ON THE LOCAL GEOMETRY OF A BIHAMILTONIAN STRUCTURE

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Abstract. We give several examples of bihamiltonian manifolds and show that under very mild assumptions a bihamiltonian structure in "general position" is locally of one of these types. This shows, in particular, that a bihamiltonian manifold in general position is always a moduli space of some kind. In the even-dimensional case it is a Hilbert scheme of a surface, in the odd-dimensional case it is a subcotangent bundle of a moduli space of rational curves on a surface.

Contents

Date: August 1992 Printed: July 13, 2000.

0. Introduction

Here we want to discuss a little bit nonstandard (but becoming now more customary) way to ask questions about an integrable system. The usual way is to consider solutions of a, say, system

$$u^{\cdot} = u''' + 6uu' + u^3, \qquad u^{\cdot} = \frac{\partial}{\partial t}u, \quad u' = \frac{\partial}{\partial x}u, \tag{0.1}$$

and to study the behavior of these solutions as functions of x and t. However, it is possible to consider a manifold M of functions $u = u(x)$ instead. In this case the equation (0.1) determines a time evolution of a point u of the manifold, i.e., a vector field V on this manifold.

In this consideration we lose the information of x-dependence of a solution: the notion of manifold includes the freedom to work in any coordinate system, and values of the function $u = u(x)$ at particular points x_i are in this approach just *some* coordinate functions on our manifold M. So the differential-geometric freedom of considering systems up to a diffeomorphism results in a big restriction on the questions we can ask about the system. However, the usual duality results in the fact that a restriction on possible questions means the possibility to give more precise answers on remaining questions.

Here we want to investigate the bihamiltonian geometry of the systems in question. Let us begin with the above example. It is known that it is possible to introduce a couple of *Poisson structures*[1] (or *Poisson brackets*) $\{,\}_1$, $\{,\}_2$ on the manifold M in such a way that the vector field V is Hamiltonian with respect to any linear

[1]Let us remind that a *Poisson structure* is a Lie algebra structure

$$f, g \mapsto \{f, g\}$$

on the set of functions on the manifold satisfying the Leibniz condition:

$$\{f, gh\} = g\{f, h\} + \{f, g\}h.$$

combination of the Poisson structures. That means that the linear combination $\{,\}_{\lambda_1,\lambda_2} = \lambda_1 \{,\}_1 + \lambda_2 \{,\}_2$ of the Poisson brackets

$$\{f,g\}_{\lambda_1,\lambda_2} = \lambda_1 \{f,g\}_1 + \lambda_2 \{f,g\}_2$$

is again a Poisson bracket, and there exist a family of *Hamiltonians* H_{λ_1,λ_2} such that for any function f on M and for almost any pair (λ_1, λ_2) (in the case of a periodical function u we can take $(\lambda_1, \lambda_2) \neq 0$)

$$Vf = \lambda_1 \{H_{\lambda_1,\lambda_2}, f\}_1 + \lambda_2 \{H_{\lambda_1,\lambda_2}, f\}_2 . \tag{0.5}$$

(The latter formula is a particular case of the notions of the *Hamiltonian flow* or Hamiltonian vector field V_φ corresponding to a function φ, which is given by the rule

$$V_\varphi f = \{\varphi, f\},$$

hence this two conditions together mean that our vector field is Hamiltonian with respect to any linear combination of Poisson structures.)

Now it is easy to see that these properties of the dynamical system in question can be cut in two: we have a condition on the Poisson structures which has no reference to the vector field V, and an additional condition on V. It is known that the former condition allows us to find the vector field V basing on the Poisson structures in an almost unique way. Hence we have two different geometrical problems: to find the local forms (up to a diffeomorphism) of pairs of Poisson brackets such that any linear combination is again Poisson, and to find all the vector fields that satisfy the condition (0.5). Let us call a pair of Poisson structures such that any linear combination is again Poisson as *compatible Poisson structures* or a *bihamiltonian structure*. In the same way we can define *k-hamiltonian structures*.

Although in what follows we do not need the explicit formulae for the Poisson structures in specific coordinate frames, we give here these formulae in the case of the system (0.1). The Poisson structures are defined as

$$\begin{aligned} \{f,g\}_1 &= \int \frac{\partial f}{\partial u(x)} \left(\frac{d}{dx} \frac{\partial g}{\partial u(x)} \right) dx, \\ \{f,g\}_2 &= \int \frac{\partial f}{\partial u(x)} \left(\left(\left(\frac{d}{dx} \right)^3 + 4u(x) \frac{\partial}{\partial x} + 2u'(x) \right) \frac{\partial g}{\partial u(x)} \right) \mathrm{dx}. \end{aligned} \tag{0.7}$$

Here f and g are functionals depending on $u = u(x)$, $\frac{\partial f}{\partial u(x_0)}$ is a density of the partial derivative of f with respect to the coordinate $u \mapsto u(x_0)$:

$$f(u + \delta u) - f(u) \approx \int \delta u(x) \frac{\partial f}{\partial u(x)} dx.$$

More precisely, the left-hand side of this formula is a linear functional in δu, hence it can be expressed as an integral of δu with some density, which we call by definition the partial derivative $\frac{\partial f}{\partial u(x)}$. In all these formulae we take the integrals over the set of definition of the function u: if we consider periodical functions—along the period, in the case of rapidly decreasing functions—along the real line.

In what follows we often interpret a Poisson structure as a bivector field, i.e., a section η of $\Lambda^2 TM$. Indeed, the Leibniz condition implies that $\{f,g\}|_x$ depends only on $df|_x$ and $dg|_x$, hence it can be written as

$$\langle \eta|_x, df \wedge dg|_x \rangle ,$$

for some $\eta \in \Gamma(\Lambda^2 TM)$. For example, $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ corresponds to a Poisson structure

$$(f,g) \mapsto \frac{\partial f}{\partial x} \cdot \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \cdot \frac{\partial f}{\partial y}.$$

Now the geometrical formulation of the problem in question is:

Problem 0.1. Find the normal forms for a manifold with two compatible Poisson structures.

Even in the particular case of the system (0.1) this problem seems to be intractable. We mean that it seems to be very difficult to determine when the local bihamiltonian structures in neighborhoods of two given points u_1, u_2 of the manifold M are diffeomorphic. We discussed some partial results and the related analytical complications in the paper [1].

However, the finite-dimensional variant is much simpler. In this case (as well as in the infinite-dimensional case, in fact, see [1, 3]) we should separate two cases: the case of an even-dimensional manifold, and the case of an odd-dimensional manifold. The reason for this is very simple: a generic skewsymmetric bilinear form on an even-dimensional space is non-degenerate, and visa versa in the odd-dimensional case. The generic approach to skew forms in an odd-dimensional vector space is to take a quotient by the kernel of the form. However, in the case of *a pair* of forms it is not generally possible, as shows the classification theorem from the section 3: the generic case of a pair of forms in an odd-dimensional space is undecomposable, hence it should be considered separately.

We separate here a simple and robust geometrical classification of bihamiltonian manifolds, that occupies the sections 1 and 2, and more subtle or more technical topics, which occupy the appendices in the sections 3–6. In turn, we separate the discussions of the even-dimensional and the odd-dimensional cases into sections 1 and 2 respectively.

In the even-dimensional case we proceed as following: in the section 1.1 we consider the trivial case of a 2-dimensional manifold. In the section 1.2 we state a particular simple case of the unfortunately-not-so-well-known theorem on linear algebra (the complete formulation of this theorem appears in the section 3). In the section 1.3 we, using the constructions from the previous two sections, decompose a simplest particular case of bihamiltonian manifold into a direct product of (canonically defined) 2-dimensional manifolds. To do this we introduce a notion of a *weak leaf* (that is a symplectic leaf of some linear combination of two hamiltonian structures) and associate to a point on $2n$-dimensional bihamiltonian manifold the n-tuple of weak leaves passing through this point. In the particular case we consider in this section the set of weak leaves breaks into n (2-dimensional) connected parts, and this n-tuple contains exactly one leaf from any parts, so we identify the manifold with the product of this n parts. In other words, in this case we can order this (generally speaking) unordered n-tuple.

In the section 1.4 we state a program of generalization of this construction to a less restricted case, where we cannot separate the set of weak leaves into n parts. Therefore instead of a direct product we need to consider a symmetric product, i.e., a quotient of a power of the set of weak leaves by the action of the symmetric group. In the section 1.5 we give the definition of a *regular* point on a bihamiltonian manifold, that is the main restriction on the the bihamiltonian structures we can classify. Under this restriction the set of weak leaves is a smooth 2-dimensional manifold with a canonically defined bihamiltonian structure. In the section 1.6 we introduce the notion of a *Hilbert scheme*, that is a particular substitute for a (singular) quotient by a symmetric group. We also define a bihamiltonian structure on the Hilbert scheme of a bihamiltonian surface. In fact up to this point we are concerned only with definitions of the objects we need, not with proving any meaningful theorem.

At last, in the concluding section 1.7 of the part 1 we combine the introduced earlier definitions and show that the natural mapping from the bihamiltonian manifold to the symmetrical power of the set of weak leaves can be lifted to the mapping from this manifold to the Hilbert scheme of the set of weak leaves. We show that this mapping is compatible with bihamiltonian structures, and is a diffeomorphism under very mild conditions. We formulate here a general theorem on the necessary conditions and a particular case when these conditions can be geometrized: the case of a regular point. In the latter part we use some flatness results, however we postpone the proof of these results until the section 4.

At this point we drop the discussion of the even-dimensional case and switch to the odd-dimensional case. This topic occupies the section 2. However, we continue this discussion in the sections 5 and 6, where we discuss more subtle properties of even-dimensional bihamiltonian manifolds.

As in the section 1, we begin the discussion of the odd-dimensional case with the corresponding part of the theorem on the linear algebra from the section 3. This occupies the section 2.1. In the section 2.2 we study the geometry of the set of weak leaves. As in the even-dimensional case, this set is 2-dimensional. However, now through a given point on the bihamiltonian manifold there passes a whole 1-parametric family of weak leaves. Therefore we can associate with a point of the bihamiltonian manifold a *rational curve* on the surface of weak leaves. Moreover, we show that a whole family of points corresponds to the same curve on this surface.

This motivates the definition of the Veronese web in the section 2.3. It is the result of gluing together the points on the bihamiltonian manifold that correspond to the same curve on the space of weak leaves. After taking this quotient we lose the information on the bihamiltonian structure, however we preserve the information on the relative position of weak leaves.[2]

Later, in the section 2.4, we show that we can reconstruct the initial bihamiltonian manifold (up to a diffeomorphism) basing on this structure on the corresponding Veronese web. This reconstruction uses the notion of the *subcotangent* bundle to a Veronese web, which is turn is constructed using basic methods in the theory of G-structures. In the same way as the cotangent bundle carries a natural Poisson structure, the subcotangent bundle carries a natural bihamiltonian structure. However, the construction of this structure is so complicated, that we omit it here. The reason for these complications is the absence of the corresponding notion of a symplectic manifold in this case, so anyone who tried once to construct a Poisson structure on the cotangent bundle without any reference to a symplectic structure would appreciate these difficulties.

There is yet another part of this theorem that we leave without a proof: the coincidence of the bihamiltonian structure on the subcotangent bundle to a Veronese web with the bihamiltonian structure we got this web from. However, this proof uses so unusual cohomological construction that we do allow ourselves to present it there with several loose ends. What remained is the definition of the *double cohomology* in the section 2.5.

In the following section 2.6 we harvest the fruits of living in the category of analytic manifolds: we exploit a notion of the *twistor transform.* To explain this notion let us consider a family of submanifolds $B_\gamma \subset B$ parameterized by a manifold $\Gamma \ni \gamma$. We can consider the universal family $A = \{(b, \gamma) \mid b \in B_\gamma\} \subset B \times \Gamma$ and two projections

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ B & & \Gamma. \end{array}$$

We can see that the notion of a family of submanifolds is a particular case of a notion of a double bundle. However, in the latter notation B and Γ appear in the same form, so we can interchange them and instead of a family of submanifolds in B parameterized by Γ we can consider a family of submanifolds in Γ parameterized

[2]It is the place to note that glued together points on the bihamiltonian manifold have the same *action* variables, but different *angle* variables. Here we use the usual in the theory of integrable systems distinction between two sets of coordinate functions: the former can be obtained in a simple geometrical way (compare with *local Hamiltonians* and the *Magri—Lenard scheme*), the later demand some additional work. In the case of bihamiltonian manifolds the Lenard scheme gives a natural way to construct the action variables. We should note that in the odd-dimensional case the number of angle variables is one less than the number of action variables.

We see here that the Veronese web reflects the geometry of the action variables. The implications of the classification theorem for the Veronese webs in the theory of integrable systems should be found elsewhere. In particular, it is possible to show the existence of a global coordinate system for the open Toda lattice such that two Poisson structures become translation-invariant. That implies, in particular, the existence of a $\frac{n(n-1)}{2}$-hamiltonian structure on the Toda lattice with n points.

by B. To a point $b \in B$ we associate a submanifold $\Gamma_b \subset \Gamma$ consisting of passing through b members of the former family.

This is still not a definition of a twistor transform, however we call a family Γ_b a twistor transform of a family B_γ if the geometry of the former family is somewhat simpler than the geometry of the latter. This is the case for Veronese webs. By definition a Veronese web is a family of hypersurfaces, so we can consider the twistor transform, that is a family of rational curves on a surface. Moreover, we show that this family is in fact a *complete family*: it contains any deformation of any curve in this family. That means that we need not specify the family: it is determined by the geometry of the surface itself. Therefore *to determine a Veronese web it is sufficient to provide a surface with a rational curve that allows a deformation.*

In the section 2.7 we discuss shortly the consequences of the twistor transform in the theory of bihamiltonian manifolds and Veronese webs, like the amount of non-diffeomorphic bihamiltonian systems and the fusion operations over Veronese webs. At last, in the sections 2.8 and 2.9 we consider the simplest non-trivial cases: 2-dimensional Veronese webs and 3-dimensional bihamiltonian manifolds.

At this point we leave the discussion of basic geometrical questions and delve into details and technicalities. As we have already mentioned, in the section 3 we state the general theorem on classification of pairs of linear mappings or bilinear forms. Even if technical, this theorem concerns the utterly basic notions that by strange reasons are dropped in the basic mathematical education. Though we state the necessary particular cases of this theorem in the places we use them, we strongly recommend the reader *to begin* with reading this appendix. Furthermore, we want to note that this theorem is a keystone in our approach to the concerned here problems, and a generalization of this theorem to the infinite-dimensional case of the forms (0.7) associated with the KdV equation was the challenge that eventually piloted us to the questions concerned here.

In the section 4 we provide the missed link in the proof of the existence of the mapping to the Hilbert scheme.

In the section 5 we consider the global geometry of an even-dimensional bihamiltonian manifold. We construct here examples of manifolds such that the topology itself defines a polyhamiltonian structure on these manifolds. Moreover, as we will see in the section 6.4, these manifolds give as example of compact manifolds such that the weak classification theorem is applicable to them in any point.

In the final section 6 we exploit the classification theorem to investigate the local geometry of a bihamiltonian manifold. We begin the section 6.1 with an example of the case when the weak classification theorem is applicable, but the strong one is not. It is an example of a non-regular point on the Hilbert scheme $S^2\mathbb{A}^2$, and we make the preparations to find the set of regular point on a generic Hilbert scheme. We give a description of the tangent space to a Hilbert scheme, but we fail to describe a bivector that corresponds to a Poisson structure. However, we can give a formula for such a bivector if we know the bivector that corresponds to some other Poisson structure.

This information is sufficient to describe the *recursion operator* of the bihamiltonian structure. We do it in the section 6.2.

In the section 6.3 we use these results to give the description of the subset of regular points on a Hilbert scheme. Here we also describe generalized weak leaves, that allows us to show in the section 6.4 that the weak classification theorem is applicable in any point of a Hilbert scheme. We also give here an example of a weak leaf with a singular closure and study a tangent cone to this closure.

In the section 6.5 we introduce a natural identification of a neighborhood of a regular point on a Hilbert scheme with an open subset in the cotangent bundle to polynomials of one variable. This determines another Poisson structure on this bundle, and it has polynomial coefficients. This identification shows also that the conditions of the Magri Classification theorem for bihamiltonian structures [6] are satisfied in this case, that shows essentially that the strong classification theorem and the Magri classification theorem are in fact applicable in absolutely the same cases (if we modify the Magri theorem a little). However, we want to note that this equivalence is obtained by providing an isomorphism of *models* of bihamiltonian manifolds, not by comparing the conditions of these theorems—they remain still absolutely unrelated.

While the Magri coordinate system gives a model in which one Poisson structure is constant and another polynomial, in the section 6.6 we introduce another coordinate system on the Hilbert scheme. In this coordinate system the first Poisson structure remains constant, but the second structure becomes *linear*. That means that the geometry in a neighborhood of a regular point is connected with a particular (finite-dimensional) Lie algebra with two cocycles. This algebra is described in the final section 6.7. Let us note that the knowledge that a bihamiltonian structure can be written in such a form could allow one to find the corresponding classification theorems by solving some standard problems of linear algebra, like: *find all the possible structure constants for a Li algebra structure for which two given bilinear forms are cocycles.* We solve the simplest non-trivial particular case of this problem in the section 6.8. The solution provides us with a rich set of examples of 4-dimensional bihamiltonian systems. In two of these examples the *set of weak leaves* $M^{(2)}$ *is non-smooth!* This example shows that we cannot drop one condition on the weak classification theorem. However, in the same section we show that in this particular case the theorem remains true if we consider a *normalization* of the Hilbert scheme instead of the Hilbert scheme itself. That gives a hope to generalize the theorem on this direction.

We want to emphasize here the unexpected similarity between the even-dimensional and odd-dimensional cases: in both cases we can reconstruct the initial bihamiltonian manifold basing on the surface of weak leaves. And in both cases the reconstruction involves taking the moduli spaces of submanifolds on this surface, though the dimensions of these submanifold are different.

We are indebted to a lot of people for fruitful discussions and inestimable help, among them A. Givental, A. Goncharov, M. Kontsevich, F. Magri, A. Radul, N. Wallach and A. Weinstein. We should express special thanks to Henry McKean who provided us with his variant of the Magri Classification theorem [7], which inspired the discussions of the even-dimensional case here, and to Vera Serganova, whose patient remarks allowed this paper to acquire its current form. The idea to use the twistor transform for the local classification of the Veronese webs is due to A. Goncharov [4].

Anywhere in this paper (if not stated otherwise) we consider analytic manifolds. However, a lot of results can be easily translated to the C^{∞}-case.

1. The even-dimensional case

1.1. A 2-dimensional example.

Example 1.1. Let us try to classify 2-dimensional bihamiltonian systems in general position. In dimension two *any* bivector field corresponds to a Poisson structure, so we should simply classify pairs of bivector fields. We can suppose that at the given point the bivector corresponding to the first Poisson structure is non-degenerate. That means that in a neighborhood of this point this Poisson structure is "an inverse" of a symplectic structure. We can chose a local coordinate system (x_1, x_2) such that this symplectic structure can be written as $dx_1 \wedge dx_2$, hence the Poisson structure can be written as $\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$.

Now we can consider a ratio of our bivector fields, which is a function on M. If the given point is not a critical point of this function, then we can chose it as the first coordinate x_1 and can still find another function x_2 such that the first Poisson structure is $\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$. Hence the second Poisson structure is $x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$.

We see that in 2-dimensional case there is essentially *one* bihamiltonian manifold, and any manifold in general position is locally isomorphic to *some* part of this manifold (that is a small difference with d'Harboux theorem, where any symplectic manifold is isomorphic to *any* part of the fixed manifold).

In the case of not general position the classification is reduced to (well-known) problem of local classification of a function on a symplectic manifold.

This example seems to be trivial, however it implies a powerful construction in a multidimensional case: if M_1 and M_2 are two bihamiltonian manifolds, then $M_1 \times M_2$ is also a bihamiltonian manifold. In this way using the example above we can construct a bihamiltonian manifold of any even dimension. Now we can ask if the universality property of 2-dimensional example is still true in this case. The answer is the following particular case of the Turiel theorem [8]:

Theorem 1.2. *Consider a point m on an $2n$-dimensional bihamiltonian manifold M. If a couple of bivectors $\eta_1|_m, \eta_2|_m \in \Lambda^2 T_m M$ is in general position, then in a neighborhood of m it is possible to choose a coordinate system in such a way that* *the bivector fields are*

$$\eta_1 = \sum_{k=1}^{n} f_k\left(x_{2k-1}, x_{2k}\right) \frac{\partial}{\partial x_{2k-1}} \wedge \frac{\partial}{\partial x_{2k}}, \quad \eta_2 = \sum_{k=1}^{n} g_k\left(x_{2k-1}, x_{2k}\right) \frac{\partial}{\partial x_{2k-1}} \wedge \frac{\partial}{\partial x_{2k}}.$$

Therefore our bihamiltonian system is represented as a product of 2-dimensional ones.

