# WEBS, VERONEZE CURVES AND BIHAMILTONIAN SYSTEMS

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Abstract. We define a special kind of multidimensional webs, connected with the Veroneze curve. For these webs the foliations in question depend not on a discrete parameter, but on the point on a projective line. For each bihamiltonian system of odd dimension in general position we construct such a web and show how to reconstruct the original bihamiltonian system basing on these data.

Contents

## Definitions of basic objects

**Veroneze webs.** Let us fix a two-dimensional linear space $\mathcal{S}$ and call it the fundamental space[1]. By $PV$ we denote the projectivization of a vector space $V$, i.e., the space of 1-dimensional subspaces in $V$.

*Veroneze curves.* Since the notion of a Veroneze curve will play a crucial rôle in what follows, we discuss it up to subtleties here.

**Definition.** A *Veroneze inclusion* in $k$-dimensional projective space is a mapping $P\mathcal{S} \hookrightarrow PV$ such that there exists an isomorphism $PV \to PS^k\mathcal{S}$ which makes the diagram

$$\begin{array}{ccc} P\mathcal{S} & \longrightarrow & PV \\ \Big\| & & \downarrow \\ P\mathcal{S} & \xrightarrow{S^k \operatorname{id}} & PS^k\mathcal{S} \end{array}$$

commutative.

Here $S^kV$ is the $k$-th symmetrical power of $V$, $S^k\varphi$ for a map $\varphi: V \to W$ denotes the corresponding map $V \to S^kW: v \mapsto \varphi(v)^k$ or the corresponding map of projectivizations.

**Example.** The more habitual definition of a Veroneze curve is the curve that in a coordinate system is parameterized as $\left(t, t^2, \dots, t^k\right)$. So a Veroneze curve on a line is the line itself, on a plane is a parabola (or any conic). The connection between this two definitions is established by the choice

$$\mathcal{S} ='' \text{ the space of homogeneous linear functions in } (y_1, y_2)''.$$

In this case

$$S^k\mathcal{S} ='' \text{ the space of homogeneous polynomials of degree } k \text{ in } (y_1, y_2)''.$$

It is clear that $S^k \operatorname{id}$ sends a linear function $(t_1y_1 + t_2y_2) \in \mathcal{S}$ to a polynomial $(t_1y_1 + t_2y_2)^k \in S^k\mathcal{S}$. Multiplication of this linear function by a constant results in multiplication of this polynomial by another constant, so this map can be reduced to a map of projectivizations.

So $(y_1 + ty_2)$ goes to

$$(y_1 + ty_2)^k = 1 \cdot \binom{k}{0} y_1^k + t \cdot \binom{k}{1} y_1^{k-1} y_2 + t^2 \cdot \binom{k}{2} y_1^{k-2} y_2^2 + \dots + t^k \cdot \binom{k}{k} y_2^k,$$

and this vector-valued function in $t$ differs from the function $t \mapsto \left(t, t^2, \dots, t^k\right) = \left(1 : t : t^2 : \dots : t^k\right)$ only by a coordinate change.

Hence any curve that in some coordinate system is parameterized by $\left(t, t^2, \dots, t^k\right)$ is a Veroneze curve in the sense of our definition.

**Example.** Another useful parameterization of a Veroneze curve is

$$t \mapsto \left((t - \lambda_1)^k, (t - \lambda_2)^k, \dots, (t - \lambda_k)^k\right).$$

To show that the curve $P\mathcal{S} \to PS^k\mathcal{S}$ allows this parameterization let us chose the "interpolation coordinate system" on the space of homogeneous polynomials of degree $k$:

$$p(y) \mapsto (p(1,0), p(-\lambda_1, 1), p(-\lambda_2, 1), \dots, p(-\lambda_k, 1)).$$

[1] We have chosen this letter (although there appears a visual conflict between two S's in the notation $S^k\mathcal{S}$ below) to accent the apparent connection of this space with a spinor space. The action of the special linear group for the space $\mathcal{S}$ (denoted as $\mathrm{SL}(\mathcal{S})$) will be one of main tools in the discussion below.

