}$ is some f_γ, then two different choices of $\{j_\delta\}$ contribute the same share in the alternation, and this share does not depend on the choice of places we inserted ones in.

we should fix the number of f_α in $\left\{\widetilde{f}_{j_i+1}, \ldots, \widetilde{f}_{j_{i+1}}\right\}$ and count the contribution.

It is clear now that the theorem is true up to a choice of coefficients in the decomposition of $\widehat{c}^{n,r}$ in $\widehat{c}^{n,r}_{(k)}$. However, since any insertion of ones give the same contribution, we should only compute the sum of coefficients at all the insertions.

The proof that the formula of the theorem gives a mapping of complexes in the case of a Poisson manifold can be carried out in the same way as we did before, without S^r. □

Corollary 3.17. *Let M be a Poisson manifold with a trace* Tr*, and $\mathcal{P}$ be the sheaf of functions with the Poisson bracket. The "shifted strange pairing" between $S^r C^n_{\mathrm{AS}}(M, \mathcal{P})$ and $C^{\mathrm{Lie}}_{n+1+2r}(\mathrm{Lie}(\Gamma(M, \mathcal{P})))$ can be routed via the pairing between $\Omega^n M$ and $\Omega^{n+2r} M$. This pairing can be written as*

$$\left\langle \omega^n, \omega^{n+2r} \right\rangle = \mathrm{Tr} \sum_k \alpha_k i\left(\eta^{n-2k}\right)\left(i\left(\eta^k\right)\omega^n \wedge i\left(\eta^{k+r}\right)\omega^{n+2r}\right)$$

for appropriate constants α_k.

4. Example: pseudodifferential symbols

4.1. **The sheaf of pseudodifferential symbols.** Here we use a synthetic approach and intertwine definitions of pseudodifferential operators and pseudodifferential symbols. However, the operators are only intermediate steps in the process of definition of symbols for us.

Definition 4.1. A function $\widetilde{A}(x, \xi)$ on $T^*\mathbb{R}^n$ is a *classical pseudodifferential symbol* of order $k \in \mathbb{Z}$ if for any given N it has a decomposition

$$\widetilde{A}(x, \xi) = \sum_{j=-N}^{k} A_j(x, \xi) + A^{(N)}(x, \xi),$$

where A_j is a smooth (outside 0 section of $T^*\mathbb{R}^n$) homogeneous in ξ function of homogeneity degree j and $A^{(N)}$ is $o\left(\xi^{-N}\right)$ locally in x when $\xi \to \infty$. We say that

$$\widetilde{A}(x,\xi) \simeq \sum_{j=-\infty}^{k} A_j(x,\xi)$$

is the asymptotic expansion of $\widetilde{A}$.

We consider two symbols the same if they have the same asymptotic expansion.

Consider an operator $A\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$. Consider the point $x_0 \in \mathbb{R}^n$, the δ-function δ_{x_0} in this point and the linear functional

$$A^*\delta_{x_0}\colon f \mapsto (Af)(x_0).$$

on $C^\infty(\mathbb{R}^n)$. Let us translate this generalized function on the vector $-x_0$

$$f(x) \mapsto f_{x_0}(x) = f(x+x_0) \mapsto (Af_{x_0})(x_0)$$

and denote it φ_{A,x_0}. For not to worry about the behavior at large x, fix a cut-off function $\omega(x)$ and denote $\omega\varphi_{A,x_0}$ by $\widetilde{\varphi}_{A,x_0}$.