It is easy to see that if not only the values, but also the derivatives of the bivector fields η_1, η_2 *at the point* m *are in general position, we can set*

$$\eta_1 = \sum_{k=1}^{n} \frac{\partial}{\partial x_{2k-1}} \wedge \frac{\partial}{\partial x_{2k}}, \quad \eta_2 = \sum_{k=1}^{n} x_{2k-1} \frac{\partial}{\partial x_{2k-1}} \wedge \frac{\partial}{\partial x_{2k}},$$

if we allow the point m *to correspond to an arbitrary point* (x_i) *and not necessary to the origin* $(x_i) = 0$.

Hence any generic bihamiltonian manifold is locally isomorphic to a product of 2-dimensional manifolds from the example.

In fact the Turiel theorem is much more powerful (it handles also some cases of not general position), however, in the following discussion we need only this particular case. It is also quite easy to prove this case (as well as the Turiel theorem in whole) using the general arguments of linear algebra.

1.2. **A theorem from linear algebra.** Indeed, the linear algebra says that a pair of skewsymmetric bilinear forms (α, β) in general position in an even-dimensional vector space V can be canonically decomposed in a direct sum of pairs of forms in 2-dimensional spaces:

$$V = \bigoplus_i V_i, \qquad \alpha = \bigoplus_i \alpha_i, \quad \beta = \bigoplus_i \beta_i,$$

where (α_i, β_i) is a pair of skewsymmetric bilinear forms in V_i (see, for example, [1]). It also says that the forms in these subspaces are proportional with some coefficients λ_i (we call them *eigenvalues*). The exact formulation of this theorem can be found below, in the appendix 3.

1.3. **A map to the set of weak leaves.** Let us apply this argument to the space $V = T_m^* M$ and to the pair of forms $\eta_1|_m$, $\eta_2|_m$ in this space. We see that the cotangent space is canonically decomposed into a direct sum of 2-dimensional subspaces. That means that the tangent space is also decomposed into a direct sum of 2-dimensional subspaces. The same is evidently true at nearby points. Hence there are n canonically defined distributions of 2-dimensional subspaces in the tangent bundle of M and n functions λ_i on M.

The next step is to prove that these distributions are integrable, i.e., are the tangent bundles to some foliations. To do this we should do the following: given a point and a fixed i-th family of subspaces we should find a 2-dimensional submanifold satisfying the following properties:

(1) it passes through the given point in the direction of the 2-dimensional subspace of the family of subspaces;
(2) the tangent space at any point of this submanifold is a subspace from the famimly.

Again, to do this it is sufficient to find a submanifold of *codimension* 2 such that the tangent space at any point of it is a sum of $n-1$ marked subspaces. Now fix a point $m \in M$ and $1 \leq i \leq n$. The bivector $\eta_1|_m - \lambda_i(m)\,\eta_2|_m$ considered as a bilinear form in the cotangent space $T^*_m M$ has a 2-dimensional kernel by the definition of λ_i. Now we need (the first time) some information of the geometric structure of a Poisson manifold.

Definition 1.3. (1) A submanifold L of a symplectic manifold M is called a *Poisson submanifold* if the restriction of $\{f, g\}$ on L is uniquely determined by the restrictions of the functions f and g on L. In this case this restriction determines a Poisson structure on L.
(2) A *symplectic leaf* in a Poisson manifold M is an imbedded Poisson submanifold L of M such that the corresponding Poisson structure on L is *nondegenerate*, i.e., corresponds to some symplectic form ω on L.[3]

Well-known theorem on the local structure of a symplectic manifold [9] claims in particular that:

Theorem 1.4. *There exists a unique symplectic leaf passing through any given point m of a Poisson manifold M. The normal space to this leaf at m coincides with the kernel of the bivector $\eta|_m \in \Lambda^2 T_m M$ considered as a bilinear form in the space $T^*_m M$.*

Let us apply this theorem to the Poisson structure $\eta_1 - \lambda_0\eta_2$ on M, where $\lambda_0 = \lambda_i(m)$. The symplectic leaf $L_{m,i}$ passing through m has a desired tangent space at m, moreover, the normal space to it at any point m' of this leaf is the kernel of $\eta_1 - \lambda_0\eta_2$. That means that $\lambda_i|_{L_{m,i}} = \lambda_0 = \lambda_i(m)$. Now we can construct the desired 2-dimensional submanifold $\widetilde{L}_{m,i}$ as an intersection of $n-1$ submanifolds $L_{m,j}$ for $j \neq i$. We can also use the constructed foliation of codimension 2 to define a local projection π_i of M to a local base M_i of this foliation.

Definition 1.5. Let us call any symplectic leaf (of non-vanishing codimension) of any linear combination $\eta_1 - \lambda\eta_2$ a *weak leaf* of the bihamiltonian structure.

It is easy to see that in our case the set of parameters of weak leaves $M^{(2)}$ is 2-dimensional and is (locally on M) a disjoint union of n submanifolds M_i corresponding to (different) eigenvalues of the pair (η_1, η_2) at the point m.

Remark 1.6. In this case we can *define* $M^{(2)}$ as the union of local bases for foliations $\{L_i\}$. However, since this space plays a crucial rôle in what follows we want to emphasize the fact that in general case this (topological) space isn't even Hausdorff.

[3]In fact, since an open subset of a symplectic leaf is again a symplectic leaf, in what follows we restrict this definition and call as a symplectic leaf only the *maximal* submanifolds with the specified properties.

If we move a point m on a manifold the pair of forms in T_m^*M changes and a pair of eigenvalues can collide. If they do it in a "civilized manner", this results in a Jordan block of the corresponding matrix (see appendix) and the dimension of the kernel does not change (this is exactly the case we are interested in below). However, a collision of eigenvalues can also result in an eigenspace of greater dimension, i.e., in a weak leaf of different dimension.

However, in the cases we study below *the space of weak leaves* $M^{(2)}$ is a smooth manifold, so we have no such complications.

Combining the above n projections we get a local identification of M and a product of n 2-dimensional manifolds M_i. What is remaining to prove is that the bivector fields are products of some bivector fields on the factors. It is sufficient to consider *one* bivector field, say, $\eta = \eta_1$. Moreover, we can suppose that the Poisson structure η is non-degenerate, since we can consider two non-degenerate linear combinations of η_1, η_2 instead of considering η_1, η_2.

What follows from the choice of the projections is that for any fixed point (say, m) on M we can represent *values* of the bivector fields (i.e., a pair of bivectors) as two products of bivectors on the factors:

$$\eta|_m = \bigoplus_i \eta_i(m), \quad \eta_i(m) \in \Lambda^2 T_{\pi_i(m)} M_i, \quad T_m M = \bigoplus_i T_{\pi_i(m)} M_i.$$

If we denote the coordinates on M_i by x_i, y_i, we can express this fact by the formula

$$\eta(x_1, y_1, \dots, x_n, y_n) = \sum_i \widetilde{\eta}_i(x, y) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}. \tag{1.5}$$

What we should prove is that the $\widetilde{\eta}_i(x, y)$ depends only on x_i, y_i, i.e., that for any two point m', m'' on $L_{m,i} = \pi_i^{-1}(\pi_i(m))$ the i-components of η are the same. That means that for any two functions φ_1, φ_2 on M that depend only on x_i, y_i the Poisson bracket also depends only on x_i, y_i.

Remark 1.7. Let us note that this property is specific to the symplectic geometry. In, say, the Riemannian case the coordinate foliations ($x_i = \text{const}$) can be orthogonal with respect to a Riemannian form that cannot be represented as a direct product.

To prove this property we will use the second theorem on the geometry of a Poisson manifold.

Theorem 1.8. (1) *The Hamiltonian vector field V_f corresponding to a function f on a Poisson manifold M preserves the Poisson structure.*
(2) *The* fundamental relation *links the operations of commutation of vector fields and the Poisson bracket on functions:*

$$V_{\{f,g\}} = [V_f, V_g].$$

Now to prove this fact it is sufficient to construct sufficiently many vector fields on M that preserve the Poisson structure and are tangent to fibers of the projection π_i. These fibers are spanned by the Hamiltonian flows corresponding to functions that depend only on x_j, y_j, $j \neq i$ (by the property (1.5)). However, we can find a function $f(x_j, y_j)$, $j \neq i$, such that the Hamiltonian flows corresponding to the this function moves the given point in an arbitrary direction tangent to a fiber of π_i. This finishes the proof of the fact that either bivector field is a product of fields on specified 2-dimensional manifolds.

Remark 1.9. We can write down the last arguments of the proof with formulae

$$\{x_j, \{\varphi(x_i, y_i), \psi(x_i, y_i)\}\} = \{\{x_j, \varphi(x_i, y_i)\}, \psi(x_i, y_i)\} + \{\varphi(x_i, y_i), \{x_j, \psi(x_i, y_i)\}\} = 0,$$

if $i \neq j$, and the same with a change of x_j to y_j (application of (1.5)). This means that $\{\varphi(x_i, y_i), \psi(x_i, y_i)\}$ is preserved by Hamiltonian flows corresponding to x_j, y_j, and, therefore, is constant along the fibers of the projection π_i. Here the non-degeneracy of η guaranties that these Hamiltonian flows span a whole fiber.

Remark 1.10. Now it is easy to see that any local automorphism of a bihamiltonian manifold in general position is coming from n diffeomorphisms of 2-dimensional factors M_i. In particular, any vector field that is Hamiltonian with respect to both Poisson structures is a product of such fields on the factors. So it is again sufficient to consider the 2-dimensional case, where such a field should preserve the ratio of the Poisson structures. Hence the Hamiltonians of this vector field are constant on the level lines of the ratio. From the other side, it is easy to see that the Hamiltonian flow of such a function is Hamiltonian with respect to any non-degenerate combination of the Poisson structures in question.

Therefore *a vector field that is hamiltonian with respect to both Poisson structures is Hamiltonian with respect to any nondegenerate linear combination of them and the Hamiltonian H of this field (with respect to any of the Poisson structures) is a sum*

$$H = H_1 + H_2 + \cdots + H_n$$

of functions H_i such that either H_i depends (maybe, multivalued) on the eigenvalue λ_i, or λ_i is constant and then H_i is constant on the i-th family of weak leaves.

Remark 1.11. In fact we have constructed a mapping

$$M \to M_1 \times M_2 \times \cdots \times M_n \hookrightarrow M^{(2)} \times M^{(2)} \times \cdots \times M^{(2)},$$

and to do this we have fixed an order of eigenvalues $\{\lambda_i\}$. If we do not fix this order, we can still construct a mapping to a factor by the action of the symmetrical group

$$M \to \underbrace{M^{(2)} \times M^{(2)} \times \cdots \times M^{(2)}}_{n \text{ times}} / \mathfrak{S}_n.$$

In fact the above considerations show that on $M^{(2)}$ there is a natural bihamiltonian structure, and if we consider a cross-product bihamiltonian structure on $M^{(2)} \times M^{(2)} \times \cdots \times M^{(2)}$, then the former mapping is a local isomorphism of Poisson manifolds. From the other hand, the taking of the quotient by $\mathfrak{S}_n$ behaves well with respect to the Poisson structure on $M^{(2)} \times M^{(2)} \times \cdots \times M^{(2)}$, and the image of the latter mapping consists of *smooth* points on $\underbrace{M^{(2)} \times M^{(2)} \times \cdots \times M^{(2)}}_{n \text{ times}} / \mathfrak{S}_n$, therefore the latter mapping is also a local isomorphism of Poisson manifolds.

We can conclude now that any (local) even-dimensional bihamiltonian manifold in general position is (locally) isomorphic to some domain in a fixed bihamiltonian manifold. Hence to specify a local diffeomorphism type of a bihamiltonian manifold in a neighborhood of a given point it is sufficient to specify a point on that manifold, i.e. a finite number of parameters.

1.4. **A case with non-trivial monodromy.** Let us consider now a local bihamiltonian manifold such that the condition of the theorem 1.2 is not satisfied. If the manifold is in sufficiently "general position" we can expect that the points where this condition *is* satisfied form a dense open subset. Let us call a point *good* if the condition of the theorem 1.2 is satisfied at this point. Let us call the subset of such points U (evidently, this subset is open).

In this case we cannot be sure that the set $U^{(2)}$ of weak leaves in U is a disjoint union of n components and that though any point of U passes a leaf from any component of $U^{(2)}$. It is possible to say which weak leaf passing through a point x' (from a vicinity of a given point $x \in U$) corresponds to a given weak leaf passing through x, but the monodromy along a closed loop in U can interchange weak leaves passing through x. However, the map

$$U \xrightarrow{\alpha} \underbrace{U^{(2)} \times U^{(2)} \times \cdots \times U^{(2)}}_{n \text{ times}} / \mathfrak{S}_n$$

is still well-defined and is a local isomorphism of Poisson manifolds.

Magri in his article [6] has shown that sometimes it is possible to classify bihamiltonian systems even in a neighborhood of a non-good point. In the case of general position which he has studied the manifold is again locally isomorphic to a piece of *one particular* bihamiltonian manifold given by explicit formulae. Here we want to give a geometrical reason for such a phenomenon, and this reason is that under mild conditions the mapping α can be extended to a mapping

$$M \xrightarrow{\alpha} \underbrace{M^{(2)} \times M^{(2)} \times \cdots \times M^{(2)}}_{n \text{ times}} / \mathfrak{S}_n, \tag{1.11}$$

and (if we modify slightly the definition of taking the quotient by $\mathfrak{S}_n$ to obtain a smooth quotient) this map is a local isomorphism. Since in the case of general position $M^{(2)}$ is a piece of *one particular* bihamiltonian plane $\left(\mathbb{A}^2, \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}\right)$, the manifold in the right-hand side of the formula (1.11) is a piece of *one particular* bihamiltonian manifold.

Here we show that conditions under which the map (1.11) exists can be made much more mild than the Magri conditions, hence we obtain here a (minor) *generalization* of the Magri result. These milder conditions are, however, not very constructive, so we give also stronger conditions of geometrical nature. These geometrical conditions are of very different origin from the Magri conditions, and the resulting description is of a bihamiltonian structures is absolutely different. We will close this gap in the concluding the paper appendix, where we show that our geometrical conditions (practically) coincide with the Magri conditions![4]

So the main goal of the discussion of the even-dimensional bihamiltonian systems is not to give a generalization of the Magri construction, but to find a geometrical framework allowing to construct a *canonical* identification of the bihamiltonian manifold with a basic example of such a manifold.

Now we want to list the conditions under which the mapping is well-defined and well-behaved.[5] First of all we want to list the obstruction for this mapping to exist.

We want to repeat it again that the variety $M^{(2)}$ in the general case can be non-smooth and even have "components" of different dimensions. If we live in a vicinity of a good point then the mapping

$$M \xrightarrow{\alpha} \underbrace{M^{(2)} \times M^{(2)} \times \cdots \times M^{(2)}}_{n \text{ times}} / \mathfrak{S}_n$$

sends a point of M to a point of the quotient that corresponds to an n-tuple of different points of M. In such points the smooth structure on the quotient is well-defined. However, if we move to the *boundary* of the good set U the eigenvalues *collide* and to a point of M can correspond an n-tuple of points of $M^{(2)}$ where some points can appear with some *multiplicity*. It is known (see the example below) that the variety $\underbrace{M^{(2)} \times M^{(2)} \times \cdots \times M^{(2)}}_{n \text{ times}} / \mathfrak{S}_n$ is *singular* at such points. Therefore we need some *desingularization* $S^n M^{(2)}$ of this variety, and this desingularization should be sufficiently small[6] to extend the Poisson structure from smooth points of $\underbrace{M^{(2)} \times M^{(2)} \times \cdots \times M^{(2)}}_{n \text{ times}} / \mathfrak{S}_n$. If this desingularization *is* sufficiently small we can expect that the map α can be raised to a map $M \to S^n M^{(2)}$, and we need this map to be a local diffeomorphism. Now *the result of Magri shows that in proper cases this program can be fulfilled*! Here we just list the "algebraic nonsense" that allows to fulfill it under very mild conditions.

Let us list here the assumptions we need to make the Magri result "*functorial*":

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[4]Frankly speaking, we cannot give a direct proof of this fact, we just show that his *canonical* form satisfies our conditions, and our canonical form (almost always) satisfies his conditions. That means that we use power of *both* classification theorems to show that the conditions coincide!

[5]In fact there is a big confidence that a suitable *algebraization* of the following discussion can help in weakening the conditions we specify, however, we want here to use a *synthetic* language and work with smooth manifolds wherever it is possible.

[6]If we consider a non-degenerate bivector field on a manifold M and a blow-up $\widetilde{M}$ of this manifold in some submanifold N, then the corresponding bivector field on $\widetilde{M}$ has a pole on the preimage of N. Therefore we cannot make any additional blow-ups on the manifold we want to construct.

(1) We need the weak leaves to have a good parameter space $M^{(2)}$ on the whole M;
(2) We need a good desingularization $S^n M^{(2)}$ of $\underbrace{M^{(2)} \times M^{(2)} \times \cdots \times M^{(2)}}_{n \text{ times}} / \mathfrak{S}_n$;
(3) We need a map $M \to S^n M^{(2)}$;
(4) We need a bihamiltonian structure on $S^n M^{(2)}$.

1.5. **A case of a regular point.** If we consider a bihamiltonian manifold in general position, then the points where the above analysis is applicable form a dense open subset. In fact the theorem on linear algebra from the appendix defines *the eigenvalues* even outside of this subset. However, we can define the eigenvalues much more simple. Indeed, a bivector field η determines a mapping $\widetilde{\eta}_m \colon T_m^* M \to T_m M$ for any point $m \in M$, and eigenvalues in question are just eigenvalues[7] of the *recursion* map

$$\widetilde{\eta}_{1,m}^{-1} \widetilde{\eta}_{2,m} \colon T_m^* M \to T_m^* M$$

that is defined anywhere where η_1 is non-degenerate. Hence the complement to this dense subset consists of points where the eigenvalues collide.

Let us consider the first question first. A passing through $m \in M$ weak leaf corresponds to a kernel of a bilinear form $\eta_1 - \lambda \eta_2$ on the space $T_m^* M$ (since, say, the theorem on a local structure of a Poisson manifold [9] applied to $\eta_1 - \lambda \eta_2$ shows that there is a weak leaf with this kernel as a normal space). So to have a good parameter space of weak leaves we need at least the leaves to have the same dimension, i.e., any linear combination $\eta_1 - \lambda \eta_2$ to have at any point $m \in M$ at most 2-dimensional kernel (that guaranties that any weak leaf is of codimension 2).[8]

The theorem on the structure of a pair of skewsymmetric bilinear forms [1] (or see in the appendix on linear algebra) shows that in this case the corresponding pair of linear mappings has only one Jordan block for any eigenvalue. It is clear that the set of pairs satisfying this condition is open and that the *stabilizer* of any such pair has the same dimension as the stabilizer of a pair in general position. Therefore it is a closest generalization of the notion of *a pair in general position.*

Definition 1.12. Let us call a pair of skew-symmetric bilinear forms in a vector space V a *regular pair*, if the stabilizer of this pair in $\mathrm{GL}(V)$ has minimal possible dimension.

Let us call a point m of bihamiltonian manifold M a *regular point*, if the corresponding pair of bilinear forms in $T_m^* M$ is a regular pair.

So a regular pair in an even-dimensional space corresponds to a pair of mappings that has exactly one Jordan block for any eigenvalue and no Kroneker blocks at all.

[7]The theorem on linear algebra shows that to any eigenvalue of a pair of forms η_1, η_2 in the above sense corresponds a *double* eigenvalue of the recursion operator.

[8]However, below we give a definnition of a generalized weak leaf that allows to drop this restriction.

It is clear that the set of regular points is open, hence any weak leaf passing through a vicinity of a regular point is of codimension 2. If any leaf intersects with the set of good points, then space of weak leaves is smooth in the corresponding point.[9] Since the eigenvalue λ is constant on a leaf, to satisfy this condition it is sufficient to demand that the set of not-good points with a given eigenvalue is of codimension at least 3.

Remark 1.13. Consider the two defined above Poisson brackets on an open subset of the set of weak leaves. If any weak leaf intersects the set of good points, then these Poisson brackets can obviously be extended to the whole space of weak leaves. It would be very interesting to understand if this fact is true in the general case (including the generalization on the case of generalized weak leaves). Compare the theorem 1.23.

1.6. **A good symmetrical power.** So the first condition is explained. What is the meaning of the second condition? The problem with a definition of $S^n M^{(2)}$ is that the quotient by an action of a group can be singular.

Example 1.14. Let us consider the action of $\mathbb{Z}_2$ on a plane (x, y) by reflection $(x, y) \mapsto (-x, -y)$. The basic invariant functions are

$$a = x^2,\ b = xy,\ c = y^2,$$

they satisfy the constraint

$$ac = b^2$$

that determines a cone in the space (a, b, c). Therefore the quotient of the plane by the action of $\mathbb{Z}_2$ is a cone.

Example 1.15. The previous example is an (antisymmetrical) component of the action of $\mathbb{Z}_2 = \mathfrak{S}_2$ on the product of the plane by itself by interchanging the factors, so it suits the situation with the symmetrical power well.