Clearly, the curve $(y_1 + ty_2)^k$ has in this coordinate system the wanted parameterization.

So any curve with such a parameterization is a Veroneze curve.

*Remark.* In fact any curve of degree $k$ in $k$-dimensional space that isn't contained in any hyperspace is a Veroneze curve (we mean that it is rational—i.e., the projective line if considered as a manifold—and any identification of this curve with $P\mathcal{S}$ is a Veroneze inclusion).

*Remark.* We have chosen this awkward definition of a Veroneze curve since it opens a possibility to speak as invariantly as it is possible. In fact the identification $\alpha: PV \to PS^k\mathcal{S}$ from the definition is uniquely defined. Indeed, the inverse image of the sheaf $\mathcal{O}(1)$ on $PV$ is the sheaf $\alpha^*\mathcal{O}(1) \simeq \left(S^k \operatorname{id}\right)^* \mathcal{O}(1) = \mathcal{O}(k)$. Hence a linear function on $V$ (i.e., a global section of $\mathcal{O}(1)$) corresponds to a function on $\mathcal{S}^*$ of homogeneity degree $k$ after a choice of the last isomorphism.

However, this isomorphism is a section of $\mathcal{O}(k) \otimes \mathcal{O}(k)^* = \mathcal{O}(0)$. So isomorphism between $\alpha^*\mathcal{O}(1)$ and $\mathcal{O}(k)$ is defined up to a constant, hence the isomorphism between $V^*$ and $S^k\mathcal{S}^* = \left(S^k\mathcal{S}\right)^*$ is defined up to a constant.

*Remark.* The previous remark shows that for any Veroneze inclusion $\alpha: P\mathcal{S} \to PV$ we can define an action of a projective linear group $\mathrm{PSL}(\mathcal{S})$ on the space $V$. Really, denote the corresponding identification of $PV$ and $PS^k\mathcal{S}$ by $i_\alpha$. If $g \in \mathrm{PSL}(\mathcal{S})$, then $\alpha \circ g$ is another Veroneze inclusion. So we can consider another identification $i_{\alpha\circ g}$ of $PV$ and $PS^k\mathcal{S}$. These two identifications differ by a map $\beta_g = i_{\alpha\circ g}^{-1} \circ i_\alpha$ in $PV$. We can consider $g \mapsto \beta_g$ as a projective action of $\mathrm{PSL}(\mathcal{S})$ on $V$.

It is known, however, that any such action can be uniquely pushed up to a linear action of a special linear group of $\mathcal{S}$ (called $\mathrm{SL}(\mathcal{S})$) on $V$. It is easy to see that another definition of this action is the inverse image of $\mathrm{SL}(\mathcal{S})$-action on $S^k\mathcal{S}$ under an identification of $V$ and $S^k\mathcal{S}$. Therefore this action is irreducible.

**Proposition.** *To determine a Veroneze inclusion in the space $PV$ it is sufficient to determine an irreducible* $\mathrm{SL}(\mathcal{S})$-*structure on $V$.*

*Proof.* Really, the image of a point $\lambda \in P\mathcal{S}$ in $PV$, i.e., an one-dimensional subspace in $V$, can be found as a highest-weight vectors subspace relative to the Borel subgroup $B_\lambda = \operatorname{Stab}\lambda$. □

*The main definition of a Veroneze web.* A Veroneze web can be defined in many different ways. Here we begin with a definition that will be used throughout *this* paper.

**Definition.** A *foliation* $\mathcal{F}$ of codimension $l$ on a manifold $X$ is a family of subspaces $\mathcal{F}_x \subset T_xX$ of codimension $l$ (for $x \in X$) such that in an appropriate coordinate system $\left(x^i\right)$, $i = 1, \ldots, n$, on $X$ the subspace $\mathcal{F}_x$ is generated by vector fields $\partial/\partial x^i$, $i = 1, \ldots, n - l$. A submanifold $L \subset X$ is called a *leaf* of the foliation $\mathcal{F}$ if $T_xL = \mathcal{F}_x$ for $x \in L$ and $L$ is a maximal connected submanifold with this property. A *tangent space to a foliation* $\mathcal{F}$ in a point $x \in X$ is by definition the space $\mathcal{F}_x$, a *cotangent space* is $\mathcal{F}_x^*$. A *tangent* (or *cotangent*) bundle to a foliation is linear bundle on $X$ with corresponding fibers.