Definition 4.2. An operator $A\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ is a *classical pseudodifferential operator with a symbol* $\widetilde{A}(x,\xi) = \sum_{j=-\infty}^{k} A_j(x,\xi)$ if the generalized function $\widetilde{\varphi}_{A,x_0}(x)$

$$\varphi_{A,x_0}\colon f \mapsto \langle \varphi_{A,x_0}, f\rangle = \omega(x)\, A\left(f(x+x_0)\right)|_{x_0}$$

has Fourier transform $F\varphi_{A,x_0}(\xi)$ with the asymptotic expansion

$$F\varphi_{A,x_0}(\xi) \simeq \sum_{j=-\infty}^{k} A_j(x_0,\xi), \qquad |\xi| \to \infty.$$

Example 4.3. The operator M_α of multiplication by the function $\alpha(x)$ is pseudodifferential with symbol $A(x,\xi) = \alpha(x)$. Indeed, in this case the generalized function φ_{x_0} is just the δ-function at 0 (this is why we shift the argument of the function f in the definition) with coefficient $\alpha(x_0)$, and the Fourier transform of the δ-function is 1.

Example 4.4. The operator $\frac{\partial}{\partial x_1}$ is pseudodifferential with symbol $i\xi_1$. Moreover, any vector field corresponds to a pseudodifferential operator and the symbol is the corresponding linear function on $T^*\mathbb{R}^n$.

Proposition 4.5. *A composition of two pseudodifferential operators is again a pseudodifferential operator and its symbol has the following asymptotic expansion:*

$$\widetilde{A \circ B} = \sum_{N\geq 0} \frac{1}{N!} \frac{\partial^{|N|}}{\partial \xi^N} \widetilde{A}(x,\xi)\, \frac{\partial^{|N|}}{\partial x^N} \widetilde{B}(x,\xi). \tag{4.1}$$

(The terms in this sum have the order that goes to infinity, therefore to compute Label equ6.3,

a component of $\widetilde{A \circ B}$ *of given homogeneity degree we need to compute a sum of a finite number of summands.)*

If the symbol of a pseudodifferential operator vanishes, then this operator is an operator with a smooth kernel $K(x,y)\,dy$*,* $x, y \in \mathbb{R}^n$*:*

$$f(x) \mapsto (Af)(x) = \int K(x,y)\, f(y)\, dy.$$

Now we want to define a notion of a pseudodifferential operator on a manifold. Consider a pseudodifferential operator P on $\mathbb{R}^n$ and a pair of cut-off functions φ and ψ defined in a neighborhood of $x \in \mathbb{R}^n$. Then $\psi P \varphi$ is the operator sending a locally defined function into a locally defined function with a compact support. It is obvious that this operator is pseudodifferential, moreover, if for any functions φ_i, φ_j from a decomposition of unity

$$\sum_i \varphi_i = 1$$

the operator $\varphi_i P \varphi_j$ is pseudodifferential then the initial operator P is also pseudodifferential. This gives a localization of the notion of a pseudodifferential operator, therefore we can define a pseudodifferential operator on a manifold. However, we want also define a notion of pseudodifferential symbol on a manifold, and this is a little bit more tricky.

We know that the operators with a smooth kernel on a manifold should form a kernel of the mapping from operators to symbols. in any local chart $M \supset U \to \mathbb{R}^n$ we can associate to the pseudodifferential operator its symbol, that is an asymptotic expansion in $T^*\mathbb{R}^n$. Consider two intersecting local charts. The symbol in one of them determines the operator up to addition of an operator with a smooth kernel, therefore it determines the symbol in the part of the other chart that corresponds to intersection of charts.

What we get is the action of "local diffeomorphisms" of $\mathbb{R}^n$ on pseudodifferential symbols. This action is difficult to describe explicitly, however, if we could do it, then we could just define the notion of a pseudodifferential symbol on a manifold without a reference to pseudodifferential operators. For convinience of the reader we want to show that this action is not a new entity, but just a corollary of the formula for the multiplication.

Indeed, consider for simplicity the differential operators on $\mathbb{R}^n$. We know how diffeomorphisms of $\mathbb{R}^n$ acts on this algebra, however, we can *deduce* this action as a corollary of the commutation law for differential operators. Indeed, we can represent a diffeomorphism as an intergral of a *flow* corresponding to some vector field. Now the change in some small time of the operator under the action of this flow is described by the commutator of the vector field and the operator. Now we can integrate these changes and get the image under this diffeomorphism. We can repeat this program literally in the case of pseudodifferential operators.