Let M be a two-dimensional Poisson manifold. Suppose first that in a vicinity of a given point the Poisson structure is nondegenerate. Then we can choose local coordinates X, Y such that the structure is $\frac{\partial}{\partial X} \wedge \frac{\partial}{\partial Y}$. On $M \times M$ we can consider the coordinates X_1, Y_1, X_2, Y_2, or $\xi = \frac{1}{\sqrt{2}}(X_1 + X_2)$, $\eta = \frac{1}{\sqrt{2}}(Y_1 + Y_2)$, $x = \frac{1}{\sqrt{2}}(X_1 - X_2)$, $y = \frac{1}{\sqrt{2}}(Y_1 - Y_2)$. The cross-product Poisson structure can be written as

$$\frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial \eta} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The functions on $M \times M/\mathfrak{S}_2$ are generated by ξ, η, and a, b, c (as above), hence the quotient is a product of a plane and a cone. Let us consider a blow-up of this manifold in the singular stratum, i.e., in the product of the plane and the vertex of the cone.

[9] In this case the parameter space of weak leaves does not change if we consider only the space of good points, and the parameter space *is* smooth in the latter case.

Since all the structures (including the Poisson) are cross-product structures, it is sufficient to consider the blow-up of a cone in its vertex.

Example 1.16. Let us consider in the situation of the example 1.14 the Poisson structure $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ on the plane (x, y). Since the Poisson bracket of two $\mathbb{Z}_2$-invariant functions is again $\mathbb{Z}_2$-invariant, we can consider the corresponding bivector field on the smooth part of the quotient-cone K. Let $\widetilde{K}$ be a blow-up of this cone in its vertex. Then on the open part of $\widetilde{K}$ a bivector field is defined. We claim that this bivector field extends to the whole $\widetilde{K}$ without singularity, and that the corresponding Poisson structure on $\widetilde{K}$ is non-degenerate.

Indeed, in a local coordinate frame (α, β) on $\widetilde{K}$, where $\alpha = y/x$, $\beta = x^2$, the corresponding 2-form $dx \wedge dy$ can be written as $-\frac{1}{2} d\alpha \wedge d\beta$, therefore

$$\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = -2 \frac{\partial}{\partial \alpha} \wedge \frac{\partial}{\partial \beta}.$$

Remark 1.17. Therefore in the situation of the example 1.15 on the (smooth) blow-up of $M \times M/\mathfrak{S}_2$ in the vertex a non-degenerate Poisson structure is defined. If the original Poisson structure on M was degenerate, we can represent it as a difference of two non-degenerate bivector fields. Both these fields can be raised to $M \times M/\mathfrak{S}_2$ without singularity. Since the correspondence between bivector fields on M and on $M \times M/\mathfrak{S}_2$ is linear, the raising of the original Poisson structures is a difference of two non-singular bivector fields, and therefore is also non-singular.

Definition 1.18. Let M be 2-dimensional manifold. Let us call the blow-up of $M \times M/\mathfrak{S}_2$ in the singular stratum *a symmetric square S^2M of M*. The above considerations show that if M is equipped with a Poisson or symplectic structure, then S^2M is also equipped with a Poisson or symplectic structure.

It is known that to define a "good" notion of a symmetrical power (anywhere beyond the notion of the symmetrical square) is difficult. However, in the case of symmetrical power of 2-dimensional manifold the notion of *the Hilbert scheme* is sufficient in many cases.

Let us remind that a *desingularization* of a given variety X is a manifold X' with a mapping $\pi\colon X' \to X$ such that π is an isomorphism over the open subset U of smooth points of X. One of the ways to describe a desingularization is to demonstrate an inclusion of U into some variety in such a way that the closure $X' = \overline{U}$ of the image of U is smooth. In this case we should yet show the existence of the mapping $X' \to X$, but usually it is not difficult. We want to describe the Hilbert scheme of a manifold M as a desingularization of the manifold $\underbrace{M \times M \times \cdots \times M}_{n \text{ times}}/\mathfrak{S}_n$.

Theorem 1.19. *Let us associate to an n-tuple of different points on a smooth two-dimensional manifold M a vector space I of functions that vanish at this points. Let V be the image of this map in the Grassmannian* Gr *of subspaces of codimension n in the ring A of functions on M. Then*

(1) *The closure* $\overline{V}$ *of* V *is a smooth subvariety of* Gr*;*
(2) *Points of the manifold* $\overline{V}$ *are exactly ideals of codimension* n*;*
(3) *The only component of codimension 1 of* $\overline{V} \smallsetminus V \subset \overline{V}$ *consists of ideals with support in* $n-1$ *points, i.e., to collision of only two points on* M*;*
(4) *The tangent space to* $\overline{V}$ *at an ideal* $I \in \overline{V}$ *is* $\operatorname{Hom}_A(I, A/I) \subset \operatorname{Hom}_{\mathbb{C}}(I, A/I)$.
(5) *If* $n = 2$, *then* $\overline{V}$ *is a blow-up of* $M \times M/\mathfrak{S}_2$ *in the singular stratum.*

This manifold is called *a Hilbert scheme* of M (on the level n). Now, if M is equipped with a Poisson structure, then on the open subset V of the Hilbert scheme a cross-product bivector field is defined. As we have seen, this bivector field has no singularity on $\overline{V} \smallsetminus V$ in the case $n = 2$, hence it has no singularity on the component of codimension 1 also in the case of arbitrary n. However, the Hartogs theorem claims that if a function has no singularity outside of a subset of codimension > 1, then it has an extension without singularity. Therefore the Poisson structure on V has an extension to the whole $\overline{V}$.

If the Poisson structure on M is non-degenerate, then the same discussion applied to the corresponding symplectic form shows that the symplectic form can be extended to the whole $\overline{V}$ without singularity. Therefore the Poisson structure on $\overline{V}$ is non-degenerate.

Remark 1.20. The above definition of the Hilbert scheme is applicable only in the case when the manifold M is affine, so the ring of functions on M is sufficiently reach. In the other case we should just consider *subsheaves* $\mathcal{I}$ of the sheaf $\mathcal{O}$ instead of ideals in $\mathcal{O}(M)$, and $\dim \Gamma(\mathcal{O}/\mathcal{I})$ instead of $\dim A/I$. However, we will abuse the notations and will work with the Hilbert scheme as if it consists of ideals even in the case of projective M.

Corollary 1.21. *It is possible to define a notion of* a symmetric power $S^n M$ *of a 2-dimensional Poisson manifold* M, *that is also a Poisson manifold. To do this it is sufficient to consider the Hilbert scheme of* M. *If the Poisson structure on* M *is non-degenerate, the corresponding Poisson structure on* $S^n M$ *is non-degenerate too. Since the correspondence between these Poisson structures is linear, the symmetrical power of a bihamiltonian manifold is a bihamiltonian manifold.*

1.7. **A mapping to the symmetrical power.** Now basing on a local $2n$-dimensional bihamiltonian manifold M in a vicinity of a regular point we have constructed a bihamiltonian manifold $S^n M^{(2)}$ and a mapping from the subset of good points on M into this bihamiltonian manifold. This mapping preserves pairs of Poisson structures. What we want to do now is to show that we can extend this mapping to the whole M.

In fact such an extension should associate an ideal of codimension n on $M^{(2)}$ to any point $m \in M$. Let us construct this mapping on the set of good points of M. If we do all constructions algebraically, then we will be able to apply them in the case of an arbitrary point of M.

Lemma 1.22. *Let m is a good point on M. Let $C \subset M \times M^{(2)}$ be the incidence set consisting of pairs (m, L) such that $m \in L$. Consider two projections π_1 and π_2 from C to M and to $M^{(2)}$. Let S be a set of weak leaves that pass through m,*

$$S = \pi_2 \pi_1^{-1} (\{m\}) .$$

Let I_m be an ideal of functions on M vanishing at m. Then the ideal $\pi_{2*}\pi_1^* (I_m)$ in the algebra of functions on $M^{(2)}$ consists of vanishing on $S \subset M^{(2)}$ functions.

Proof. Though this fact is absolutely standard in algebraic geometry, we give here a proof.

We want to associate to a good point m of M an ideal on $M^{(2)}$ with zeros in the weak leaves passing through m. We can "pass objects on M through correspondence C": we can consider the inverse image with respect to the first projection π_1 (this gives us an object on C) and after that a direct image with respect to the second projection π_2.

Now $\pi_1^* I_m$ is the ideal generated by lifts of functions from the ideal I_m, i.e., by lifts the equations of the point m. So if the point m has equations $z_k = 0$, were z_k are coordinates on M, then this ideal is generated by the functions z_k considered as functions on C. If the point m is good, then the projection π_1 is locally a nonramified covering, hence the ideal consists of functions that vanish on all n points in $\pi_1^{-1} (m)$.

The direct image of the ideal consists of functions such that their inverse image is in the ideal. In our case if m is a good point, then the corresponding functions on $M^{(2)}$ should vanish at the points $\pi_2 \pi_1^{-1} (m)$. □

Therefore the described algorithm $m \mapsto \pi_{2*}\pi_1^* (I_m)$ is indeed what we need, at least at good points. Consider it at an arbitrary point now. In fact we have constructed a mapping that to any point of M associates an ideal on $M^{(2)}$. What we need to prove is the fact that to any point of M we associate an ideal *of codimension n indeed.* That signifies that the number of weak leaves passing through a given point of M (and taken with proper multiplicities) does not depend on the point of M we choose. On the algebraic language this is denoted by the words *the map is flat.* So the only thing we need to do now is to prove that the projection $\pi_1 \colon C \to M$ is flat.

Theorem 1.23. *If the mapping $\pi_1 \colon C \to M$ is flat over a neighborhood of $m \in M$, then for m' in this neighborhood* $\operatorname{codim} \pi_{2*}\pi_1^* (I_{m'}) = n$*, and the mapping*

$$M \to S^n M^{(2)} \colon m' \mapsto \pi_{2*}\pi_1^* (I_{m'})$$

is compatible with bihamiltonian structures. Both the Poisson structures can be extended from an open subset of $M^{(2)}$ to the whole space $M^{(2)}$ if $M^{(2)}$ is smooth. Moreover, if one of the Poisson structures on M is nondegenerate at m, and the space $M^{(2)}$ is smooth, then this mapping is a local isomorphism of bihamiltonian manifolds. Here we consider a bihamiltonian structure on $S^n M^{(2)}$ defined in the previous section.

Proof. The mapping $M \to S^n M^{(2)}$ preserves the Poisson structures on an open dense subset of good points of M. Therefore it preserves the Poisson brackets everywhere.

Suppose that $M^{(2)}$ is smooth and a Poisson structure there has a singularity on a curve L. Then the corresponding 2-form has a zero on this curve. However, it can have a singularity on some other curve L', so consider a point on $L \smallsetminus L'$. The discussion above shows that the corresponding 2-form on an open subset of $S^k M^{(2)}$ is non-singular and degenerate on a hypersurface. Consider a generic point on the intersection of the image of M and the corresponding hypersurface in $S^n M^{(2)}$. A neighborhood of this point is a direct product of $S^k M^{(2)}$ and $S^{n-k} M^{(2)}$ for an appropriate k, and the 2-form is a direct product of a non-singular form on $S^k M^{(2)}$ and some (possibly singular) form on $S^{n-k} M^{(2)}$. Consider two functions on $S^k M^{(2)}$ and corresponding functions on $S^n M^{(2)}$. The Poisson bracket of these functions has a pole on a hypersurface, but no zero nearby. Therefore the Poisson bracket of the corresponding functions on M is singular, what is impossible.

Now to prove that this mapping is a *local isomorphism* we should only note that if one of the Poisson structures on the bihamiltonian manifold M is nondegenerate, then by construction the corresponding Poisson structure on the set of weak leaves $M^{(2)}$ is also nondegenerate, therefore the corresponding Poisson structure on $S^n M^{(2)}$ is nondegenerate. Since the map $M \to S^n M^{(2)}$ preserves the Poisson structures, the Jacobian of this map is non-vanishing, therefore this map is a local isomorphism. □

Remark 1.24. The previous theorem is adapted for a classification of bihamiltonian structures in a neighborhood of a regular point, as in a corollary below. However, it can be generalized a lot after introduction of a new definition.

Let us call a closure of a weak leaf of codimension 2 a *generalized* weak leaf, and let us extend this definition by taking a limit: call a submanifold a generalized weak leaf if it can be approximated by closures of weak leaves:

Definition 1.25. A submanifold L_0 of a bihamiltonian manifold M is called a generalized weak leaf if there exists a locally close submanifold $\mathcal{L} \subset M$ such that in a neighborhood of any point there exists a function $\psi\colon \mathcal{L} \to \mathbb{C}$ such that $\psi^{-1}(t) \stackrel{\text{def}}{=} L'_t$ is a closure of a weak leaf of codimension 2 if $t \neq 0$ and is L_0 if $t = 0$.

Amplification 1.26 (The weak classification theorem)**.** *Let $M^{(2)\prime}$ be a set of generalized weak leaves and C' be the corresponding incidence set:*

$$C' = \{(m, L) \mid m \in L, L \textit{ is a generalized weak leaf}\} \subset M \times M^{(2)\prime}.$$

Suppose that $\pi_1\colon C' \to M$ is flat. Then the conclusions of the theorem 1.23 remain true, if we change $M^{(2)}$ to $M^{(2)\prime}$.

Corollary 1.27 (The strong classification theorem)**.** *Let m be a regular point on a bihamiltonian manifold M of dimension $2n$. Suppose that any weak leaf on M intersects the set of good points. Consider a (partial) mapping*

$$M \to S^n M^{(2)}$$

defined on the set of good points. Then this mapping extends onto a whole neighborhood of m and is a local isomorphism of bihamiltonian manifolds.

Therefore to any such manifold we associate a canonically defined 2-dimensional bihamiltonian manifold $M^{(2)}$, and we canonically identify the initial manifold with the Hilbert scheme of this 2-dimensional manifold.

The proof is already completed modulo the flatness result. As usual, the proofs of flatness of particular maps are absolutely straightforward and a little bit dull. We postpone it until the appendix in the section 4.

Let us consider the conditions of the weak classification theorem. It is possible to construct an example of a bihamiltonian manifold with non-smooth set of weak leaves (see the section 6.8). This shows that we cannot drop the restriction of smoothness in the theorem. Moreover, this example shows in fact that we cannot drop this condition even in the case of the strong classification theorem. However, in the particular case of this example the theorem remains true if we consider a *normalization* of the Hilbert scheme instead of the Hilbert scheme itself. This shows that there exist some potential for generalization of the theorem.

We cannot drop the condition of non-degeneracy either. Indeed, consider a 2-dimensional bihamiltonian manifold such that the two bivector fields have common zeros of second order. Then in some points on the corresponding Hilbert scheme the bivector fields also have zeros of the second order, therefore we can pull these bivector fields up to a blow-up of this point. Now, if we consider this blow-up, we can see that the mapping to the Hilbert scheme of the set of weak leaves coincides with the mapping of this blow-up, therefore not an isomorphism.

2. The odd-dimensional case

We have seen that in the case of even dimension the set of weak leaves is 2-dimensional and the original manifold can be canonically reconstructed basing on this set. Let us try to proceed with this program as far as we can in the odd-dimensional case.

2.1. **Facts from the linear algebra.**

First we want to give a more vivid picture of a pair of bilinear forms in general position in an odd-dimensional case. The theorem from the appendix gives us a good picture in the case of pairs of mappings. In the section 1.2 we have already warned the reader that while the reading of the appendix in the section 3 was not nessecary, it was highly recommended. This warning is still effective here, where we interpret what this theorem says in a coordinate form. We strongly recommend to read the appendix on linear algebra now, at least those concerning the Kroneker pairs and the odd-dimensional case.

Let us introduce a basis $x_l = r_1^l r_2^{k-l}$ in the space $S^k R$ (here R is spanned by two vectors r_1, r_2) and a basis $y_l = r_1^l r_2^{k-1-l}$ in the space $S^{k-1}R$. Then the two mappings

$\widetilde{\alpha}$, $\widetilde{\beta}$ of the Kroneker pair K_k^+ can be written as

$$y_l \overset{\widetilde{\alpha}}{\mapsto} x_{l+1}, \quad y_l \overset{\widetilde{\beta}}{\mapsto} x_l.$$

However, for the Kroneker pair K_k^- we want to use a different description. Let us consider a pair of *dual* mappings to the mappings K_k^+. It is clear that this pair is undecomposable, therefore isomorphic to the pair K_k^- (see the theorem). In the dual basis it can be written as

$$x_l^* \overset{\widetilde{\alpha}^*}{\mapsto} y_{l-1}^*, \quad x_l^* \overset{\widetilde{\beta}^*}{\mapsto} y_l^*.$$

Here we set $y_{-1}^* = y_k^* = 0$.

That means that basing on the theorem 3.1 we can describe a pair of skew-symmetric forms α, β (in general position) in an $(2k+1)$-dimensional vector space V as following: there is a basis $\left(x_0, \dots, x_k, y_0^*, \dots, y_{k-1}^*\right)$ in this space and the only non-vanishing basic pairings are

$$\alpha\left(x_{i+1}, y_i^*\right) = 1, \quad i = 0, \dots, k-1$$

and

$$\beta\left(x_i, y_i^*\right) = 1, \quad i = 0, \dots, k-1,$$

(so if $k = 0$ both pairings (in 1-dimensional!) space with the basis x_0 vanish). The kernel of the combination $\alpha - \lambda\beta$ is spanned by the vector $x_0 + \lambda x_1 + \lambda^2 x_2 + \dots + \lambda^k x_k$. In accordance with what the theorem says, these kernels evidently span the space W_1 generated by (x_i), $i = 0, \dots, k$, and form in the projectivization of this space an image of the *Veronese inclusion*: $\mathbb{P}^1 \to \mathbb{P}^k$: $(1 : \lambda) \mapsto \left(1 : \lambda : \lambda^2 : \dots : \lambda^k\right)$.

2.2. **The space of weak leaves.** Let us consider now a (local) $(2k+1)$-dimensional bihamiltonian manifold M and the space $M^{(2)}$ of weak leaves in it.[10] Suppose again that *the values of bivector fields* $\eta_1|_m$, $\eta_2|_m$ at the given point $m \in M$ are in general position (then they are in general position also in some neighborhood of m). First of all we want to show that we don't misuse the notation here:

Lemma 2.1. *The space $M^{(2)}$ is 2-dimensional.*

Proof. Indeed, consider again the incidence set $C \subset M \times M^{(2)}$ of pairs (x, L), $x \in L$. The dimension of this set is

$$\dim C = \dim M^{(2)} + \dim L = \dim M + \delta,$$

where δ is the dimension of the set of weak leaves containing a given point $m \in M$. However, in the odd-dimensional case *any linear combination of the bilinear forms* has a kernel, and this kernel is 1-dimensional (in the case of general position). Hence $\dim L = \dim M - 1$, δ is the dimension of the Veronese curve, i.e., $\delta = 1$. Hence $\dim M^{(2)} = 2$. ☐

[10]Let us remind that a weak leaf is a symplectic leaf of some linear combination of two Poisson structures.

So we see a big contrast with the even-dimensional case. A whole 1-parametric family of weak leaves is passing through a given point. That means that to a given point corresponds a (rational) curve on the surface $M^{(2)}$.

However, the greatest difference with the even-dimensional case is that *the kernels do not span the whole cotangent space at the given point*, but a subspace of codimension k. (Here again we consider a bivector as a bilinear form in the cotangent space.) From the other side, the kernel is a normal space to the corresponding symplectic leaf, therefore the intersection of all the weak leaves passing through a given point is not that point, but a submanifold of dimension k passing through this point. Indeed, the sum of normal spaces to a family of subspaces is the normal space to the intersection of this family. This means that the intersection of the *tangent spaces* to weak leaves passing through m is k-dimensional. Now we need to prove that this is true not only on the level of tangent spaces, but in a neighborhood of m.

To do this we can note that it is sufficient to consider a sum of k kernels corresponding to $k+1$ different values $(\lambda_0, \dots, \lambda_k)$ of λ, since this sum coincides with the whole subspace W_1 spanned by all the kernels. Therefore the tangent space to the (evidently transversal) intersection $L_{m,\lambda_0,\dots,\lambda_k}$ of $k+1$ corresponding weak leaves passing through m coincides with the tangent space to the intersection of all the weak leaves passing through m. Now *integration* gives us that *any* weak leaf passing through m contains $L_{m,\lambda_0,\dots,\lambda_k}$. Submanifolds $L_{m',\lambda_0,\dots,\lambda_k}$ for different points m' form a foliation of dimension k on M. We can conclude that any weak leaf that intersects some leaf of this foliation should contain it. It is clear that this foliation does not depend on the particular values of $(\lambda_0, \dots, \lambda_k)$. Let us call this foliation $\mathcal{L}$, and call a leaf of this foliation $\mathcal{L}_m = L_{m,\lambda_0,\dots,\lambda_k}$.