Sometimes we will introduce a foliation by the set of it's leaves.

As usual, for linear subspace $W \subset V$ the subspace $W^\perp \subset V^*$ is the orthogonal complement to $W$.

*Remark.* Let us consider a foliation $\mathcal{F}$ of codimension 1 on $X$. By definition the leaves of this foliation can be determined as $f = \mathrm{const}$, $f$ being a function on $X$. So $\mathcal{F}_x = (df)^\perp = (g\,df)^\perp$ for an arbitrary non-zero function $g$. Note that if $\omega = g\,df$, then $\omega \wedge d\omega = 0$.

Inversely, let $\omega$ be an 1-form on $X$ such that

$$\omega \wedge d\omega = 0.$$

It is known that we can find (locally) a function $\varphi$ on $X$ such that $d(\varphi\omega) = 0$. So $\omega = \varphi^{-1}df$, and $\mathcal{F}_x = (\omega)^\perp$ is a foliation with the leaves $\{f = \mathrm{const}\}$.

**Definition.** A *Veroneze web* on a $(k+1)$-dimensional manifold $X$ is a family of codimension 1 foliations $\mathcal{F}_\lambda$ parameterized by $\lambda \in P\mathcal{S}$ such that in every point $x \in X$ the orthogonal complements $\mathcal{F}_{\lambda,x}^{\perp}$ to leaves of foliations $\mathcal{F}_\lambda$ at $x$ correspond to Veroneze inclusion $P\mathcal{S} \hookrightarrow PT_x^*X$, $\lambda \mapsto \mathcal{F}_{\lambda,x}^{\perp}$.

*Equivalent definitions.* The definition from the previous section being useful in the connection with bihamiltonian structures, in other domains there can be useful different definitions of this object.

**Proposition.** *For a given Veroneze web on $X$ there exists (locally) a family of 1-forms $\omega_\lambda$ on $X$, $\lambda \in \mathcal{S}$, such that $\mathcal{F}_{\lambda,x} = \omega_\lambda(x)^{\perp}$ and*

$$\omega_\lambda \wedge d\omega_\lambda = 0,$$

*these forms are homogeneous polynomials in $\lambda \in \mathcal{S}$ of degree $k$ and are non-degenerate in the sense that for any vector $v \in T_xX$ $\langle v, \omega_\lambda \rangle \not\equiv 0$ in $\lambda$. Inversely, such a family on a manifold $X$ determines (uniquely) a Veroneze web on $X$. Two families differing only on multiplication by function $\varphi$ not dependent on $\lambda$*

$$\widetilde{\omega}_\lambda(x) = \varphi(x)\,\omega_\lambda(x),$$

*determine the same Veroneze web.*

*Proof.* The only non-evident assertion is the polynomial dependence of $\omega_\lambda$ in $\lambda$. The polynomial map $S^k\,\mathrm{id}\colon \mathcal{S} \to S^k\mathcal{S}$ sends $\mathcal{S}$ onto the preimage of the Veroneze curve under the projection $S^k\mathcal{S} \to PS^k\mathcal{S}$. Taking an identification of $S^k\mathcal{S}$ and $T_x^*X$ that depends smoothly in $x$ we get what is claimed. □

**Theorem.** *Consider $2k+1$ foliations $\mathcal{F}^i$ of codimension 1 on $X$ such that*

1. *The orthogonal complements $\mathcal{F}^{i\perp}$ to the foliations considered as points on $PT_x^*X$ are different;*
2. *for any two points $x, y \in X$ there exists a projective map $PT_x^*X \to PT_y^*X$ sending this set of $2k+1$ points on $PT_x^*X$ to the corresponding set for the point $y$;*[2]
3. *and that in one (or in any) point $x \in X$ this set lies on an appropriate Veroneze curve.*