Corollary 4.6. *Consider a 1-parametric group of diffeomorphisms h_t of $\mathbb{R}^n$ corresponding to a vector field V. It can be raised to $T^*\mathbb{R}^n$, so it determines a group of diffeomorphisms h'_t of $T^*\mathbb{R}^n$ and a vector field V' on $T^*\mathbb{R}^n$. Consider a pseudodifferential symbol P_0 and the equation*

$$-\frac{d}{dt}P_t = V \circ P_t - P_t \circ V.$$

Call a solution of this equation the translation of P by the flow h_t.

The leading terms of $[V, P]$ and of the Lie derivative of the symbol P with respect to the field V' coincide, hence the leading term of P_t moves with the flow h'_t. Moreover, in the equation above we can restict our attention to any fixed number of terms in the symbol P, since the commutator with V preserves degree. Hence if $P = \sum P_j$, and

$$P_j^{(t)} = (h'_t)^* P_{j,t}$$

then the equation on $P_j^{(t)}$ is upper-triangular:

$$\frac{d}{dt}P_j^{(t)} = \sum_{k>j} \alpha_k \left(P_k^{(t)}\right).$$

Here α are some differential operators. Therefore the solution always exists, its leading term is a translation of the leading term of P_0 by the action of h'_t, and any term of the translation depends only on the values of the terms with the same of higher order in the preimage of a given point on $T^*\mathbb{R}^n$.

Consider a manifold M and an operator $A\colon C^\infty(M) \to C^\infty(M)$. We call this operator a *pseudodifferential operator* on M if it is locally of such type, i.e., if for a local chart $h\colon M \supset U \to \mathbb{R}^n$ it acts on functions with compact support in U as some psuedodifferential operator in $\mathbb{R}^n$. This means that for a cut-off function σ with support on U the corresponding operator

$$h^{-1*} \circ M_\sigma \circ A \circ M_\sigma \circ h^*\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$$

is pseudodifferential. It is easy to see that we can consider the symbol of this operator in this coordinate frame and that the highest order term of this symbol is correctly defined function on T^*M. We can consider a complete symbol of A as an asymptotic expansion of a function on T^*M with a "twisted" transformation law under chart changes on M: only the highest term is just transferred by the flow, to the lower terms some additional terms (depending on the higher order terms) are added.

However, we can see that if in one chart the symbol of the operator A is 0 when ξ goes to infinity inside a given open conic subset of T^*M, then this condition is satisfied in any other coordinate chart. The composition law (4.1) shows that a product of such operator with any other operator is again of this type. This means that the restriction of the symbol of the product to an open conic subset is uniquely determined by values of the symbols of factors on the given subset.

Therefore we can consider the set $\Psi\mathrm{DS}(M)$ of pseudodifferential symbols on M, define the multiplication law of such symbols and transformation laws under diffeomorphisms. It easy to see that this ring has a natural structure of a sheaf of rings over the "infinity in the cotangent bundle".

So consider a projective (or better, spherical) completion $\mathcal{P}T^*M$ and the infinity PT^*M in this completion. We can consider a symbol on M as a function on the "punctured infinitesimal neighborhood of PT^*M in $\mathcal{P}T^*M$". We call this (formal) manifold DT^*M. It is fibered over PT^*M with a "punctured disk of infinitesimally small radius" as a fiber. The fiber has two connected components, corresponding to the positive part of the disk and the negative one.

Here we want to show that the cyclic cohomogy of this ring is exhausted by the "topological type" cocycles defined above. To do this we use the description of the cyclic cohomology obtained in the papers [?BryGet, ?Wod] and compare this description with the image of the mapping $\mathcal{I}$.

4.2. **Cohomology of symbols: the Poincaré lemma.** In the section on Poisson algebras we have shown that the strange pairing determines an inclusion of cohomology in the case of germs of functions on a symplectic manifold. Here we want to show the same fact in the case of germs of pseudodifferential symbols.