Corollary 2.2. *To any point $m \in M$ corresponds a rational curve on the space of weak leaves $M^{(2)}$. Points on the same leaf of $\mathcal{L}$ correspond to the same rational curve on $M^{(2)}$, points on different leaves correspond to different curves.*

2.3. **Veronese webs.** We see that in contrast with the even-dimensional case the natural correspondence between M and $M^{(2)}$ glues together points on a leaf of the foliation $\mathcal{L}$. Therefore we cannot directly reconstruct M basing on $M^{(2)}$, but only the *local base* of the foliation $\mathcal{L}$. Let us call this $(k+1)$-dimensional base X_M. Here we want to describe some geometrical structure on this manifold. We will be able to construct $M^{(2)}$ basing on this structure alone. Moreover, this base and this structure on it can be canonically reconstructed basing on $M^{(2)}$. After that the correspondence

$$(M, \eta_1, \eta_2) \mapsto M^{(2)}$$

can be passed via X_M:

$$(M, \eta_1, \eta_2) \mapsto X_M \mapsto M^{(2)},$$

and the natural correspondence between X_M and $M^{(2)}$ does not glue any two points on X_M.[11]

Since any weak leaf (of codimension 1) either contains a leaf of $\mathcal{L}$, or does not intersect it, it corresponds to a submanifold of codimension 1 of X_M. For a fixed λ the symplectic leaves of $\alpha - \lambda\beta$ (of codimension 1) form a foliation on M, that can be pushed down to a foliation on X_M. Hence we have a parameterized by $\lambda \in \mathbb{P}^1$ family of foliations on X_M. For a given point $x \in X_M$ to any $\lambda \in \mathbb{P}^1$ we can associate the normal subspace to the passing through x leaf of the foliation with parameter λ. We can consider this subspace as a point in the projectivization $PT_x^* X_M$ of the cotangent subspace at x. The results above show that this mapping $\mathbb{P}^1 \to PT_x^* X_M$:

$$\lambda \mapsto \text{ a normal space to the projection of the symplectic leaf for } \alpha - \lambda\beta$$

is (in an appropriate coordinate system) isomorphic to the *Veronese inclusion* $\lambda \mapsto (1 : \lambda : \cdots : \lambda^k)$. Such an object has so beautiful geometry that it is worthy a name.

Definition 2.3. A *Veronese curve* is an inclusion of $\mathbb{P}^1$ into a projective space isomorphic to a Veronese inclusion $\mathbb{P}^1 \hookrightarrow \mathbb{P}^k$.

Definition 2.4. A *Veronese web* is a $(k+1)$-dimensional manifold X with a parameterized by $\lambda \in \mathbb{P}^1$ family of foliations $\{\mathcal{F}_\lambda\}$ of codimension 1 on X such that given a point $x \in X$ the normal lines $N_x \mathcal{F}_{\lambda,x} \subset T_x^* X$ to the leaves $\mathcal{F}_{\lambda,x}$ of foliations passing through x form a *parameterized by* $\lambda \in \mathbb{P}^1$ *Veronese curve*

$$\lambda \mapsto N_x \mathcal{F}_{\lambda,x}$$

in $PT_x^* X$.

2.4. **Reconstruction of the bihamiltonian manifold basing on a Veronese web.** Now we can (canonically) associate a Veronese web X_M to any odd-dimensional bihamiltonian system M in general position. We say that this bihamiltonian manifold is a *bihamiltonian structure over* X_M. A remarkable fact is that *this correspondence is invertible up to a diffeomorphism*:

Theorem 2.5 ([2])**.** *Let X be a (local) Veronese web. Basing on X we can construct a bihamiltonian manifold M_X with a natural projection to X. The Veronese web X_{M_X} constructed basing on M_X is naturally isomorphic to X.*

If we consider analytic manifolds and if the Veronese web X corresponds in the described above way to a bihamiltonian manifold M, i.e., $X = X_M$, then the bihamiltonian manifold $M_X = M_{X_M}$ is locally isomorphic to M and the map $M \to X_M$ corresponds under this isomorphism to the map $M_{X_M} \to X_M$ (however, this isomorphism is not canonical).

[11] In fact it is possible to reconstruct (M, η_1, η_2) itself basing on X_M (at least locally), however not canonically but only up to a (local) diffeomorphism (see [2]). We show how to reconstruct M (without Poisson structures) in the section 2.4.

Here we do not want to discuss a proof of this theorem, however, we want to explain briefly the construction of the bihamiltonian manifold M_X as a plain manifold (i.e., we cannot explain here the construction of two Poisson structures on M_X). Consider the cotangent bundle T^*X. We have a Veronese inclusion of the same projective line $\mathbb{P}^1$ in the projectivization of any vector space of this bundle. Let us consider the Veronese inclusion $\mathbb{P}^1 \to \mathbb{P}^k$. It is easy to see that for any projective transformation of $\mathbb{P}^1$ we can find a (unique) projective transformation of $\mathbb{P}^k$ that in the restriction to the image of $\mathbb{P}^1$ gives the given transformation of $\mathbb{P}^1$. Hence the same is true for any Veronese curve, in particular for any point $x \in X$. Let us denote by $\mathcal{S}$ the 2-dimensional coordinate vector space (so $P\mathcal{S} = \mathbb{P}^1$).[12]

Hence with any volume-preserving linear transformation of $\mathcal{S}$ (i.e., an element of $\mathrm{SL}_2 = \mathrm{SL}(\mathcal{S})$) we can associate a volume-preserving transformation of T^*_xX (i.e., an element of $\mathrm{SL}(T^*_xX)$). Therefore we have an SL_2-structure on the cotangent bundle of X. Now we are going to do the following (usual in the theory of vector bundles) trick: to any vector bundle with an action of a group we can associate a *principal bundle* for this group over X, and to any representation of this group we can associate another vector bundle over X. It is easy to see that the cotangent bundle on X considered as an SL_2-bundle corresponds in this consideration to the representation of $\mathrm{SL}(\mathcal{S})$ in the k-th symmetrical power $S^k\mathcal{S}$ of $\mathcal{S}$.

Now we can define M_X as a total space of the vector bundle corresponding to the *previous* symmetrical power $S^{k-1}\mathcal{S}$. Since this argument is a little bit misleading, we want to give a more direct definition. Let us consider the vector bundle $T^*X \otimes \mathcal{S}$ over X. The action of the group $\mathrm{SL}(\mathcal{S})$ on the fibers of this bundle decomposes canonically into a direct sum of two representations: one $(k+2)$-dimensional, another k-dimensional. Now M_X is a total space of the vector bundle over X corresponding to the second component with respect to the action of SL_2.

Definition 2.6. We call M_X a *subcotangent bundle* for X and denote it by $T^{*(-1)}X$.

In the same way as one can define a Poisson structure on the cotangent bundle to a manifold, it is possible to define a family of Poisson structures on the subcotangent bundle parameterized by the vector space $\mathcal{S}$, i.e., a bihamiltonian structure. However, this definition in the present form [2] is rather ugly. The situation is very similar to a try to define a Poisson structure on the cotangent bundle without a reference to the symplectic structure[13] on this manifold: it is possible, but we do not know a "direct" way to do it.

2.5. **The double complex.** Another remaining question is why if we start with a bihamiltonian manifold M, construct a Veronese web X_M basing on it and construct

[12] If the Veronese web is associated with a bihamiltonian manifold, we can identify $\mathcal{S}$ with the space of linear combinations of two Poisson structures.

[13] I.e., without the use of the operation of inversion of a matrix.

the bihamiltonian structure $T^{*(-1)}X_M$ basing on this web, we get two locally isomorphic bihamiltonian structures M and $T^{*(-1)}X_M$. Here we also do not know a direct proof, in fact, in the C^∞-case we even do not know if this is true.

However, the algebraic formalism involved in the proof is so exotic that we want to provide some details of this proof here (risking to annoy the reader with an absence of precise definitions).

The proof is based on the consideration of an analogue of the de Rham complex on X. This complex is associated with the vector bundle $T^{*(-1)}X \to X$ instead of $T^*X \to X$, i.e., it is the complex of sections of $\Lambda^\bullet T^{*(-1)}X$. In the same way as it is possible to define a differential of degree 1 on $\Omega^\bullet = \Gamma\left(\Lambda^\bullet T^*X\right)$, we can define *two* differentials d_1, d_2 on

$$\widetilde{\Omega}^\bullet = \Gamma\left(\Lambda^\bullet T^{*(-1)}X\right),$$

any linear combination of which is again a differential. The last condition means

$$d_1^2 = d_2^2 = d_1d_2 + d_2d_1 = 0.$$

(The only difference of this situation and of the definition of a bicomplex is that we have only $\mathbb{Z}$-grading, but not $\mathbb{Z}^2$-grading.)

It is possible to show that any section φ of $T^{*(-1)}X = \widetilde{\Omega}^1$ satisfying

$$d_1d_2\varphi = 0 \in \widetilde{\Omega}^3$$

gives rise to some bihamiltonian manifold $M_{X,\varphi}$ over X. Any bihamiltonian manifold over X can be obtained in this way.

However, if φ_1 and φ_2 satisfy the above differential equation and

$$\varphi_1 - \varphi_2 = d_1\psi_1 + d_2\psi_2 \in \widetilde{\Omega}^1, \quad \varphi_{1,2} \in \widetilde{\Omega}^0 = \Gamma\left(\mathcal{O}\left(X\right)\right),$$

we can construct an isomorphism $M_{x,\varphi_1} \to M_{X,\varphi_2}$ over X and visa versa. Therefore the local isomorphism classes of bihamiltonian structures over X correspond to *double cohomology* classes

$$\mathbb{H}^1\widetilde{\Omega}^\bullet = \frac{\operatorname{Ker} d_1d_2 \colon \widetilde{\Omega}^1 \to \widetilde{\Omega}^3}{\left(\operatorname{Im} d_1 \colon \widetilde{\Omega}^0 \to \widetilde{\Omega}^1\right) + \left(\operatorname{Im} d_2 \colon \widetilde{\Omega}^0 \to \widetilde{\Omega}^1\right)}.$$

At least in the local holomorphic case we can prove that any space of *double cohomology* $\mathbb{H}^i\widetilde{\Omega}^\bullet$, $i \geq 1$, vanishes, and $\mathbb{H}^0\widetilde{\Omega}^\bullet = \mathbb{C}$. That finishes our sketch of the proof of the theorem.

2.6. **The Kodaira theorem and Veronese webs.** We have "shown" the relation between odd-dimensional bihamiltonian manifolds in general position and Veronese webs. However, we began this discussion in connection with the question: can we reconstruct the bihamiltonian structure basing on the set of weak leaves $M^{(2)}$, i.e., in the same spirit as when dealing with even-dimensional structures. It is clear that we cannot hope to reconstruct more information than one contained in the corresponding

Veronese web, since the mapping $M \mapsto M^{(2)}$ goes via the Veronese web (since any weak leaf on M corresponds to a leaf of some foliation on $M^{(2)}$).

So let us call any leaf of any marked foliation on the Veronese web X *a weak leaf on* X. Call the set of weak leaves $X^{(2)}$ (The legality of this notation is guaranteed by the isomorphism $M^{(2)} \simeq X^{(2)}$ if X is associated with a bihamiltonian manifold M.) There is a natural mapping $\pi\colon X^{(2)} \to \mathbb{P}^1$, that send any leaf of the foliation $\mathcal{F}_\lambda$ to λ. To any point $x \in X$ there corresponds a section Γ_x of this bundle: to any λ we can associate the leaf of the foliation $\mathcal{F}_\lambda$ passing through x.

However, in the holomorphic case we can indeed reconstruct the Veronese web X basing on the set $X^{(2)}$. In fact in contrast with the even-dimensional case where we needed the bihamiltonian structure on $M^{(2)}$ we do not need any additional information here:

Theorem 2.7. *In the case of analytic manifolds if X is a (local) Veronese web, then any section of the projection $X^{(2)} \to \mathbb{P}^1$ corresponds to a point on X.*

Proof. Let us note that in the case when X is a germ in neighborhood of $x \in X$ the 2-dimensional manifold $X^{(2)}$ is a germ in a neighborhood of the curve $\Gamma_x \subset X^{(2)}$. Since we are working with germs of manifolds, it is sufficient to show that the dimension of the set of section of the projection $X^{(2)} \to \mathbb{P}^1$ that are deformations of the curve Γ_x is equal to the dimension of X. The Kodaira theorem [5] says that it is sufficient to show that *the degree* of the normal bundle to the (rational) curve Γ_x equals $\dim X - 1$. Fix a point (x, λ) on Γ_x. The normal space $N_{(x,\lambda)}\Gamma_x$ coincides with the normal space to the leaf $\mathcal{F}_{\lambda,x}$ of the foliation $\mathcal{F}_\lambda$ at x. Therefore *the normal bundle for Γ_x coincides with the tautological bundle for the Veronese inclusion*

$$\lambda \mapsto N_x\mathcal{F}_{\lambda,x},$$

and the degree of this bundle can be computed without any difficulty. □

We came to the following

Construction . Consider a germ of an (analytic) surface $\mathcal{X}$ in a neighborhood of a rational curve $\Gamma \subset \mathcal{X}$. Suppose that the degree of the normal bundle to Γ is $k \geq 0$. Then by the Kodaira theorem the curve Γ can be included in the maximal ($(k+1)$-parametric) family of rational curves on $\mathcal{X}$ parameterized by some $(k+1)$-dimensional complex manifold X. (Since $\mathcal{X}$ is a germ only, X is a germ in neighborhood of $\Gamma \in X$.)

Fix a mapping $\pi\colon \mathcal{X} \to \mathbb{P}^1$. Let us define the foliations $\mathcal{F}_\lambda$, $\lambda \in \mathbb{P}^1$. A leaf of the foliation $\mathcal{F}_\lambda$ on X consists of those curves on $\mathcal{X}$ (i.e., points of X) that pass through a fixed point on $\pi^{-1}(\lambda) \subset \mathcal{X}$.

Theorem 2.8. *Any mapping $\pi\colon \mathcal{X} \to \mathbb{P}^1$ that is an isomorphism on Γ corresponds in the specified above way to a canonical structure of a Veronese web on X.*

Proof. The manifold X with a family of foliations $\mathcal{F}_\lambda$ is already defined in the theorem. What we need to do is to show that at any given point $x \in X$ the normal lines to the fibers of the foliations form a Veronese curve. However, *any curve that is a deformation of a Veronese curve is again a Veronese curve*, since any smooth curve of degree k in $\mathbb{P}^k$ that spans the whole space $\mathbb{P}^k$ is a Veronese curve. Therefore it is sufficient to show that in one fixed point x of X this curve (corresponding to the inclusion $i\colon \mathbb{P}^1 \hookrightarrow PT_x^*X$) is a Veronese curve, i.e., it spans the whole space PT_x^*X and has degree k. Of course, we choose the point $\Gamma \in X$ as x.

However, a generic curve on $\mathcal{X}$ that is a small deformation of Γ intersects Γ in k points, therefore a generic tangent to X vector at x lies in a tangent space to a leaf of $\mathcal{F}_\lambda$ for k different values of λ. Hence a generic hyperplane intersects the image of inclusion $i : \mathbb{P}^1 \hookrightarrow PT_x^*X$ in k points.

If this image is contained in a hyperplane, then tangent spaces to leaves of $\mathcal{F}_\lambda$ contain a common vector v. However, this vector corresponds to an infinitesimal deformation of Γ that intersects Γ in any point of Γ, therefore to a 0 section of a normal bundle to Γ. By the Kodaira theorem the tangent space to X at x is *identified* with the space of sections of this normal bundle, therefore $v = 0$. □

2.7. **Assembling the pieces of the puzzle.** Now the correspondence

$$M \overset{\alpha}{\mapsto} M^{(2)}$$

(between odd-dimensional bihamiltonian manifolds and 2-dimensional manifolds with a projection onto $\mathbb{P}^1$ and a section of this projection) can be broken into a chain

$$M \overset{\alpha_1}{\mapsto} X_M \overset{\alpha_2}{\mapsto} (X_M)^{(2)} = M^{(2)},$$

where the object in the middle is a Veronese web. The previous theorem says that in the local case the last arrow can be canonically inverted (i.e., inverted up to a canonically defined isomorphism), so it is an equivalence of corresponding categories. However, even in the local case the first arrow can be canonically inverted only from the right, i.e., although the mapping

$$X \overset{\beta_1}{\mapsto} T^{*(-1)}X$$

satisfies the relation $\beta_1\alpha_1(M) \simeq M$, $\alpha_1\beta_1(X) \simeq X$, only the latter isomorphism can be chosen canonically, i.e., compatibly with isomorphisms.

We can explain it in a more concrete way: a bihamiltonian manifold has much more automorphisms than the corresponding Veronese web. In fact, the set of automorphisms of a bihamiltonian manifold that commute with the mapping[14] $M \to X_M$ is similar to the described above (trivial) classification of bihamiltonian manifolds over a Veronese web: it coincides with the vector space

$$\left(\operatorname{Im} d_1\colon \widetilde{\Omega}^0 \to \widetilde{\Omega}^1\right) \cap \left(\operatorname{Im} d_2\colon \widetilde{\Omega}^0 \to \widetilde{\Omega}^1\right) = \operatorname{Ker} d_1 \oplus d_2\colon \widetilde{\Omega}^1 \to \widetilde{\Omega}^2 \oplus \widetilde{\Omega}^2.$$

[14]I.e., automorphisms of the bihamiltonian manifold that induce the trivial automorphism of a Veronese web.

It is possible to show that the number of such automorphisms coincides with a $k-1$ times the number of functions of two variables. We consider a particular case $k=2$ in the section 2.9.

Remark 2.9. Now we can see that to give a local classification of bihamiltonian manifolds it is sufficient to describe 1-*dimensional nonlinear bundles* $X^{(2)}$ over $\mathbb{P}^1$ of a given degree. Here a degree $\deg X^{(2)}$ of a bundle denotes the degree of its linearization: to have a degree a bundle should have a section Γ, and the vertical tangent bundle to this bundle at this section should be of the given degree. Moreover, we should consider only local objects of this sort, i.e., only germs of 2-dimensional manifolds in a neighborhood of the section Γ. A simple calculation shows that the set of isomorphism classes of such objects is parameterized (essentially—compare the remark 2.12 below) by $\deg X^{(2)}$ germs (at 0) of holomorphic functions of two variables. Therefore to describe a Veronese web of dimension k up to an isomorphism we need to provide $k-1$ functions of two variables.

Remark 2.10. What we get in the previous remark is in fact a remarkable fact: *the dimension of the parameter space for isomorphism classes of odd-dimensional bihamiltonian manifolds (almost) does not depend on the dimension of these manifolds!* The total amount of 5-dimensional bihamiltonian manifolds is equal to the total amount of pairs of 3-dimensional manifolds and so on! Indeed, in the 5-dimensional case the space of parameters is a pair of functions of two variables, that is twice as much as in the 3-dimensional case.

Moreover, it is possible to show that in the holomorphic case there is an operation that associates a (local) k-dimensional Veronese web to a given (local) $(k-1)$-dimensional Veronese web and a (local) 2-dimensional Veronese web. Any k-dimensional Veronese web can be uniquely obtained in this way. So eventually we can obtain any given Veronese web using this operation and starting from 2-dimensional Veronese webs. (Unfortunately, we do not have a place here for a discussion of this beautiful construction.) Note that this is compatible with the calculation of the size of of the moduli space for Veronese webs: to get k-dimensional Veronese web we should fuse $k-1$ Veronese webs of dimension 2, and to describe these webs we need $k-1$ functions of two variable.

2.8. 2-dimensional webs.

Another remarkable fact is the possibility to simplify a lot the definition of 2-dimensional Veronese webs.

Lemma 2.11. *A 2-dimensional Veronese web is uniquely determined by any three different foliations from the $\mathbb{P}^1$-parameterized family. Moreover, any three foliations on a surface such that the three tangent lines at any point are different correspond to some Veronese web.*

The proof is trivial, since a Veronese inclusion $\mathbb{P}^1 \to \mathbb{P}^1$ is just an isomorphism, that is uniquely determined by the image of any three different points. Therefore any three families of curves on a surface in general position determine a Veronese web.

This is a reason why we use the name *web* here, since a web on a plane is exactly three families of curves in general position. So to any web on a plane we can associate some 3-dimensional bihamiltonian manifold. However, the given above description of this manifold can be simplified a lot in this particular case.

2.9. **3-dimensional bihamiltonian structures.** Let us consider a 3-dimensional bihamiltonian manifold M with a mapping $M \xrightarrow{\pi} X_M$. In this case the Veronese web X_M is 2-dimensional, therefore we can consider instead of $\mathbb{P}^1$-parameterized family of foliations only three foliations corresponding to values $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{P}^1$. These foliations can be defined as level lines of three functions

$$f_1 = \text{const}\,,\; f_2 = \text{const}\,,\; f_3 = \text{const}\,.$$

We can suppose that $\lambda_1 = (1:0)$, $\lambda_2 = (0:1)$, $\lambda_3 = (1:1)$. Then the functions $f_i \circ \pi$ are the Casimir functions[15] on M respective to the Poisson structures $\{,\}_1$, $\{,\}_2$ and $\{,\}_1 + \{,\}_2$ correspondingly.