*Then there exists (uniquely determined) Veroneze web $\mathcal{F}_\lambda$, $\lambda \in \mathcal{S}$, on $X$ and $2n+1$ points $\lambda_m$ on $P\mathcal{S}$ such that $\mathcal{F}^m = \mathcal{F}_{\lambda_m}$. These objects are determined up to a simultaneous projective map $P\mathcal{S} \to P\mathcal{S}$.*

*Proof.* It is easy to see that the Veroneze curve containing the points in question is uniquely determined. Therefore it is possible to construct in this case a family of 1-forms $\omega_\lambda$ that depends polynomially in $\lambda \in \mathcal{S}$ and sweeps for any $x \in X$ the this Veroneze curve. The expression

$$\omega_\lambda \wedge d\omega_\lambda$$

is 0 as an homogeneous polynomial of degree $2k$ vanishing in $2k+1$ points on a projective line (since for $2n+1$ different values of $\lambda$ the form $\omega_\lambda$ determines one of the original $2k+1$ foliations). □

*Examples.* In what follows we will use only the first of these examples. However, the second shows that the object we have introduces it quite classical.

**Example.** Let $i\colon P\mathcal{S} \hookrightarrow PV^*$ be a Veroneze inclusion. For $\lambda \in P\mathcal{S}$ consider a foliation on $V$ with hyperplane leaves orthogonal to $i(\lambda)$. It is clear that this family of foliations form a Veroneze web. A *flat* Veroneze web is a web that is diffeomorphic to this one.

[2]We can express this by the phrase that *the configurations* formed by these points on $PT_x^*X$ and on $PT_y^*Y$ are isomorphic.

**Example.** Consider the previous theorem in the case $k = 1$. The only nontrivial condition on foliation is the first one—the condition of transversality. Hence any 3 transversal families of curves on a plane determine a Veroneze web.

Let us rephrase the theorem in this special case. Fix three points $\lambda_1, \lambda_2, \lambda_3 \in P\mathcal{S}$. Let $\mathcal{F}_{\lambda_1}$, $\mathcal{F}_{\lambda_2}$, $\mathcal{F}_{\lambda_3}$ be three arbitrary foliations (i.e., families of curves) on two-dimensional manifold $X$. Then $\mathcal{F}^{\perp}_{\lambda_1,x}, \mathcal{F}^{\perp}_{\lambda_2,x}, \mathcal{F}^{\perp}_{\lambda_3,x} \in PT^*_xX$ (being three points on a projective line) determine a projective coordinate system on $PT^*_xX$, i.e., an identification $i_x: P\mathcal{S} \to PT^*_xX$. For any $\lambda \in P\mathcal{S}$ the distribution of lines $i_x(\lambda)^{\perp} \subset T_xX$, $x \in X$, is integrable (as any distribution of lines), therefore it defines a Veroneze web.

*Remark.* The latter example shows that the ordinary definition of a web (as of 3 families of curves) is just a special case of our definition. In subsequent papers we are going to show that this transition to continuous families of foliations from a discrete family (that consist of three foliations) is very natural. In particular, the 6-angle invariant of the web obtained from Veroneze web by choosing three points on $P\mathcal{S}$ *does not depend* essentially on the choice of these points. It is this connection with ordinary web that motivated us to choose this name.

*Remark.* We can see that on two-dimensional manifolds there exist non-flat Veroneze webs (e.g., webs on a plane with non-vanishing 6-angle invariants). The interest of this definition is connected in particular with existence of non-flat Veroneze webs for higher dimensions and the rich geometry of these objects.

**Bihamiltonian structures.** A bihamiltonian structure is a pair of Poisson structures with an additional condition. The definition of a Poisson structure is well known (see, e.g., [6]).

**Definition.** A Poisson structure on a manifold $Y$ is a bivector field $\eta \in \Gamma\left(\Lambda^2 TY\right)$ such that a skew-symmetrical bracket

$$\{,\}: (f,g) \mapsto \{f,g\}, \qquad \{f,g\}(y) = \langle \eta(y), (df \wedge dg)(y) \rangle$$

satisfy the Jacoby identity

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0.$$

Here $f$, $g$, $h$ (and $\{f,g\}$) are functions on $Y$.