The sheaf of pseudodifferential symbols lives on the formal manifold DT^*M, which is a product of a punctured formal infinitesimal disk and the spherization of the cotangent bundle. Therefore the cohomology of the base is the product of cohomology of the spherization and cohomology of the punctured disk. A punctured disk looks like a circle homotopically, therefore the cohomology should be 1-dimensional in degrees 0 and 1. The corresponding cocycles in the de Rham complex are 1 and dz/z. The corresponding representatives in the Alexander—Spanier complex are $f(z) = 1$ and $g(z_1, z_2) = \log \frac{z_2}{z_1}$. Let us note that we can write the second cocycle as $1 \wedge \log z$ if we allow $\log z$ as an additional function on the disk. The fact that $\log z$ is outside the ring of functions we consider ensures the non-triviality of this cocycle.

The trace on pseudodifferential symbols is correctly defined on symbols with compact support along the spherization. Therefore we get a mapping from the Alexander—Spanier complex of DT^*M with complex support along the spherization to the Lie-algebraic complex for the Lie algebra of pseudodifferential symbols. This is in a complete analogy with what we did in the case of Poisson algebra on a symplectic manifold.

Example 4.7. Consider a small (convex) open conic subset C of T^*M and the Lie algebra of symbols of pseudodifferential operators in this subset. Taking the coordinates x^i on M we get the corresponding coordinates x^i, ξ_i on T^*M. We can suppose that C is a neighborhood of $x^i = 0$, $i > 0$, $\xi_j = 0$, $j > 1$, $\xi_1 > 0$.

Consider a step function $s(y)$ on $\mathbb{R}$, $s' \neq 0$ only in a small neighborhood of the $y = 0$. We can consider now two Alexander—Spanier cochaines on C:

$$1 \wedge s\left(x^1\right) \wedge \cdots \wedge s\left(x^n\right) \wedge s\left(\xi_2/\xi_1\right) \wedge \cdots \wedge s\left(\xi_n/\xi_1\right)$$

and

$$1 \wedge \log \xi_1 \wedge s\left(x^1\right) \wedge \cdots \wedge s\left(x^n\right) \wedge s\left(\xi_2/\xi_1\right) \wedge \cdots \wedge s\left(\xi_n/\xi_1\right).$$

(We can understand $1 \wedge \log z$ as an entity or as an exterior product with $\log z$ added to the ring of functions.) The same reasons as in the case of a Poisson algebra show that these cochains are cocycles and have a compact support along the spherization. Therefore they define two Lie-algebraic cocycles for the Lie algebra of pseudodifferential symbols in C via the strange pairing.

On the other hand, we can provide two Lie-algebraic *chains* for the same algebra:

$$\frac{1}{\xi_1} \wedge x^2 \wedge \cdots \wedge x^n \wedge \xi_1 \wedge \cdots \wedge \xi_n \text{ and } 1 \wedge x^1 \wedge \cdots \wedge x^n \wedge \xi_1 \wedge \cdots \wedge \xi_n.$$

Again, the simple calculation shows that these chains are cycles and that they have a nondegenerate strange pairing with the above Alexander—Spanier cocycles. This shows that all four (co)cycles are nontrivial and the pairing is nontrivial.

Corollary 4.8. *Consider a small (convex) conic subset C of T^*M. The strange pairing defines a mapping from the Hochschild—Alexander—Spanier complex (with compact support along S^*M) of the neighborhood of infinity in C to the Hochschild complex of the ring of pseudodifferential symbols in C. This mapping is a quasi-isomorphism. The same is true with the mapping from the cyclic Alexander—Spanier complex into the cyclic complex.*

Proof. Let us proof the claim about the cyclic complexes first. It is known that in this case the cyclic cohomology forms a free module over $K[S]$ with two generators in degrees $2n$ and $2n+1$ [?Wod, ?BryGet]. (Let us remind that the operation S has degree 2.) From the other side, the description of cyclic Alexander—Spanier cohomology shows that it is a free module over $K[S]$ in degrees $2n-1$ and $2n$. Since two actions of S on two complexes in question are compatible, it is sufficient to show that the generators of cyclic Alexander—Spanier cohomology go to non-trivial cyclic cocycles. Therefore it is sufficient to provide two cyclic cycles with nontrivial strange pairing with these basic cyclic Alexander—Spanier cocycles.