Locally we can represent M as a product of X_M and a line. Choose a coordinate z along this line. We can choose functions $x = f_1$ and $y = f_2$ as two coordinates on X_M and write $f_3 = F(x,y)$. Since the function x is a Casimir function with respect to the bracket $\{,\}_1$, the bivector field that corresponds to that bracket can be written as $\varphi_1(x,y,z)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$, $\varphi_1 \neq 0$. In the same way the second bivector field can be written as $\varphi_2(x,y,z)\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z}$, $\varphi_2 \neq 0$. Now the condition that the function $f_3 = F(x,y)$ is a Casimir function with respect to the bracket $\{,\}_1 + \{,\}_2$ gives us

$$\left(\varphi_1(x,y,z)\frac{\partial}{\partial y} + \varphi_2(x,y,z)\frac{\partial}{\partial x}\right) F(x,y) = 0,$$

or

$$\frac{\varphi_1}{\varphi_2} = -\frac{F_x}{F_y}.$$

We have yet an arbitrariness in a choice of a function z. It is easy to understand (this is a variant of the d'Harboux theorem) that by a change of the function z we can change φ_1 in an arbitrary way. In particular, we can choose $\varphi_1 = -F_x$, so $\varphi_2 = F_y$. So to a web

$$x = \text{const}\,,\; y = \text{const}\,,\; F(x,y) = \text{const}$$

we associate a bihamiltonian manifold with coordinates (x,y,z) and two Poisson brackets

$$\{f(x,y,z), g(x,y,z)\}_1 = -F_x f_y g_z + F_x f_z g_y,$$
$$\{f(x,y,z), g(x,y,z)\}_2 = F_y f_x g_z - F_y f_z g_x.$$

Remark 2.12. It is easy to see that in the example above we associated to a 2-dimensional Veronese web a function $F(x,y)$. Let us find the arbitrariness in the definition of this function.

[15] I.e., say, $\{f_1, g\}_1 = 0$ for any function g on M.

It is easy to see that the function F is defined up to a change of the form $F_1 = \varphi(F)$, $x_1 = \psi(x)$, $y_1 = \chi(y)$. Therefore this function is in fact a mapping from a product of two 1-dimensional manifolds to a third 1-dimensional manifold. Let us denote them by L_1, L_2, L_3. If we work in the local situation we have a marked point, therefore the coordinate changes φ, ψ, and χ send 0 to 0. The correspondences

$$L_1 \to L_3\colon x \mapsto F(x,0), \qquad L_2 \to L_3\colon y \mapsto F(0,y)$$

determine identifications of L_1 and L_2 with L_3. So let $L_1 = L_2 = L_3 = L$. Consider a coordinate system on L. Now basing on a Veronese web we have constructed a function $\widetilde{F}(x,y)$ up to a change of the form $\widetilde{F}_1(x,y) = \phi\left(\widetilde{F}\left(\phi^{-1}(x), \phi^{-1}(y)\right)\right)$ with restrictions $\widetilde{F}(x,0) = \widetilde{F}(0,x) = x$.

Consider a function $G(x) = \widetilde{F}(x,x)$. It is defined up to a change $G_1(x) = \phi\left(\widetilde{G}\left(\phi^{-1}(x)\right)\right)$, and $\frac{dG}{dx}|_{x=0} = 2$. We can find ϕ such that $G_1(x) = 2x$. Now *the only changes of* ϕ *that preserve this restriction* are linear changes. This gives us *a canonically defined function* $\overline{F}$ with conditions $\overline{F}(x,0) = \overline{F}(0,x) = x$, $\overline{F}(x,x) = 2x$ up to a change $\overline{F}_1(x,y) = \alpha^{-1}\overline{F}(\alpha x, \alpha y)$. Therefore we can write

$$\overline{F}(x,y) = x + y + xy(x-y)\mathring{F}(x,y),$$

where the function $\mathring{F}$ is arbitrary and defined up to a change

$$\mathring{F}(x,y) \mapsto \mathring{F}_1(x,y) = \alpha^2\mathring{F}(\alpha x, \alpha y), \quad \alpha \neq 0.$$

Therefore we have proved the following

Theorem 2.13. (1) *The set of germs of 2-dimensional Veronese webs (or 3-dimensional bihamiltonian manifolds) up to isomorphism can be identified with the set of germs of functions of two variables up to a change*

$$\mathring{F}_1(x,y) = \alpha^2\mathring{F}(\alpha x, \alpha y), \quad \alpha \neq 0.$$

(2) *The corresponding to* $\mathring{F}$ *web can be written in an appropriate coordinate system as three foliations given by equations*

$$x = \text{const}, \quad y = \text{const}, \quad x + y + xy(x-y)\mathring{F}(x,y) = \text{const}$$

correspondingly. This coordinate system is determined uniquely up to a homotety if $\mathring{F} = 0$ *and uniquely otherwise.*

(3) *The brackets of the corresponding to* $\mathring{F}$ *bihamiltonian manifold can be written in an appropriate coordinate system as*

$$\{f(x,y,z), g(x,y,z)\}_1 = \left(1 + xy(x-y)\mathring{F}_x + y(2x-y)\mathring{F}\right)(-f_y g_z + f_z g_y),$$

$$\{f(x,y,z), g(x,y,z)\}_2 = \left(1 + xy(x-y)\mathring{F}_y + x(x-2y)\mathring{F}\right)(f_x g_z - f_z g_x).$$

This coordinate system is determined uniquely up to a transformation

$$x_1 = x, \quad y_1 = y, \quad z_1 = z + \beta(x, y)$$

if $\mathring{F} \neq 0$ *and up to a transformation*

$$x_1 = \alpha x, \quad y_1 = \alpha y, \quad z_1 = \alpha^{-1} z + \beta(x, y)$$

otherwise.

3. Appendix on linear algebra

Consider a pair of skewsymmetric bilinear forms α, β in a (finite-dimensional) vector space V. Call a pair of forms *decomposable* if there exist two supplementary non-zero subspaces V_1, V_2 such that both forms α and β are direct sums of their restrictions on V_1 and V_2, i.e., the subspaces V_1 and V_2 are skeworthogonal[16] with respect to both forms. Any pair of forms in a vector space V can be decomposed in a direct sum of *undecomposable* pairs in subspaces V_i. We want to describe pairs of forms in a vector space up to an isomorphism (i.e., a coordinate change in the space V). It is sufficient to describe undecomposable pairs.

As it turns out, it is useful to describe such pairs basing on other objects of linear algebra: pairs of linear mappings from one vector space to another. It is clear what is a direct sum of two such pairs. Let us call a pair *undecomposable* if it cannot be represented as a direct sum.

Theorem 3.1 ([1]). (1) *The list of undecomposable components (up to an isomorphism) of a pair of skewsymmetric bilinear forms is uniquely defined, the same is true for a pair of linear mappings;*

(2) *If a pair of skewsymmetric bilinear forms α, β in a (finite-dimensional) vector space V is undecomposable, then the vector space V can be represented as a direct sum of two subspaces W_1 and W_2, where*
   (a) *Both W_1 and W_2 are isotropic with respect to both forms α and β;*
   (b) *The pairings α and β determine two mappings $\widetilde{\alpha}, \widetilde{\beta}\colon W_1 \to W_2^*$, and this pair of mappings from one vector space to another is undecomposable in the above sense;*

(3) *On the other side, any undecomposable pair of mappings $\widetilde{\alpha}, \widetilde{\beta} : W_1 \to W_2^*$ determines an undecomposable pair of skewsymmetric bilinear forms α, β in the vector space $W_1 \oplus W_2$ by the rule*

$$\alpha((w_1, w_2), (w_1', w_2')) = \langle \widetilde{\alpha}(w_1), w_2' \rangle - \langle \widetilde{\alpha}(w_1'), w_2 \rangle,$$
$$\beta((w_1, w_2), (w_1', w_2')) = \left\langle \widetilde{\beta}(w_1), w_2' \right\rangle - \left\langle \widetilde{\beta}(w_1'), w_2 \right\rangle.$$

(4) *Any undecomposable pair of mappings from a vector space X_1 to a vector space X_2 is isomorphic to exactly one pair from the list:*

[16] I.e., orthogonal with respect to a skew form.

(a) *The Jordan case* J_k^λ, $k \geq 1$ *with* eigenvalue λ*: here* $X_1 = X_2$, $\dim X_1 = k$, $\widetilde{\alpha} = \mathrm{id}_{X_1}$, $\widetilde{\beta}$ *is a mapping from* X_1 *to* X_1 *with exactly one Jordan block (of size* k*) with eigenvalue* λ*;*
(b) *The Jordan case* J_k^∞, $k \geq 1$ *with* eigenvalue ∞*: here* $X_1 = X_2$, $\dim X_1 = k$, $\widetilde{\alpha}$ *is a mapping from* X_1 *to* X_1 *with exactly one Jordan block (of size* k*) with eigenvalue 0,* $\widetilde{\beta} = \mathrm{id}_{X_1}$*;*
(c) *The Kroneker case* K_k^+, $k \geq 1$*: here* $X_1 = S^{k-1}R$, $X_2 = S^kR$ *(i.e., the symmetrical powers),* R *is a 2-dimensional vector space with a basis* r_1, r_2, $\widetilde{\alpha} = M_{r_1}$, $\widetilde{\beta} = M_{r_2}$, *where* M_r *is the mapping of multiplication by* r *from* $S^{k-1}R$ *to* S^kR*;*
(d) *The Kroneker case* K_k^-, $k \geq 1$*: here* $X_1 = S^kR$, $X_2 = S^{k-1}R$ *(i.e., the symmetrical powers),* R *is a 2-dimensional vector space with a basis* r_1, r_2, $\widetilde{\alpha} = D_{r_1}$, $\widetilde{\beta} = D_{r_2}$, *where*

$$D_{r_1} = \frac{\partial}{\partial r_1}, D_{r_2} = \frac{\partial}{\partial r_2}\colon S^kR \to S^{k-1}R;$$

(e) *The trivial Kroneker case* K_0^+: $\dim X_1 = 0$, $\dim X_2 = 1$, $\widetilde{\alpha} = \widetilde{\beta} = 0$*;*
(f) *The trivial Kroneker case* K_0^-: $\dim X_1 = 1$, $\dim X_2 = 0$, $\widetilde{\alpha} = \widetilde{\beta} = 0$*;*

(5) *If a pair of skewsymmetric bilinear forms is in general position, then*
(a) *if the space* V *is even-dimensional all the undecomposable components are 2-dimensional, canonically defined and correspond to the pairs of mappings* J_1^λ, $\lambda \in \mathbb{C} \cup \{\infty\}$*;*
(b) *if the space* V *is odd-dimensional,* $\dim V = 2k - 1$, *then there is only one undecomposable component (so the pair is undecomposable), corresponding to the Kroneker case* K_k^- *(or* K_k^+, *since* K_k^+ *and* K_k^- *lead to isomorphic pairs of skew-symmetric bilinear form);*

(6) *If an undecomposable pair of skewsymmetric bilinear forms in an odd-dimensional vector space* V *corresponds (as above) to the Kroneker pair of mappings* $K_k^-\colon W_1 \to W_2^*$, *then the subspace* $W_1 \subset V$ *is canonically defined. It is spanned by 1-dimensional kernels (i.e., by the vectors which are orthogonal to the whole space) of linear combinations* $\alpha - \lambda\beta$ *of forms* α *and* β*. These kernels considered as points in the projectivization* PW_1 *of the space* W_1 *form* a Veronese curve, *i.e., a curve of minimal possible degree (equal to* $\dim PW_1$*) spanning the whole space* PW_1.

We see that there is a close relation between pairs of linear mappings and pairs of skewsymmetric bilinear forms. If we consider a pair of bilinear forms in a vector space V as a pair of mappings $V \to V^*$, then this pair of mappings becomes a sum of two (dual) pairs of mappings, and the original pair of forms can be reconstructed (up to an isomorphism) basing on any one of these dual pairs.[17] If the pair of forms

[17]It is the place to note that the analogue of this theorem for *symmetric* bilinear forms is wrong, as shows an example of a pair of forms on 1-*dimensional* vector space. In fact in the symmetric

is undecomposable, the corresponding pairs of mappings is undecomposable too, and visa versa. We can call eigenvalues and sizes of Jordan blocks in the corresponding pair of mappings *eigenvalues and sizes of blocks* for a pair of skewsymmetric forms.

4. Appendix on flats and maps

We want to prove here that if we consider two projections π_1, π_2 from the incidence set $C \subset M \times M^{(2)}$ on the factors in the case of an even-dimensional M, then for any regular point $m \in M$ the ideal $\pi_{2*}\pi_1^* I_m$ on $M^{(2)}$ is of codimension $n = \frac{1}{2} \dim M$. Here I_m is the corresponding to $m \in M$ ideal in $\mathcal{O}(M)$, $M^{(2)}$ is the set of weak leaves in M. We have already shown that this is true on an open dense subset of good points.

First of all we want to prove that the mapping π_1 *is flat* on the set of regular points. That means (in this case) that the codimension of $\pi_1^* I_m$ does not depend on the point m, hence is indeed n. We will use the following properties of flat maps:

1. An isomorphism is flat;
2. If a map $Y \to X$ is flat *locally* on X, then it is flat;
3. If the space of functions on Y is a finitely generated free module over the ring of functions on X, then the mapping $Y \to X$ is flat;
4. A composition of flat mappings is flat;
5. If a mapping $\alpha\colon Y \to X$ is flat and there is a mapping i from X' to X, then the *inverse image* of α
$$X' \times_X Y \xrightarrow{i^*\alpha} X'$$
is also flat.

Let us consider instead of π_1 some closely related mapping $p_1 : C' \to M$. To define it we begin with a definition of C'. Denote by $\operatorname{Gr}_2 T_m M$ the space of 2-dimensional subspaces in the $T_m M$, let $\operatorname{Gr}_2 TM = \bigcup_{m \in M} \operatorname{Gr}_2 T_m M$. Let $C' \subset \operatorname{Gr}_2 TM$ be the subset consisting of tangent spaces to weak leaves and p_1 be a natural projection.

There is a natural map from the space C to C' that sends a pair (m, L), $m \in L$, to the subspace $T_m L \subset T_m M$. It is easy to see that this mapping is an isomorphism, so it is sufficient to show that p_1 is flat. However, the structure of the map p_1 is much simpler, *since this mapping is an inverse image from the space of pairs of skewsymmetric bilinear forms.*

Indeed, consider a local coordinate system on M. It identifies M with a piece of a vector space, call it V^*. Now all the cotangent spaces at different points of M are identified with V. Then to any point $m \in M$ corresponds a regular pair of skewsymmetric forms in the vector space V. Let $U \subset \Lambda^2 V^* \times \Lambda^2 V^*$ be a subset of regular pairs. Consider a subset $\mathcal{C}$ of $U \times \operatorname{Gr}_2 V$ consisting of triples (α, β, S) such that S is a kernel of some non-zero linear combination $\lambda\alpha + \mu\beta$. It is easy to see now

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case there is one additional series of undecomposable pairs that includes this example. All other undecomposable pairs can be constructed basing on pairs of linear mappings.

that the mapping $C' \to M$ is an inverse image of the mapping $\mathcal{C} \to U$ with respect to the map $M \to U$.

So what remains to prove is the flatness of the mapping $\mathcal{C} \to U$. Now we want to consider yet another mapping $\mathcal{D} \to U$. Here $\mathcal{D}$ is a subset $U \times \mathbb{C}$ consisting of triples (α, β, λ) such that λ is an eigenvalue of the pair α, β, i.e., such that the form $\alpha - \lambda\beta$ is degenerate. Since the pair (α, β) is regular, the kernel of the form $\alpha - \lambda\beta$ is 2-dimensional, therefore there is a natural isomorphism $\mathcal{D} \to \mathcal{C}$. Now it is sufficient to prove that $\mathcal{D} \to U$ is flat.

The basic example of a flat mapping is the mapping $A_n \to B_n$, where B_n is a set of all polynomials P of degree n with the leading coefficient 1, and A_n is a set of solutions, $A_n = \{(x, P) \in \mathbb{C} \times B_n \mid P(x) = 0\}$. The flatness of this mapping is equivalent to a fact that any polynomial of degree n has exactly n solutions, if counted with multiplicity. Using the above facts, we can prove the flatness of this map by the note that any function f on A_n can be uniquely represented in the form

$$f(x, P) = f_0(P) + x f_1(P) + \cdots + x^{n-1} f_{n-1}(P),$$

therefore the space of functions on A_n is indeed a free module over the ring of functions on B_n.

Now we can apply this example to the proof of flatness of the mapping $\mathcal{D} \to U$. Let us consider the characteristic polynomial $P_{\alpha,\beta} = \det(\alpha - \lambda\beta)$ of the pair (α, β). The theorem on linear algebra shows that this polynomial can be represented as a square of a polynomial $Q_{\alpha,\beta}$. We can normalize Q to get a polynomial with the leading coefficient 1. In this way we have defined a map $U \to B_n$. It is clear that the map $\mathcal{D} \to U$ is an inverse image of the map $A_n \to B_n$, so it is flat. Therefore, the map $C \to M$ is indeed flat.

We proved that the codimension of $\pi_1^* I_m$ is n. What remains to prove is that the codimension of $\pi_{2*}\pi_1^* I_m$ is equal to the codimension of $\pi_1^* I_m$. Speaking nonformally, this a consequence of the fact that the ideal $\pi_1^* I_m$ "lives on the submanifold $m \times M^{(2)} \subset M \times M^{(2)}$", and this submanifold projects isomorphically on $M^{(2)}$. To give a formal proof let us consider the inclusion $C \hookrightarrow M \times M^{(2)}$ and the ideal I_C consisting of vanishing on C functions. Let us consider this picture locally. The ring $\mathcal{O}(C)$ is the quotient of $\mathcal{O}\left(M \times M^{(2)}\right)$ by I_C. Let P_1 and P_2 be two projections from $M \times M^{(2)}$ to the factors. Then $\pi_1^* I_m \subset \mathcal{O}(C)$ is just $P_1^* I_m / (I_C \cap P_1^* I_m)$, hence $\mathcal{O}(C) / \pi_1^* I_m$ coincides with $\mathcal{O}\left(M \times M^{(2)}\right) / (I_C + P_1^* I_m)$.

However, the last ring can be written as

$$\left(\mathcal{O}\left(M \times M^{(2)}\right) / P_1^* I_m\right) / \left((I_C + P_1^* I_m) / P_1^* I_m\right) = \mathcal{O}\left(m \times M^{(2)}\right) / (I_C + P_1^* I_m).$$

A function on $M^{(2)}$ is in the ideal $\pi_{2*}\pi_1^* I_m$ if its inverse image on $M \times M^{(2)}$ is in the ideal $\pi_1^* I_m$, i.e., the image of this function in the ring $\mathcal{O}\left(m \times M^{(2)}\right) / (I_C + P_1^* I_m)$ is 0. Since the ring $\mathcal{O}\left(m \times M^{(2)}\right)$ is isomorphic to the ring $\mathcal{O}\left(M^{(2)}\right)$, the ideal $\pi_{2*}\pi_1^* I_m$ corresponds under this isomorphism to the ideal $(I_C + P_1^* I_m) / P_1^* I_m$, therefore has the same codimension. This completes the proof.

5. Appendix on global bihamiltonian geometry

We have seen in the section on the Kodaira theorem that to construct a local odd-dimensional bihamiltonian structure we should just have a nonlinear bundle of rank 1 over a projective line. However, to do the same in the even-dimensional case we need several 2-dimensional bihamiltonian systems. Therefore in the odd-dimensional case we should have only the information of "topological" origin (i.e., a complex structure on a given topological object), and not of the differential-geometrical origin, as in the even-dimensional case.

However, in the global case the picture can be very similar to that "topological paradise" even in even-dimensional case, as shows the following

Lemma 5.1. *Let M be a manifold such that*

$$\dim \Gamma\left(\Lambda^2 TM\right) = k < \infty, \ \dim \Gamma\left(\Lambda^3 TM\right) = 0.$$

Then on M there is a canonical (up to a linear change) k-hamiltonian structure.

Proof. Let us remind that a Poisson structure is a bracket on the set of functions that satisfies the Leibniz and Jacobi condition. As we have seen, any bracket satisfying the Leibniz condition can be written as

$$\{f, g\}|_x = \langle \eta|_x, (df \wedge dg)|_x \rangle$$

for an appropriate bivector field η on M (i.e., a section of $\Lambda^2 TM$). Let us consider the Jacobi condition for this bracket. It is easy to see that the number

$$\mathcal{H}(f, g, h)|_x = (\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\})|_x$$

depends only on the differentials of functions f, g, h in the point x, and is skewsymmetric with respect to these differentials, so it can be written as

$$\mathcal{H}(f, g, h)|_x = \langle \mathcal{H}|_x, (df \wedge dg \wedge dh)|_x \rangle$$

for some 3-vector field $\mathcal{H}$ on M (i.e., a section of $\Lambda^3 M$). Therefore the condition of the lemma implies that any global bivector field on M gives rise to a Poisson structure.

That means that we have k-dimensional vector space of Poisson brackets on M, i.e., a k-hamiltonian structure. □

Remark 5.2. In fact this k-hamiltonian structure is defined canonically up to a linear change of the k basic Poisson structures, but this is *the object* people usually work with.

Example 5.3. Let us consider a 2-dimensional case. Then $\dim \Gamma(\Lambda^3 M)$ is 0 for sure, so we should bother only with $\Gamma(\Lambda^2 M)$. Let us consider the behavior of the bivector field $\frac{d}{dx} \wedge \frac{d}{dy}$ on the infinity in $\mathbb{P}^2$. The corresponding 2-form $dx \wedge dy$ has a pole of the third order on infinity: in the coordinates $(x : y : 1) = (1 : z : t)$ it can be written as $d\frac{1}{t} \wedge d\frac{z}{t} = -\frac{1}{t^3} dt \wedge dz$. Therefore the bivector field has a zero of the third order on infinity (since in local frames the bivector field and the 2-form have mutually inverse coefficients).

This means that the global sections of $\Lambda^2 T\mathbb{P}^2$ can be written as $P_3(x,y)\,\frac{d}{dx}\wedge\frac{d}{dy}$, where P_3 is a cubic polynomial. Therefore *on the projective plane a natural* 10-*hamiltonian structure is defined.*

Example 5.4. Since we are primary interested in *bihamiltonian* structures, we give here another example. Consider two different cubic curves on $\mathbb{P}^2$. They intersect one another in 9 points of the plane. Consider a blow-up M of the plane in these 9 points. A bivector field η on M corresponds to a bivector field $\widetilde{\eta}$ on $\mathbb{P}^2$ at least outside of these points. However, the Hartogs theorem implies that the bivector field $\widetilde{\eta}$ can be extended to the whole $\mathbb{P}^2$.

We have shown already that a nondegenerate polyvector on a 2-dimensional manifold gets a pole when raised to a blow-up. This implies that the bivector field $\widetilde{\eta}$ on $\mathbb{P}^2$ has zeros in these 9 points on the plane. Therefore the corresponding to $\widetilde{\eta}$ cubic polynomial on the plane has zeros in these points, therefore is a linear combination of equations of initial cubic curves. That means that *a canonical* 2-*bihamiltonian structure is defined on* M.

Remark 5.5. It is easy to see that this bihamiltonian structure is *in general position* in a neighborhood of any point on M. However, the space of cubic polynomials vanishing in given 8 points (in general position) on the plane is also 2-dimensional. That means that instead of blowing-up 9 points it were sufficient to blow-up only 8 points of these 9—on the resulting manifold there is a natural bihamiltonian structure.

There is a remarkable algebro-geometrical construction on a plane that to a 8-tuple of points on $\mathbb{P}^2$ in general position associates a 9th point: any cubic passing through these 8 points passes through this 9th point. (Therefore we cannot get an arbitrary 9-tuple of points as an intersection of two cubics!)

Now consider the result M' of blowing up the plane at these 8 points. This manifold has two independent global bivector fields, and they both vanish at the 9th point.

Therefore the bihamiltonian structure is not in general position in a neighborhood of this point.

Now, when we have constructed a 2-dimensional manifold M with a canonically defined bihamiltonian structure, we can consider the Hilbert scheme $S^n M$ (the definition of the Hilbert scheme can be found in the section 1). As it was shown in the section 1.4, the bihamiltonian structure on M determines a bihamiltonian structure on $S^n M$. We can show that if we are sufficiently lucky this bihamiltonian structure on $S^n M$ is also canonically defined.

Lemma 5.6. *Consider a connected* 2*-dimensional manifold* M *such that* $\dim\Gamma(TM) = 0$, $\dim\Gamma(\mathcal{O}(M)) = 1$. *Then any bivector field on the Hilbert scheme* $S^n M$ *corresponds (in the specified above way) to a bivector field on* M. *Moreover,* $\dim\Gamma(\Lambda^3 TS^n M) = 0$.

Proof. Fix $n-1$ different point $m_1, m_2, \dots, m_{n-1}$ on M and consider another (variable) point $m_0 \in M$. A neighborhood of the point $\{m_0, m_1, m_2, \dots, m_{n-1}\}$ on $S^n M$

can be considered as a direct product of neighborhoods of points m_0, m_1, $m_2, \dots$. This decomposition associates to any bivector at $\{m_0, m_1, m_2, \dots, m_{n-1}\} \in S^2M$ a set of bivectors, one at any point m_0, m_1, $m_2, \dots$, m_{n-1}, and a set of elements of tensor products $T_{m_i}M \otimes T_{m_j}M$. Consider a global bivector field on S^nM and the first component of the value of this bivector field at $\{m_0, m_1, m_2, \dots, m_{n-1}\}$, that is a bivector in the point $m_0 \in M$. We defined a bivector field on $M \smallsetminus \{m_1, m_2, \dots, m_{n-1}\}$. However, by the Hartogs theorem again, this field can be extended to the whole M.

This bivector field is a linear combination of basic bivector fields on M. The coefficients of this combination depend on $m_1, m_2, \dots, m_{n-1}$. However, if we consider a coefficient as a function of, say, m_1, we see that it is defined anywhere outside of $m_2, \dots, m_{n-1}$, therefore it can be extended to a global function and is constant. That means that we defined a global bivector field on M basing on a global bivector field on S^nM. It is easy to see now that this is an inverse map to the construction of bivector field on S^nM basing on a bivector field on M.

Moreover, let us fix a cotangent to M vector at m_i for a fixed i. Consider the component in $T_{m_i}M \otimes T_{m_0}M$ of the value of the bivector field and the "scalar product" of this tensor with the fixed covector at m_i. In this way we get a tangent vector at m_0. This determines a vector field on $M \smallsetminus \{m_1, m_2, \dots, m_{n-1}\}$, that again can be extended to the whole M. Therefore the corresponding vector field is 0. Hence the off-diagonal components of the bivector field vanish, so it is determined by the diagonal components.

The same argument shows that any 3-vector field on S^nM should be 0. □

In the above 2-dimensional examples of bihamiltonian manifolds there is no global vector fields and global functions, therefore the corresponding Hilbert schemes are also equipped with canonically defined bihamiltonian structures. So:

Theorem 5.7. *Fix 8 point on a plane in general position (here that means that they are not on the same conic and no 5 point subsets is on the same line). Consider a 9-th point that is the only other point of intersection of two cubics passing through these 8 points. Denote the blow-up of the plane in these 8 points by M_1, in all 9 points by M_2. Then on the Hilbert schemes S^nM_1, S^nM_2 the spaces of global bivector fields are 2-dimensional, and any bivector field determines a Poisson structure. Therefore on both these $2n$-dimensional manifolds a canonical bihamiltonian structure is defined, the set of (generalized[18]) weak leaves is isomorphic to the corresponding 2-dimensional manifold M_1 or M_2 and these bihamiltonian structures can be reconstructed basing on the canonically defined bihamiltonian structures on M_1, M_2.*

Remark 5.8. As we will see in the section 6.4, a neighborhood of any point on these examples of bihamiltonian manifolds is also an example of a local bihamiltonian manifold to which we can apply a weak classification theorem from the section 1.7. Therefore we found examples of global bihamiltonian manifolds that are classifiable

[18]See section 1.7.

in a neighborhood of any point. Of course, in these cases the classification theorem shows only that we can reconstruct M basing on $S^n M$.

6. Appendix on the local geometry of a bihamiltonian Hilbert scheme

6.1. **Preliminaries.** We have shown above that a bihamiltonian structure in a neighborhood of a regular point[19] is isomorphic to a bihamiltonian structure on a Hilbert scheme (under a mild assumption). However, the given point goes under this correspondence to a regular point of the Hilbert scheme. It is easy to see (even in 4-dimensional example above) that not any point of the Hilbert scheme is a regular point.

Example 6.1. Consider the coordinates X, Y on the plane M with Poisson structures $\frac{d}{dX} \wedge \frac{d}{dY}$ and $X \frac{d}{dX} \wedge \frac{d}{dY}$ and corresponding coordinates

$$\xi = \frac{1}{\sqrt{2}} (X_1 + X_2), \eta = \frac{1}{\sqrt{2}} (Y_1 + Y_2), x = \frac{1}{\sqrt{2}} (X_1 - X_2), y = \frac{1}{\sqrt{2}} (Y_1 - Y_2)$$

on the $M \times M$. We know that $S^2 M$ is a blow-up of $M \times M/\mathfrak{S}_2$ in the diagonal. Consider a point on $S^2 M$ on the intersection of the preimage of the diagonal and of the preimage of $x = 0$. Then ξ, η, $\widetilde{\alpha} = x/y$ and $\widetilde{\beta} = y^2$ form a coordinate frame in this point. (In the example in the section 1.6 we considered coordinates $\alpha = y/x$, $\beta = x^2$ in the remaining points of the exceptional divisor.)

The first Poisson structure on $M \times M$ corresponds to the bivector field

$$\frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial \eta} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},$$

hence the first Poisson structure on $S^2 M$ corresponds to the bivector field

$$\frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial \eta} + 2 \frac{\partial}{\partial \widetilde{\alpha}} \wedge \frac{\partial}{\partial \widetilde{\beta}}.$$

To find the expression of the second structure it is more suitable to work with the corresponding symplectic structure $\frac{1}{X} dX \wedge dY$. We need to express $\frac{1}{X_1} dX_1 \wedge dY_1 + \frac{1}{X_2} dX_2 \wedge dY_2$ in the frame ξ, η, $\widetilde{\alpha}$, $\widetilde{\beta}$.

A tedious computation shows that this form coincides with

$$\frac{\sqrt{2}}{\xi^2 - \widetilde{\alpha}^2 \widetilde{\beta}} \left(\xi \, d\xi \, d\eta - \widetilde{\alpha} \widetilde{\beta} \, d\widetilde{\alpha} \, d\eta - \frac{\widetilde{\alpha}^2}{2} \, d\widetilde{\beta} \, d\eta - \frac{\widetilde{\alpha}}{2} \, d\xi \, d\widetilde{\beta} + \frac{\xi}{2} \, d\widetilde{\alpha} \, d\widetilde{\beta} \right).$$

[19] Let us remind that the regular point is a point where two tensors of Poisson structures form a regular pair, i.e., the dimension of the stabilizer of this pair in $\mathrm{GL}(T_m M)$, $m \in M$ is the minimal possible.

The corresponding Poisson structure can be calculated by inversion of this "matrix" and is equal to

$$\frac{1}{\sqrt{2}}\left(\xi\frac{\partial}{\partial\xi}\wedge\frac{\partial}{\partial\eta}-\widetilde{\alpha}^2\frac{\partial}{\partial\xi}\wedge\frac{\partial}{\partial\alpha}+2\widetilde{\alpha}\widetilde{\beta}\frac{\partial}{\partial\xi}\wedge\frac{\partial}{\partial\widetilde{\beta}}-\widetilde{\alpha}\frac{\partial}{\partial\eta}\wedge\frac{\partial}{\partial\widetilde{\alpha}}+2\xi\frac{\partial}{\partial\widetilde{\alpha}}\wedge\frac{\partial}{\partial\widetilde{\beta}}\right).$$

Therefore the corresponding recursion operator has in the basis $d\xi$, $d\eta$, $d\widetilde{\alpha}$, $d\widetilde{\beta}$ the matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\xi & 0 & \widetilde{\alpha} & 0\\ 0 & \xi & -\widetilde{\alpha}^2 & 2\widetilde{\alpha}\widetilde{\beta}\\ \widetilde{\alpha}\widetilde{\beta} & 0 & \xi & 0\\ \widetilde{\alpha}^2/2 & \widetilde{\alpha}/2 & 0 & \xi\end{pmatrix}.$$

The characteristic polynomial is $\left((\xi-\lambda)^2-\widetilde{\alpha}^2\widetilde{\beta}\right)^2$. When $\widetilde{\alpha}^2\widetilde{\beta}\neq 0$, the corresponding pair of forms is decomposable, since the multiplicity of eigenvalues of the recursion operator is only 2. However, if $\widetilde{\alpha}=0$, then the recursion operator is diagonal, therefore the corresponding pair of forms are proportional, therefore this pair is not regular. Therefore the set of regular points on S^2M coincides with the set $\widetilde{\alpha}\neq 0$.

So the reasonable question is to describe all the regular points on the Hilbert scheme. We can note first that it is sufficient to consider points of the Hilbert scheme (i.e., ideals on the 2-dimensional manifold) that do not come from products of previous Hilbert schemes (i.e., to consider ideals contained in exactly one maximal ideal). So let A be a ring of functions on a neighborhood U of the given point $m\in M$ and I be an ideal such that A/I is a local ring with support at m.

Let us compute the tangent space to the Hilbert scheme S^nU of U at the point $I\in S^nU$. Let us remind that the Hilbert space is sitting inside the Grassmannian $\operatorname{Gr}_n(A)$ of subspaces of codimension n in A and that it consists of subspaces that are ideals in A. Hence the tangent space to (the smooth submanifold) S^nU is sitting inside the tangent space to this Grassmannian. So we get the inclusion

$$T_I S^nU\hookrightarrow T_I\operatorname{Gr}_n(A)=\operatorname{Hom}_{\mathbb{C}}(I,A/I).$$

A standard theorem on the geometry of a Hilbert scheme (see section 1.6) shows that the image of this inclusion coincides with the set of A-homomorphisms, i.e., with

$$\operatorname{Hom}_A(I,A/I)=\operatorname{Hom}_A\left(I/I^2,A/I\right)=\operatorname{Hom}_{A/I}\left(I/I^2,A/I\right).$$

Now the analysis of the section on Hilbert schemes shows that to any Poisson structure $\{,\}$ on M there corresponds a Poisson structure on the Hilbert scheme, and if the initial Poisson structure is nondegenerate, the Poisson structure on the Hilbert scheme is also nondegenerate. A value at $I\in S^nM$ of the bivector field on S^nM that corresponds to this Poisson structure is an element of $\Lambda^2\operatorname{Hom}_{A/I}(I/I^2,A/I)$. However, it is very difficult to write this element explicitly.

We will use the fact that any Poisson structure $\{,\}$ on 2-dimensional manifold M with local coordinates x and y is proportional to a *standard* Poisson structure $\{,\}_0$

on it:

$$\{f,g\}\,|_x = \varphi(x)\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}\right) \overset{\text{def}}{=} \varphi(x)\,\{f,g\}_0\,.$$

Let us denote the corresponding to $\{,\}_0$ bivector in $\Lambda^2 \operatorname{Hom}_{A/I}(I/I^2, A/I)$ by Φ_0, the corresponding to $\{,\}$ bivector by Φ. What we are going to do is to express Φ basing on Φ_0 and φ.

We note first that the vector space $\operatorname{Hom}_{A/I}(I/I^2, A/I)$ is in fact an A/I-module. We want to consider the space $\Lambda^2 \operatorname{Hom}_{A/I}(I/I^2, A/I)$ as a subspace of

$$\operatorname{Hom}_{A/I}\left(I/I^2, A/I\right) \otimes \operatorname{Hom}_{A/I}\left(I/I^2, A/I\right).$$

We can define a structure of an A/I-module on the latter space, where $a\cdot(\alpha\otimes\beta) = (a\cdot\alpha)\otimes\beta$. That action does not preserve the subspace $\Lambda^2 \operatorname{Hom}_{A/I}(I/I^2, A/I)$, however,

Lemma 6.2. *Consider the element* $\Phi_0 \in \operatorname{Hom}_{A/I}(I/I^2, A/I) \otimes \operatorname{Hom}_{A/I}(I/I^2, A/I)$.

1. *For any element* $a \in A/I$ *the tensor* $a\Phi_0$ *is skewsymmetric;*
2. *The bivector* Φ *considered as an element of*

$$\operatorname{Hom}_{A/I}\left(I/I^2, A/I\right) \otimes \operatorname{Hom}_{A/I}\left(I/I^2, A/I\right)$$

is equal to $\phi\Phi_0$.

Proof. Let us consider first the case when the ideal I of codimension n is a product of n different maximal ideals m_i. In this case the lemma is trivial, since we can identify I/I^2 with the direct sum of 2-dimensional vector spaces $\oplus_i m_i/m_i^2$ and can identify A/I with the direct sum of 1-dimensional algebras $\oplus_i A/m_i \simeq \oplus_i \mathbb{C}$ acting in the first direct sum diagonally. Since the bivectors Φ and Φ_0 are diagonal with respect to this decomposition and the diagonal blocks are proportional with coefficient $\phi(m_i)$, the lemma is true in this particular case.

In the general case we can represent any ideal as a limit of a family of ideals of the considered above type, hence the lemma remains true. □

6.2. **The description of the recursion operator.** Let us remind the definition of the *recursion operator*. We have two bivectors η_1, η_2 in a point m of manifold M. We can consider them as elements of $T_mM \otimes T_mM = \operatorname{Hom}(T_m^*M, T_mM)$. The recursion operator is defined in the case when the bivector η_1 corresponds to an invertible operator. We define it as $r = \eta_2\eta_1^{-1} \in \operatorname{End}(T_mM)$. In the general case, when the bivector η can be non-invertible, we can consider the defined by this formula object as a *relation* in the space T_mM, i.e., a subspace in $T_mM \times T_mM$ (when r is an operator, this subspace is the graph of this operator).

Corollary 6.3. *Consider two Poisson structures* $\{,\}_1$, $\{,\}_2$ *on* 2*-dimensional manifold* M *given by following formulae:*

$$\{,\}_i = \varphi_i\,\{,\}_0\,, \quad i = 1,2.$$

(Here $\{,\}_0$ is a nondegenerate Poisson structure on M, φ_1 and φ_2 are two functions on M.) Then on the Hilbert scheme S^nM we can consider two corresponding Poisson structures. The corresponding recursion operator (or relation) in the tangent space to the Hilbert scheme S^nM at $I \in S^nM$ coincides with the operator (or relation)

$$\mu_{\varphi_2}\mu_{\varphi_1}^{-1} \in \operatorname{End}\left(\operatorname{Hom}_{A/I}\left(I/I^2, A/I\right)\right),$$

where μ_φ is the operator of multiplication by $\varphi \in A$ in the A/I-module $\operatorname{Hom}_{A/I}\left(I/I^2, A/I\right)$.

6.3. **The set of regular points on S^nM.** Now we have sufficient information to describe the regular points of the bihamiltonian structure on the Hilbert scheme.

Theorem 6.4. *Let the point $I \in S^nM$ is regular with respect to the bihamiltonian structure on S^nM corresponding to the pair of Poisson structures on M.*

(1) *If the support of the ideal I on M consists of several points, then the ideal I is a product of k ideals $I_1, \dots, I_k$ with supports at k different points of M, any ideal I_l, $l = 1, \dots, k$, is regular on the corresponding Hilbert scheme, and the values of the ratio of Poisson structures on M at these points are different;*
(2) *If the support of the ideal I on M is one point $m \in M$, then there is a smooth curve $C \ni m$ on M such that I is a direct image of an ideal on C, i.e.,*

$$f \in I \iff \left(\frac{d^l}{dt^l} f|_C\right)|_m = 0,\ l = 0, \dots, \operatorname{codim} I - 1.$$

Here t is a coordinate on C. Moreover, if we write the second Poisson structure on M as a multiple of the first one:

$$\{,\}_2 = \varphi\{,\}_1,$$

then m is a regular point of $\varphi|_C$, i.e., $\frac{d}{dt}\varphi|_C \neq 0$ at m.
(3) *The previous conditions on I are sufficient for I being a regular point.*

Proof. The first part of the theorem is trivial, since the corresponding ideal is coming from $S^{n'}M \times S^{n''}M$, and the bihamiltonian structure is a direct product. Hence we can consider only the case when the ideal I has support at one point $m \in M$.

We are free to change the second Poisson structure $\{,\}_2$ on M to the linear combination $\{,\}_2 - \alpha\{,\}_1$, $\alpha = \varphi(m)$, that is degenerate at m. After this change $\varphi(m) = 0$. It is clear that in that case all eigenvalues of r are 0. Let us remind that all Jordan block of r appear by pairs. Therefore the condition of regularity is equivalent to the fact that the recursion operator r has only 2-dimensional kernel (hence it has only two Jordan blocks of size n). By the corollary this is equivalent to a fact that the operator of multiplication by $\varphi \in A$ in $\operatorname{Hom}_{A/I}\left(I/I^2, A/I\right)$ has only 2-dimensional kernel, or that the operator of multiplication by φ^{n-1} does not vanish. We want to show that in this case there is an element $g \in I$ such that m is a regular point of g, i.e., $dg|_m \neq 0$ (we can take g as an equation of the curve C).

There is a natural decreasing filtration in A/I consisting of images of functions on M with increasing orders of zero at m:

$$F_k(A/I) = m^k A/I,$$

here m is identified with a maximal ideal in A. We can consider also the filtration in I:

$$F_k(I) = m^k \cap \mathrm{I},$$

We need only to prove that $F_1(I) \neq F_2(I)$, or what is the same, that $\dim F_1(A/I)/F_2(A/I) < 2$. We claim that $\dim F_k(A/I) = n - k$.

Indeed, we know that multiplication by φ^{n-1} is non-zero in $\mathrm{Hom}_{A/I}(I/I^2, A/I)$, hence $\varphi^{n-1} \neq 0$ in A/I. Therefore I is generated by φ^n and some element g with $dg|_m \neq 0$, that proves the necessary conditions of the theorem.

To prove the last part of the theorem we should only inverse the previous discussion. We should show only that multiplication by x in $\mathrm{Hom}_{A/I}(I/I^2, A/I)$ has two Jordan blocks if $I = (x^n, y)$. An element α in $\mathrm{Hom}_{A/I}(I/I^2, A/I)$ is uniquely determined by $\alpha(x^n)$ and $\alpha(y)$, and these two elements of A/I can be arbitrary. Hence $\mathrm{Hom}_{A/I}(I/I^2, A/I)$ as A/I-module is isomorphic to a direct sum of two copies of A/I, and x acts in A/I as a Jordan block. □

Remark 6.5. Let us compare this description with the above example of S^2M. On the latter Hilbert scheme any ideal corresponds to a pair of points or to a double point on some curve on M (as the description with the blow-up shows). Therefore the only condition of the theorem is that this pair is not on one level set or this curve is transversal to the level sets of the ratio of two Poisson structures. It is easy to see that this condition coincides with the condition $\widetilde{\alpha} \neq 0$ from the above example.