However, the Alexander—Spanier complex is a subcomplex of the cyclic Alexander—Spanier complex, and the Lie-algebraic homological complex is a subcomplex of the cyclic homological complex, therefore the above example gives us the necessary ingredients. Now the proof for the case of the Hochschild complex is trivial, because in both the topological and algebraic situation the Hochschild complex and the cyclic complex are related by a long exact sequence. □

Remark 4.9. In the above argument we used the calculations with Lie-algebraic complexes. The irony of the situation is that we can nevertheless give no description of the Lie cohomology of the algebra in question.

4.3. **The global cohomology.** In the previous section we gave a simple example of cocycles in the situation of the Poincaré lemma. We exploited the fact that the cohomology in question is known to show that the strange pairing is a quasi-isomorphism in this case. Here we exploit the fact our description of complexes and of the strange pairing is functorial to show that it is a quasi-isomorphism in the general case too.

Consider a manifold M and a sheaf $\mathcal{O}$ of K-algebras over X. Then we can consider a (Hochschild) complex of presheaves $X \supset U \mapsto CH_*(\Gamma(U,\mathcal{O}))$ and the associated complex of sheaves $\mathcal{CH}_*(\mathcal{O})$. In the same way we can consider the cyclic complex $\mathcal{CC}(\mathcal{O})$ and the Lie-algebraic complex $\mathcal{C}_{\mathrm{Lie}}(\mathrm{Lie}(\mathcal{O}))$. We can consider hypercohomology of such a complex and compare it with the corresponding cohomology of the algebra $\Gamma(X,\mathcal{O})$ of global sections.

There is a natural mapping

$$CH(\Gamma(X,\mathcal{O})) \to \Gamma(X,\mathcal{CH}(\mathcal{O}))$$

and analoguous mappings in the cases of cyclic and Lie-algebraic complexes. In the following we use the following example: as X we consider the spherization S^*M of the cotangent bundle T^*M, and as $\mathcal{O}$ we consider the sheave of pseudodifferential symbols over M. It is known [**?**BryGet] that in this case the above mapping is a quasi-isomorphism.

On the other hand, we have a strange pairing between the (say) cyclic Alexander—Spanier complex with compact support and the cyclic complex, and this pairing is correctly defined for any open subset $U \subset X$. Therefore we get a mapping from the cyclic complex of sheaves into the complex of sheaves that is dual to cyclic Alexander—Spanier complex with compact support. In the considered above case we know already that this mapping is a quasi-isomorphism of *complexes of sheaves*, since the corresponding mapping on sections is a quasi-isomorphism in the case of a small open subset.

Now the proof is almost at hand. Consider the spectral sequences associated with these two complexes of sheaves. The E^1 terms are (????)

$$E^1_{pq} = H^p(X,\mathcal{HH}_{-q}(\mathcal{O})) \text{ and } H^p(X,H^q(D)),$$

and the strange pairing induces an isomorphism of these two complexes. However, we know that the first spectral sequence converges to the homology of the algebra of global sections, therefore the strange pairing is indeed nondegenerate in the Hochschild case. The same proof works in the cyclic case. We proved

Theorem 4.10. *Consider a manifold M and the ring of global pseudodifferential symbols $\Psi\mathrm{DS}(M)$ on M. Then the strange pairings between $C^\bullet_{HAS_c}(DT^*M)$ and*

$CH^{\bullet}(\Psi\mathrm{DS}(M))$ or between $C^{\bullet}_{cAS_c}(DT^{*}M)$ and $CC^{\bullet}(\Psi\mathrm{DS}(M))$ induce nondegenerate pairings on cohomology. Moreover, the same is true if we change $T^{*}M$ to an open conic subset in $T^{*}M$, or if we consider pseudodifferential symbols with compact support and Alexander—Spanier chains with arbitrary support.

References

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