Remark 6.6. We see that in the case $n = 2$ the condition that the ideal corresponds to some curve is trivial. However, already in the case $n = 3$ this is not so. On the 6-dimensional manifold S^3M there is a whole 2-parametric subset corresponding to 3-tuples of collided points on M such that the collision was "from different directions". If this 3-tuple collides in the point $m \in M$ then the corresponding ideal consists of functions on M with trivial 1-jet in m, i.e., to

$$f(m) = 0, \quad df|_m = 0.$$

It is easy to see that this is exactly three conditions on a function f.

Of course, any ideal of codimension 3 with support in m can be described as a result of collision of three points on M. If three points were moving along the same curve C, then the corresponding ideal comes from C, as in the theorem. We can see that this is a generic case of an ideal with support at m: there is a 2-parametric family of such ideals. However, if we cannot approximate the movement of these three points by some common curve, then the resulting ideal is the described above. Therefore we came to a very strange fact: a collision in general position results in a special ideal, and some special collisions results in ideals in general position.

Remark 6.7. Another consequence of the description of the bivector field in terms of the ideal is a possibility to describe weak leaves of codimension 2. Consider an ideal $I \in S^n M$ and a weak leaf L passing through the point I. Let L be a symplectic leaf for $\{,\}_1 - \lambda\{,\}_2$. We can represent I as a product of relatively prime ideals I_0 and I_1 such that I_0 has the support on the curve $\varphi_1 - \lambda\varphi_2 = 0$, I_1 has the support outside of this curve. Now we can see that the above arguments have already proved the following

Proposition 6.8. *We can compute* $\operatorname{codim} L$ *as*

$$\dim \operatorname{Ker} \mu_{\varphi_1 - \lambda\varphi_2} \colon \operatorname{Hom}_{A/I_0}\left(I_0/I_0^2, A/I_0\right) \to \operatorname{Hom}_{A/I_0}\left(I_0/I_0^2, A/I_0\right).$$

Here we use the notations of the previous section. Therefore $\operatorname{codim} L = 2$ *is equivalent to* I_0 *being a regular point of* $S^k M$, $k = \dim A/I_0$. *Therefore* I_0 *has support in* $m_0 \in M$ *such that* $\varphi_1(m_0) = \lambda\varphi_2(m_0)$, *and the closure* $\overline{L}$ *of* L *consists of ideals inside the maximal ideal* I_{m_0}.

6.4. **The compact case revisited.** Now we have made all the preparations for a look on the known examples of compact bihamiltonian systems from the point of view of classification theorems. Consider a Hilbert scheme of a compact surface with two global Poisson structures. We want to show that though not any point of these manifold is a regular point, the weak classification theorem from the section 1.7 *is applicable* in any point of these manifolds.

Indeed, we know all the weak leaves of codimension 2 on $S^n M$: a closure of such a leaf consists of ideals that are supported in some maximal ideal I_m, $m \in M$. Moreover, if $n > 1$, then this point m can be any point but a common zero of two Poisson structures. Therefore, if M is connected and two Poisson structures are linearly independent, then the closure of the incidence set from the weak classification theorem coincides with the natural incidence set

$$C'' = \{(m, I) \mid I \subset I_m\} \subset M \times S^n M.$$

Now we only need to show that the natural projection $C'' \to S^n M$ is a flat mapping, what is a standard fact of the theory of Hilbert schemes.

Remark 6.9. Now we finished a circle in the description of the bihamiltonian systems. First, in the section 1.7 we showed that under some mild conditions a point of a bihamiltonian manifold can be described as a point on a Hilbert scheme of some canonically defined surface. Then in the section 5 we constructed examples of compact bihamiltonian manifolds as Hilbert schemes of compact surfaces. At last, in this section we show that these Hilbert schemes satisfy indeed the conditions of the classification theorem. Therefore, first, we cannot weaken the conditions of the weak classification theorem, and second, the conclusions are sufficiently weak to be true on a compact manifold.

Remark 6.10. Now we can also see how the generalized weak leaves look like. They are of two different types: either closures of a weak leaf—i.e., the ideals in a given (generic) maximal ideal; or the ideals in the maximal ideal such that both Poisson structures vanish in the corresponding point. We see that in the case of the Hilbert scheme of a plane with 9 blown-up points there is no leaves of the second kind, but they do exist if we blow up lesser number of points.

However, we can see that if we forget about Poisson structures, both this types of generalized weak leaves look the same. Here we want to consider an example of possible singularity on a closure of a weak leaf. In fact what we are doing here is to investigate

$$L_m = \{I \in S^n M \mid I_m \supset I\}$$

for a fixed $m \in M$.

Example 6.11. We have seen in the section 1.6 that in the case $n = 2$ the submanifold L_m is smooth (and equal to the blow-up of M in m). Let us consider the case $n = 3$.

It is easy to understand that the only point I_0 on L_m that can be singular is the result of a generic collision of a triple of points to m. We can consider a local frame such that m is a solution of $x = y = 0$. The corresponding ideal is (x^2, xy, y^2). Consider a nearby ideal I. It should contain a function that is near to x^2, the same for xy and y^2. The transversality allows us to correct these monoms by terms of the form $Ax + By + C$ to get an element if I. Let us denote the corresponding functions

$$\begin{aligned}&x^2 + ax + by + \alpha\\ &xy + cx + dy + \beta\\ &y^2 + ex + fy + \gamma.\end{aligned}$$

From the other side, a tangent vector to $S^3\mathbb{A}^2$ at I_0 is a mapping from

$$\begin{aligned}&\operatorname{Hom}_{\mathbb{C}[x,y]/(x^2,xy,y^2)}\left(\left(x^2,xy,y^2\right)/\left(x^4,x^3y,\dots,y^4\right),\mathbb{C}\left[x,y\right]/\left(x^2,xy,y^2\right)\right)\\ &\qquad = \operatorname{Hom}_{\mathbb{C}[x,y]/(x,y)}\left(\left(x^2,xy,y^2\right)/\left(x^3,x^2y,xy^2,y^3\right),(x,y)/\left(x^2,xy,y^2\right)\right).\end{aligned}$$

Therefore, if we denote $T_0\mathbb{A}^2$ by V, then this tangent space is just $\operatorname{Hom}_{\mathbb{C}}\left(S^2V, V\right)$. Hence the functions a, b, c, d, e, f form a good coordinate system in a neighborhood of I_0.

We want to find the equations of the subset L_0 of $S^3\mathbb{A}^2$ consisting of the contained in (x, y) ideals. This subset is of dimension 4, and it easy to see that I_0 is a singular point of this manifold. Indeed, the $\mathrm{SL}(2)$-action shows that there is only one invariant subspace of dimension 4 in the tangent space to $S^3\mathbb{A}^2$ at I_0, and this subspace obviously consists of triples with the center of mass at the origin. (The complimentary 2-dimensional invariant space consists of translations of I_0.) From the other side, the tangent cone to L_0 at I_0 is $\mathrm{SL}(2)$-invariant, therefore if it were smooth, it would conincide with that subspace, what is obviously wrong. However, it is not so difficult to write the equation for a tangent cone to this subset at I_0 explicitely.

Indeed, if $(x,y) \supset I$, then $\alpha=\beta=\gamma=0$. We claim that there are 2 ways to get a homogeneous element of degree 2 in I. First, we can take a linear combination of the above elements with a vanishing linear part, what is

$$(cf-de)\,x^2+(be-af)\,xy+(ad-bc)\,y^2 \in I.$$

Second, we can use the relation $x^2\cdot y^2=(xy)^2$ and substitute instead of quadratic monoms the congruent linear functions, what gives

$$\left(ae-c^2\right)x^2+(af+be-2cd)\,xy+\left(bf-d^2\right)y^2 \in I.$$

Compatibility gives us equations of L_0:

$$\frac{cf-de}{ae-c^2}=\frac{be-af}{af+be-2cd}=\frac{ad-bc}{bf-d^2}$$

(it is easy to see that these conditions are sufficient for the ideal

$$\left(x^2+ax+by,\,xy+cx+dy,\,y^2+ex+fy\right)$$

to be of codimension 3).

We see that even in the simplest possible case the tangent cone in a singular point is given by rather complicated equations.

Remark 6.12. We have seen in the previous remark that a generalized weak leaf looks exactly as the closure of the weak leaf if we forget about the Poisson structure on it. Therefore the singular points on it have the same geometry. However, it is a union of weak leaves of codimension ≥ 4, therefore it is interesting to investigate how these leaves are positioned in a neighborhood of the singular point.

So suppose that the origin is a common zero for both Poisson structures on $\mathbb{A}^2$. Then the considered above subset

$$L_0=\left\{I\in S^3\mathbb{A}^2 \mid I\subset(x,y)\right\}$$

is a generalized weak leaf. A closure of a weak leaf of generic position inside L_0 consists of ideals of codimension 3 inside the ideal $(x,y)\cdot(x-x_0,y-y_0)$, where $x_0\neq 0$ or $y_0\neq 0$. The equations of this subset in the coordinates $a,\dots,f$ are

$$\begin{aligned} x_0^2+ax_0+by_0&=0\\ x_0y_0+cx_0+dy_0&=0\\ y_0^2+ex_0+fy_0&=0, \end{aligned}$$

therefore these submanifolds are flat sections of the cone in question. These sections miss the vertex of the cone, are flat and isomorphic to the blow-up of the plane at the origin and at the point (x_0,y_0).

The others weak leaves of dimension 2 are limits of the above ones when the point (x_0,y_0) goes to the origin. So consider the limit of $(\varepsilon x_0,\varepsilon y_0)$ when $\varepsilon\to 0$. The corresponding equations in the coordinates $a, \dots, f$ are

$$ax_0 + by_0 = 0$$
$$cx_0 + dy_0 = 0$$
$$ex_0 + fy_0 = 0,$$

we can suppose $x_0 = 0$. The equations of the weak leaf become $b = d = f = 0$, and the equation of the cone after this restriction become

$$ae - c^2 = 0.$$

We see that these 2-dimensional weak leaves (that are exeptions!) have simpler singularities than the 4-dimensional leaves (that correspond to the case of general position!).

6.5. **The Magri subset.** Now, when we know the set of regular points in $S^n M$, we want to show what this set is already described in the Magri work. First, we suppose that M is $\mathbb{C}^2 = T^*\mathbb{A}^1$, the first Poisson structure is the standard one $\frac{d}{dx} \wedge \frac{d}{dy}$ and the second is $x\frac{d}{dx} \wedge \frac{d}{dy}$ (any generic bihamiltonian surface can be reduced locally to such a form by a coordinate change and a change $\{,\}_i \mapsto \alpha_{i1}\{,\}_1 + \alpha_{i2}\{,\}_2$ with some constant α_{ij}). Here we are going to introduce the coordianate system on the Hilbert scheme that establishes a connection between the subset of regular points and the Magri coordinate system on a bihamiltonian manifold in general position.

Consider the subset U of $S^n M$ consisting of regular points I on $S^n M$ such that the first Poisson structure is non-degenerate in these points. Any such point satisfies the following condition: if $\{m_1, m_2, \dots, m_k\}$ is the support of the ideal I, then all the x-coordinates $x(m_1), x(m_2), \dots, x(m_k)$ of these points are (finite and) distinct. In this case the factors I_l, $l = 1, \dots, k$, of the ideal I at points m_l, $l = 1, \dots, k$, determine some n_l-jets of curves in these points that are transversal to the level sets $x = \text{const}$. That means that we can find a curve $C = \{(x, y) \mid y = f(x)\}$ with given jets in points m_i. Since

$$\sum_{l=1}^{k} (n_l + 1) = n,$$

we can in fact choose f to be a polynomial of degree $n - 1$, and this condition determines the curve C in the unique way. We call this polynomial f_I.

If we know the curve C, then to determine the ideal I it is sufficient to find the corresponding ideal in the ring of functions on C. (We should remind that the ideal I is by definition a direct image of an ideal on C.) However, the projection x on $\mathbb{A}^1$ identifies this ring with the ring of functions of x.

Any ideal in the ring of functions on line is uniquely determined by its support (considered as a finite subset of $\mathbb{C}$ with multiplicities). In turn, this subset $\langle x_1, x_2, \dots, x_n \rangle$ is uniquely determined by the values of the elementary symmetric functions on it.

Here we want to show that *the Magri coordinate system is associated with a particular choice of the set of symmetric functions,* with $s_l = \sum x_i^l$, $l = 1, \dots, n$, This choice identifies $S^n\mathbb{A}^1$ with a subset in the dual space to the vector space of polynomial of degree $\leq n$ by

$$(x_1, \dots, x_n) \mapsto \left(P \mapsto \sum_i P(x_i) \right).$$

Indeed, we see that to determine the ideal $I \in U \subset S^n M$ it is sufficient to provide the corresponding polynomial f of degree $n-1$ in x and a linear functional

$$l_I \colon P \mapsto \sum_i P(x_i)$$

on the vector space $\mathcal{P}_n$ of polynomials of degree n. Since l_I sends 1 to n, it depends essentially only on the derivative $P' \in \mathcal{P}_{n-1}$:

$$l_I(P) = nP(x_0) + \widetilde{l}_I(P').$$

Let us consider instead the corresponding functional on $\mathcal{P}_{n-1}$:

$$\widetilde{l}_I \colon P \mapsto \sum_i \int_{x_0}^{x_i} P(t)\, dt.$$

A change of the constant x_0 results only in an addition of an independent of I functional on $\mathcal{P}_{n-1}$, i.e., the translation of the image of $T^*\mathbb{A}^1$ in $\mathcal{P}^*_{n-1}$, what is irrelevant in what follows. We put $x_0 = 0$.

Hence we identified U with $\mathcal{P}_{n-1} \times \mathcal{P}^*_{n-1} = T^*\mathcal{P}^*_{n-1}$. On the latter vector space there is a natural symplectic 2-form

$$((f_1, \varphi_1), (f_2, \varphi_2)) = \langle f_1, \varphi_2 \rangle - \langle f_2, \varphi_1 \rangle, \tag{6.33}$$

that determines a translation-invariant symplectic or Poisson structure. Let us show that this structure coincides with the first Poisson structure on $S^n M = T^*\mathbb{A}^1$. It is sufficient to show this on an open dense subset, hence we can consider the subset of U where all n points $x_1, \dots, x_n$ are different.

If $n = 1$, then the ideal I (i.e., a point $(x_1, y_1) \in M$) goes to a constant function $f_I(x) = y_1$ and a functional $\widetilde{l}_I \colon 1 \mapsto x_1$, so the claim is evident in this case. In general case let I correspond to $\{(x_i, y_i)\}$, $i = 1, \dots, n$, in (x, y)-representation. We can represent any tangent vector $\{(\delta x_i, \delta y_i)\}$ at I as $\delta x_i = P(x_i)$, $\delta y_i = Q(y_i)$ with appropriate $P, Q \in \mathcal{P}_{n-1}$, and the bracket of two such vectors with respect to the *symplectic* structure is

$$\left\{ (P, Q), \left(\widetilde{P}, \widetilde{Q} \right) \right\}_1 = \sum_i \left(P\widetilde{Q} - \widetilde{P}Q \right)(x_i).$$

On the other side, consider $\left(f, \widetilde{l}\right)$-representation. Since $f(x_i) = y_i$, the $\mathcal{P}_{n-1}$-component δf of this tangent vector is

$$Q - \sum_i P(x_i) f'(x_i) T_i,$$

where T_i is the only polynomial of degree $n-1$ with zeros in x_j, $j \neq i$, and a value 1 in x_i. The $\mathcal{P}^*_{n-1}$-component is

$$p \mapsto \sum_i P(x_i) \left(\left(\frac{d}{dx}\right)^{-1} p\right)'(x_i) = \sum_i P(x_i) p(x_i),$$

so the symplectic structures coincide indeed.

We can see now that we have identified an open subset U of the set of regular points on $S^n(T^*\mathbb{A}^1)$ with $T^*(S^n\mathbb{A}^1)$ (here, in 1-dimensional case, the Hilbert scheme $S^n\mathbb{A}^1$ coincides with $(\mathbb{A}^1)^n/\mathfrak{S}_n$), and the first Poisson structure on U goes to the natural Poisson structure on the cotangent bundle.

It is also easy to see now that the second Poisson structure can be also described easily in terms of $\mathcal{P}$ and $\mathcal{P}^*$. It is slightly easier to work with symplectic structures again, so consider the open subset $x_i \neq 0$, $i = 1, \dots, n$, where the second Poisson structure is non-degenerate. Working with symplectic structures allows as consider the pairing of tangent vectors instead of cotangent, and the bracket of the above tangent vectors with respect to this (second) symplectic structure is

$$\left\{(P, Q), \left(\widetilde{P}, \widetilde{Q}\right)\right\}_2 = \sum_i \left(P\widetilde{Q} - \widetilde{P}Q\right)(x_i)/x_i.$$

Therefore, if we denote by $M_{\{x_i\}}$ a linear operator in the space[20] $\mathcal{P}_{n-1}$ such that

$$\left(M_{\{x_i\}} f\right)(x_l) = f(x_l)\, x_l, \quad l = 1, \dots, n,$$

then the corresponding symplectic form in $\mathcal{P}_{n-1} \times \mathcal{P}^*_{n-1}$ at $\left(f, \widetilde{l}\right)$, where $\widetilde{l}$ corresponds to $\{x_i\}$, is

$$((\delta f_1, \delta\varphi_1), (\delta f_2, \delta\varphi_2)) = \left\langle M^{-1}_{\{x_i\}} \delta f_1, \delta\varphi_2 \right\rangle - \left\langle M^{-1}_{\{x_i\}} \delta f_2, \delta\varphi_1 \right\rangle.$$

That means that the corresponding Poisson pairing (given by the inverse pairing matrix) can be written as

$$((df_1, d\varphi_1), (df_2, d\varphi_2)) = \left\langle M^*_{\{x_i\}} df_1, d\varphi_2 \right\rangle - \left\langle M^*_{\{x_i\}} df_2, d\varphi_1 \right\rangle. \tag{6.40}$$

Here $df_{1,2}$ are linear functionals on $\mathcal{P}_{n-1} = T^*_{\widetilde{l}}\mathcal{P}^*_{n-1}$, $d\varphi_{1,2}$ are linear functionals on $\mathcal{P}^*_{n-1}$.

[20] That is the vertical tangent space for the cotangent bundle to $\mathcal{P}^*_{n-1}$.

Let us note now that $M^*_{\{x_i\}}$ depends polynomially on the point $\widetilde{l}_{\{x_i\}} \in \mathcal{P}^*_{n-1}$. Indeed, $M_{\{x_i\}}$ is essentially the multiplication by x corrected by a term killing the coefficient at x^n:

$$M_{\{x_i\}} f = xf - (\text{the leading coefficient of } f) \cdot P_{\{x_i\}}, \tag{6.41}$$

where $P_{\{x_i\}}$ is the only polynomial of degree n with the leading coefficient 1 andzeros in x_i, $i = 1, \dots, n$. The coefficients of $P_{\{x_i\}}$ are another set of elementary symmetric functions of x_i,

$$P_{\{x_i\}}(x) = x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \dots \pm \sigma_n.$$

However, the variables σ_k depend polynomially on the variables s_l, i.e., coordinates in the vector space $\mathcal{P}^*_{n-1}$, therefore the operator $M_{\{x_i\}}$ depends polynomially on the point $\widetilde{l}_{\{x_i\}} \in \mathcal{P}^*_{n-1}$, therefore the second Poisson structure on the linear space $T^*(S^n \mathbb{A}^1) = T^* \mathcal{P}^*_{n-1}$ is polynomial.

Formally speaking, we proved that these formulae are true only on the open dense subset $x_i \neq 0$, however we can extend them anywhere by continuity. We get the fact that the identification of the open subset U of the set of regular points on $S^n(T^* \mathbb{A}^1)$ with the vector space $T^* \mathcal{P}^*_{n-1}$ transforms the first Poisson structure into a constant one, and the second Poisson structure into a polynomial Poisson structure. In the following section we consider another coordinate system that will simplify this situation yet further.

However, the formulae we get coincide literally with the formulae for a bihamiltonian structure in the Magri's coordinate system. Let us consider the Magri's hypothesis. He considered the characteristic polynomial of the *recursion operator.* As we have seen, this polynomial is an exact square. Consider a mapping from the bihamiltonian manifold to the set of polynomials that sends a point to a square root of this polynomial. We call this mapping the *Magri mapping.*[21] The Magri theorem claims that if this mapping is a submersion, then in an appropriate coordinate system there is a local normal form of the Poisson structures on the manifold (that coincides[22] with the formulae (6.33), (6.40), (6.41)). From these formulae (or the formulae of the Magri's paper) we can see that any such a point is a regular point of bihamiltonian structure, and the non-degeneracy condition is satisfied. This shows that in fact our conditions are equivalent to the Magri's ones.

Therefore combining two local classification theorems, the Magri's one and the our, we get the following

Corollary 6.13. *The following two conditions on a point on a bihamiltonian manifold are equivalent:*

[21]In fact the Magri considered a slightly different mapping: instead of considering the coefficients of the square root, that are the elementary symmetric functions σ_i, he considered the symmetric functions s_i, exactly as we here. However, the conditions of submersion are equivalent for these two mappings, so we permitted ourselves to interchange these two mappings.

[22]To see this we can note that s_i are exactly the local Hamiltonians in the original Magri mapping.

(1) *The point is a regular point, any weak leaf passing through it intersects the set of good points, and the first Poisson structure is locally non-degenerate;*
(2) *The first Poisson structure is locally non-degenerate and the Magri mapping is a submersion.*

It is not clear how to get this corollary more directly.

6.6. **Another coordinate system.** Here we want to show that if instead of considering the elementary symmetric functions s_i on $S^n\mathbb{A}^1$ we consider the elementary symmetric functions σ_i, then the formulae of the previous section can be simplified a lot. Considering a particular set of functions on $S^n\mathbb{A}^1$ is just a way to introduce a coordinate system on this set, so to rewrite the formulae of the previous section in another coordinate system we want first to give a coordinate-independent description of two Poisson structures.

It is very easy with the first Poisson structure, since it is just a canonical Poisson structure on $T^*(S^n\mathbb{A}^1)$, therefore we can easily rewrite it in any coordinate system. However the description (6.40) of the second Poisson structure uses the decomposition of the tangent space to a point in $T^*(S^n\mathbb{A}^1)$ into a horizontal and a vertical parts, what is much more difficult to rewrite. Here we give another description of the second Poisson structure on $T^*(S^n\mathbb{A}^1)$.

In the previous section we defined an endomorphism $M_{\{x_i\}}$ of the linear space $\mathcal{P}_{n-1}$, and considered it as an endomorphism of the cotangent space to $S^n\mathbb{A}^1 = \mathcal{P}^*_{n-1}$ at the point $\{x_i\} \in S^n\mathbb{A}^1$. Here we want to consider this family of mapping of cotangent spaces as a universal mapping $M\colon T^*S^n\mathbb{A}^1 \to T^*S^n\mathbb{A}^1$.

Proposition 6.14. *Consider a subset $W = \{x_i \neq 0 \mid i = 1, \dots, n\}$ of $S^n\mathbb{A}^1$. The restriction of M on T^*W is a diffeomorphism, and the second Poisson structure on T^*W is a direct image of the first structure under the action of this diffeomorphism:*

$$\{\varphi_1, \varphi_2\}_2 = \{\varphi_1 \circ M, \varphi_2 \circ M\}_1 . \tag{6.43}$$

Proof. It is sufficient to prove this on an open dense subset of configurations of different points. Locally on this subset $S^n\mathbb{A}^1$ is isomorphic to $(\mathbb{A}^1)^n$, so $T^*(S^n\mathbb{A}^1)$ is locally isomorphic to a direct product $(T^*\mathbb{A}^1)^n$. Both Poisson structures, as well as the mapping M can be written as direct products, so it is sufficient to consider the case $n = 1$, that is obvious. □

Remark 6.15. The fact that the second Poisson structure can be written by both the formulae (6.40) and (6.43) requires very special properties of the mapping M. These properties are insured by the following nice, simple, and totally unexpected lemma that expresses symplectic properties of the dependence of σ_i on s_i. To formulate it we need to repeat some definitions.

Consider the coordinate system s_i, $i = 1, \dots, n$, on $S^n\mathbb{A}^1$. It essentially identifies $S^n\mathbb{A}^1$ with a dual space to the vector space of polynomials of degree $\leq n$ with a zero at the origin:

$$\{x_i\} \mapsto \left(p(x) \mapsto \sum_i p(x_i)\right).$$

The differentiation identifies this space of polynomials with $\mathcal{P}_{n-1}$. Denote the corresponding mapping $S^n\mathbb{A}^1 \to \mathcal{P}^*_{n-1}$ by S.

Consider the coordinate system σ_i, $i = 1, \dots, n$, on $S^n\mathbb{A}^1$. It essentially identifies $S^n\mathbb{A}^1$ with the space of polynomials of degree n with a leading coefficient 1:

$$\{x_i\} \mapsto T(x), \quad T(x_i) = 0,\, i = 1, \dots, n.$$

The translation by $-x^n$ identifies the latter space with $\mathcal{P}_{n-1}$. Denote the corresponding mapping $S^n\mathbb{A}^1 \to \mathcal{P}_{n-1}$ by Σ.

Lemma 6.16. *Consider the mapping*

$$S \times \Sigma \colon S^n\mathbb{A}^1 \to \mathcal{P}^*_{n-1} \times \mathcal{P}_{n-1} = T^*\mathcal{P}^*_{n-1}.$$

The image of this mapping is a lagrangian submanifold, and the corresponding 1*-form on* $\mathcal{P}^*_{n-1}$ *is* $-\frac{ds_{n+1}}{n+1}$. *Here we consider the elementary symmetric function* s_{n+1} *as a function of* $s_1, \dots, s_n$.

Now, when we know the coordinate-independent expressions for the Poisson structures in question, we can write them down in the coordinate system σ_i on $S^n\mathbb{A}^1$. The only thing we need to do is to write down the expression of the operator M in the new coordinate system.

Lemma 6.17. *Denote the dual to* σ_i *coordinates on* $T^*\mathcal{P}_{n-1}$ *by* Σ_i. *Then the matrix of the operator* $M_{\{x_i\}}$ *in this basis is*

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_{n-1} & \sigma_n \\ -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix} = M_{ij}.$$

The second Poisson structure can be written as

$$\{f, g\}_2 = \sum_{ij} M_{ji}\left(\frac{\partial f}{\partial \sigma_i}\frac{\partial g}{\partial \Sigma_j} - \frac{\partial g}{\partial \sigma_i}\frac{\partial f}{\partial \Sigma_j}\right) + \sum_{ij} N_{ij}\frac{\partial f}{\partial \Sigma_i}\frac{\partial g}{\partial \Sigma_j},$$

where

$$N_{ij} = \begin{pmatrix} 0 & \Sigma_2 & \dots & \Sigma_n \\ -\Sigma_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\Sigma_n & 0 & \dots & 0 \end{pmatrix}.$$

We see that in this coordinate system two Poisson brackets in question are of the simplest possible form: the first is constant, the second is linear. In fact we defined a pair of affine Poisson brackets on $T^*\mathcal{P}_{n-1}$.

6.7. **The corresponding Lie algebra.** Let us remind the usual description of affine Poisson brackets:

Lemma 6.18. *Call a Poisson bracket on a linear space V an affine bracket, if the bracket of two linear functions is a linear (nonhomogeneous) function. There is a $1-1$ correspondence between affine Poisson brackets on V and pairs $([,],c)$, where $[,]$ is a structure of Lie algebra on V^*, and c is a 2-cocycle on V^*.*

Proof. Define the Lie operation on V^* as a linear part of the Poisson bracket:

$$[\varphi_1,\varphi_2] = \text{the linear part of } \{\varphi_1,\varphi_2\},$$

and the cocycle as

$$c(\varphi_1,\varphi_2) = \{\varphi_1,\varphi_2\}|_0.$$

The inverse operation is the consideration of the corresponding to c central extension $\widetilde{V}^*$ of V^*, and the identification of V and the subspace of $\left(\widetilde{V}^*\right)^*$ passing through $c \in \left(\widetilde{V}^*\right)^*$. Due to this identification the Lie—Kirillov bracket on $\left(\widetilde{V}^*\right)^*$ defines a Poisson bracket on V. □

The formulae of the previous section show that any linear combination

$$\lambda\{,\}_1 + \{,\}_2$$

of brackets on $T^*(S^n\mathbb{A}^1)$ is affine in the coordinate system σ_i, and the linear parts of these brackets coincide. That means that on the dual space to $T^*\mathcal{P}_{n-1}$ there is a structure of Lie algebra. Moreover, there are two cocycles c_1, c_2 for this algebra, and the Poisson brackets $\lambda\{,\}_1+\{,\}_2$ are associated with the sums $\lambda c_1 + c_2$.

To write down this Lie algebra structure let me remind that we write a generic polynomial $p \in \mathcal{P}_{n-1}$ as

$$-\sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \dots \pm \sigma_n$$

(so σ_i are linear coordinate functions on $\mathcal{P}_{n-1}$, $\sigma_i \in \mathcal{P}_{n-1}^*$), and we call the dual coordinates on $\mathcal{P}_{n-1}^*$ by Σ_i, $\Sigma_i \in \mathcal{P}_{n-1}$ (in fact $\Sigma_i = (-1)^i x^{n-i}$).

Lemma 6.19. *The only non-zero brackets of basic elements for the Lie algebra structure on $\mathcal{P}_{n-1}\times\mathcal{P}_{n-1}^*$ associated with the second Poisson structure are*

$$[\Sigma_1,\Sigma_i] = \Sigma_i,\ i \neq 1, \quad \textit{and} \quad [\Sigma_1,\sigma_k] = -\sigma_k.$$

The only non-zero coordinates of the cocycle c_1 are

$$c_1(\sigma_i,\Sigma_i) = 1, \quad i = 1,\dots,n.$$

The only non-zero coordinates of the cocycle c_2 are

$$c_2(\sigma_i,\Sigma_{i+1}) = 1, \quad i = 1,\dots,n-1.$$

Remark 6.20. It is interesting to find some algebraic conditions on this algebra that make it appear in this geometrical situation. It is easy to recognize the Jordan case of undecomposable pairs of bilinear forms in this pair of cocycles. One conjectural description would be that this is a generic case of a Lie algebra structure on a vector space such that this pair of forms is a pair of cocycles. See the next section for a discussion of a simplest example $n = 2$.

In that section we show that in this particular case the set of compatible Lie algebra structures has two irreducible components of maximum dimension, and that any of these components contains an open orbit of the group of automorphisms of the pair of forms. Moreover, though two Lie algebra structures corresponding to these components are noon-isomorphic, the corresponding local bihamiltonian structures *are* isomorphic (so this isomorphism is non-linear). We will see also that the considered here structure corresponds to one of these two components indeed.

6.8. **Examples of linearizations and non-smooth spaces $M^{(2)}$.** The discussion in the previous section allows as to formulate the following

Problem 6.21. Consider a pair of skewsymmetric bilinear forms α, β in a vector space V. Find all Lie algebra structures in V such that α and β are cocycles.

We have seen that such a structure determines a bihamiltonian structure in the space V^*. It is especially interesting to consider this problem in the case when α and β form an undecomposable pair of forms. In this section we give the solution of this problem in the first non-trivial case, when $\dim V = 4$ and α, β form a pair that corresponds to a Jordan block.

Theorem 6.22. *Consider a pair of skew forms*

$$a^* \wedge \alpha^* + b^* \wedge \beta^*, \quad a^* \wedge b^*$$

in 4-dimensional vector space V with a basis a, b, α, β. These forms are 2-cocycles for the following Lie algebras:

(1) $[\alpha, \beta] = \beta$, $[\alpha, a] = a$, $[\alpha, b] = -b$;
(2) $[\alpha, \beta] = 2\beta$, $[\alpha, a] = a$, $[\alpha, b] = -b$, $[\beta, b] = a$;
(3) $[a, \alpha] = \alpha$, $[b, \beta] = \beta$;
(4) $[a, \alpha] = \beta$, $[b, \alpha] = \alpha$, $[b, \beta] = \beta$;
(5) $[b, \beta] = \beta$;
(6) $[b, \alpha] = \beta$;
(7) $[\bullet, \bullet] = 0$.

In this list we write only non-zero brackets. Moreover, any Lie algebra structure for which these forms are cocycles can be transformed to one of these forms by a linear transformation of V which preserves this pair of forms.

Remark 6.23. Consider the set $\mathfrak{L}$ of all Lie algebra structures on V such that the above forms are cocycles. The theorem claims that the group G of automorphisms of this pair of forms acts on $\mathfrak{L}$ with 7 orbits. Denote by H the subgroup of G consisting of elements which preserve vectors α and β. Note that $G = \mathrm{SL}_2 \ltimes H$. Then the first two orbits are principal homogeneous spaces for G, the stabilizer of the third is $\mathbb{Z}/2\mathbb{Z} \ltimes H$, the stabilizers of the fourth and fifth are H, the stabilizer of the sixth is $\mathbb{Z}/3\mathbb{Z} \ltimes H$, and the seventh orbit consists of one point. Here the generator of $\mathbb{Z}/2\mathbb{Z}$ is the element

$$\alpha \mapsto -\beta, \quad \beta \mapsto \alpha, \quad a \mapsto -b, \quad b \mapsto a,$$

of SL_2, and the generator of $\mathbb{Z}/3\mathbb{Z}$ is the element

$$\alpha \mapsto \mu\alpha, \quad \beta \mapsto \mu^{-1}\beta, \quad a \mapsto \mu^{-1}a, \quad b \mapsto \mu b,$$

here $\mu^3 = 1$.

It is interesting also to understand which of these orbits are adjacent. Unfortunately, the simple analysis leading to the theorem 6.22 could not give the answer on this question. A cumbersome and absolutely straightforward calculation shows that the picture is as the following:

$$\begin{array}{ccccc} (1) & & (2) & & (3) \\ & \searrow & \downarrow & & \\ (4) & \rightarrow & (6) & \rightarrow & (7) \\ & & \uparrow & & \\ & & (5) & & \end{array}$$

However, it is unclear how to check that this calculation contains no error, so one should handle this statement with some care. One check is the compatibility with the classification of bihamiltonian systems. The description below shows that there is no immediate contradiction with the geometric intuition.

If we accept the above statement, we can see that $\mathfrak{L}$ contains 5 irreducible components in two connected components. The bihamiltonian structure considered in the previous section corresponds to Lie algebra structure that is a point in an open subset of one of two irreducible components of maximal dimension. We will see that the points in another orbit of maximal dimension correspond to the same (local) bihamiltonian structure.

It is easy to understand how to write the Poisson bracket $\{,\}_\lambda = \lambda\{,\}_1 + \{,\}_2$ that corresponds to any particular case of the theorem. We want to investigate this bracket in the third case of the theorem.

Example 6.24. On V^* we can consider coordinates a, b, α, β, and the basic brackets are:

$$\{a, \alpha\}_\lambda = \alpha + \lambda, \quad \{b, \beta\}_\lambda = \beta + \lambda, \quad \{a, b\}_\lambda = 1.$$

The conditions that two cocycles form a Jordan pair imply that the origin is a regular point on V^* with a double eigenvalue 0. Consider the space of weak leaves.

The Pfaffian of the corresponding to $\{,\}_\lambda$ bivector is $(\alpha + \lambda)(\beta + \lambda)$, therefore this bivector is degenerate in two cases: $\alpha = -\lambda$ and $\beta = -\lambda$. Let $\alpha = -\lambda$. Under this restriction a bracket of α or $e^{a}(\beta-\alpha)$ with any other function is 0. Therefore

$$\alpha=\alpha_0,\quad e^{a}(\beta-\alpha)=\beta_0$$

are equations of weak leaves. In the same way $\beta=-\lambda$ gives the second set of weak leaves:

$$e^{-b}(\beta-\alpha)=\alpha_1,\quad \beta=\beta_1.$$

However, if $\beta_0=\alpha_1=\beta_1-\alpha_0=0$, then these two families of equations give the same weak leaf. Hence the parameter space of weak leaves is a union of two planes intersecting by a line.

Therefore we get an example of a regular bihamiltonian structure that has a non-smooth parameter space of weak leaves! Two Poisson structures on this space are given by

$$\{\alpha_0,\beta_0\}_1=-\alpha_0\beta_0,\quad \{\beta_1,\alpha_1\}_1=\beta_1\alpha_1.$$
$$\{\alpha_0,\beta_0\}_2=-\beta_0,\quad \{\beta_1,\alpha_1\}_2=\alpha_1.$$

Any one of these two Poisson structures corresponds to bivector fields on the intersecting planes. We can see that the bivector fields on these planes vanish on the intersection line with opposite linear parts.

Remark 6.25. Let us list the descriptions of bihamiltonian structures in the remaining examples. The first two examples lead to the same bihamiltonian structure as the considered in the previous section (for dim $=4$). The first one leads to the same coordinate system as before, the second one to a different coordinate system. The remarkable property of the latter coordinate system is the fact that not only one of the Poisson structures is constant and the other one affine, but also the Lie derivative by a constant vector field $\frac{\partial}{\partial a}$ transforms the linear one into the constant one! We do not know if it is possible to do the same in the case dim ≥ 6.

We have already considered the third case. In the fourth case we get an example of a bihamiltonian structure with bivector fields forming a Jordan pair at any point. Turiel introduced the multidimensional generalization of this example in the paper [8].

In the fifth example we get again a pair a planes intersecting by a line as a parameter space of weak leaves. However, in this case one Poisson structure is as above, the other Poisson structure vanishes on one plane, and on the second one it has a zero of the second order on the intersection line.

In the sixth example the pair of Poisson structure can be transformed to a translation-invariant form in the coordinate system $a, b-a^2/2, \beta, \beta a+\alpha$. In the seventh example the pair of forms is already translation-invariant.

Remark 6.26. We want to explain here how to interpret the above example of non-smooth $M^{(2)}$ using the language of Hilbert schemes. If $M^{(2)}$ *is* smooth, then M can be identified with a piece of the Hilbert scheme $S^kM^{(2)}$ (here $k=2$). We want to analyze here what can be a possible generalization of this fact to a case of non-smooth manifold. We know already both M and $M^{(2)}$, below we compute $S^2M^{(2)}$ and see that $S^2M^{(2)}$ is non-smooth, but normal, and M is an irreducible component of the *normalization* of $S^2M^{(2)}$.

Consider a union M of two planes π_1 and π_2 in 3-dimensional (projective) space. Consider a Hilbert scheme of this variety on the level 2. Consider three open subsets on S^2M: the first consists of pairs of different points, one on each of planes; and the other two consist of pairs of different points on either one of planes. The closures of these open subsets form three irreducible components of S^2M. The last two components are clearly smooth. We are going to study the geometry of the remaining component.

To any pair of different points on M we can associate a line passing through these points. It is easy to see that we can extend this mapping to a mapping from the Hilbert scheme to the set of lines in the space. The preimage of a line consists of one point of the Hilbert scheme excepting the case when this line is inside M. In the latter case the preimage is a 1-dimensional manifold naturally identified with the line in question, except the case when this line is $\pi_1 \cap \pi_2$, when this preimage is the symmetric square of this line.

Since two subsets Π_1, Π_2 consisting of lines inside π_i intersect transversely in the set of lines in the space, we can consider a blow-up L of the latter space in these two subvarieties. The order of two blow-ups is irrelevant because of the transversality. The preimage of $\Pi_1 \smallsetminus \Pi_2$ consists of lines in π_1 with a marked point, the same for $\Pi_2 \smallsetminus \Pi_1$, the preimage Π_{12} of $\Pi_1 \cap \Pi_2$ consists of ordered subsets of two points on $\Pi_1 \cap \Pi_2$.

We see that the first irreducible component of S^2M can be identified with the quotient of L by the action of the symmetric group $\mathfrak{S}_2$ on the submanifold Π_{12}. The only non-smooth points on this quotient are the points on the image of Π_{12}. Consider a point on this image. In a local coordinate system Π_{12} is given by the equations $x = y = 0$ and $\mathfrak{S}_2$ is acting by $(0, 0, z, t) \mapsto (0, 0, -z, t)$. We can split off the variable t and consider the 3-dimensional manifold with coordinates x, y, z and an action of $\mathfrak{S}_2$ on $x = y = 0$ by $(0, 0, z) \mapsto (0, 0, -z)$.

The basic coordinate functions on the quotient are (x, y, xz, yz, z^2). We can see that $\sqrt{z^2} = \frac{xz}{x}$ is an element in both the integer closure and the field of ratios of this ring, therefore the *normalization* of the quotient is the initial 3-dimensional space.[23] Hence *the normalization of the Hilbert scheme is smooth.* One of three connected

[23]The analogous 2-dimensional example where $\mathfrak{S}_2$ is acting on $x = 0$ as $(0, y) \mapsto (0, -y)$ corresponds to the famous *Whitney's umbrella.* Indeed, the basic coordinate functions on the quotient are

$$a = x, \quad b = xy, \quad c = y^2,$$

and the relation is $b^2 - a^2c = 0$.

In this example it is easy to draw the corresponding picture and to see that the result of the normalization (i.e., of the separating of two intersecting sheets of the umbrella) is the initial plane.

components of this normalization is the discussed above blow-up of the space of lines.

Now to a Poisson structure on M we can associate a Poisson structure on an open subset of S^2M. However, we cannot apply the proof from the section 1.6 to extend this Poisson structure to the whole S^2M: there are additional hypersurfaces where the corresponding bivector field can have a pole. They are two exceptional divisors on the blow-up. In fact a simple calculation shows that this bivector field *has* a pole unless the bivector fields on the components of M have opposite linear parts on the intersection. (These bivector field *should* vanish on the intersection for the Poisson bracket of two functions to be a function on M.)

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