**Definition.** A bihamiltonian structure on a manifold $Y$ is a pair of Poisson structures $\eta_1$, $\eta_2 \in \Gamma\left(\Lambda^2 TY\right)$ such that any linear combination $\mu_1\eta_1 + \mu_2\eta_2$ is also a Poisson structure.

Let us recall that in the tensor notations $t_{mnkl,i}$ denotes the derivative of the component $t_{mnkl}$ of the tensor field $t$ in the direction of $i$.

*Remark.* Let us note that the condition imposed on a bivector $\eta$ to be a Poisson structure (in a local coordinate frame $y^i$ on $Y$ it can be written as

$$\operatorname*{Alt}_{ijk} \eta^{il} \eta^{jk}_{,l} = 0)$$

is quadratic in $\eta$. (Here $\mathrm{Alt}_{ijk}$ denotes the alternation operation:

$$\operatorname*{Alt}_{i_1 i_2 \dots i_l} X_{i_1 i_2 \dots i_l} = \sum_{\sigma \in \mathfrak{S}_l} (-1)^{\sigma} X_{\sigma(i_1)\sigma(i_2)\dots\sigma(i_l)},$$

where $\mathfrak{S}_l$ is a symmetrical group.) Hence an arbitrary linear combination of $\eta_1$, $\eta_2$ will be Poisson provided some fixed non-trivial combination (for example, $\eta_1 + \eta_2$) is Poisson.

**Definition.** Denote by $[\eta, \eta']$ the vector-valued symmetrical bilinear form

$$\eta, \eta' \in \Gamma\left(\Lambda^2 TY\right) \mapsto [\eta, \eta'] \in \Gamma\left(\Lambda^3 TY\right)$$

that corresponds to the quadratic form

$$\eta \mapsto [\eta, \eta] = \operatorname*{Alt}_{ijk} \eta^{il} \eta^{jk}_{,l}.$$

This is a canonically defined map $S^2\left(\Gamma\left(\Lambda^2 TY\right)\right) \to \Gamma\left(\Lambda^3 TY\right)$, which is called *Schouten—Nijenhuis bracket.*

*Remark.* For convenience in what follows we will understand the bihamiltonian structure as two-dimensional linear family of Poisson structures parameterized by the fundamental space $\mathcal{S}$, $\eta.\colon \mathcal{S} \mapsto \Gamma\left(\Lambda^2 TY\right)$. The connection with previous (ordinary) definition arises after a choice of a basis in $\mathcal{S}$,

$$\eta_{(\lambda_1,\lambda_2)} = \lambda_1\eta_1 + \lambda_2\eta_2, \qquad (\lambda_1,\lambda_2) \in \mathcal{S}.$$

The interest of bihamiltonian structure lies in the theory of integrable systems, where a wide family of such structures arises in a natural way [1]. Inversely, the bihamiltonian structure can produce an "integrable system"[3] by the Lennard scheme [1], which makes it possible to construct sufficiently many Hamiltonians in involution.

The geometry of a bihamiltonian structure has been deeply investigated in the works of Magri [2] and Gelfand—Dorfman [3]. Unfortunately, these considerations were based on the hypothesis that one of the 2-vector[4] fields $\eta_1$, $\eta_2$ (say, $\eta_1$) is non-degenerated. However, in a lot of interesting examples (as in the one of the simplest one—the periodical Korteweg—de Vries system!) any linear combination of this two Poisson structures is degenerate. The linear algebra arising in *one* cotangent space in such a case was examined in [4]. It differs radically from the situation in the non-degenerated case and resembles greatly the situation in an odd-dimensional space. (It is clear that in the last case *any* 2-vector is degenerate.)

In this example we consider a bivector in $T_yY$ as a skewsymmetric bilinear form on $T_y^*Y$. In the case of Korteweg—de Vries system $Y$ is the space $V$ of functions with the period 1. We will identify the space $T_y^*Y$ with the same space using the scalar product

$$(f,g) \mapsto \int_0^1 f(x)\, g(x)\; dx.$$

After these preparations we can write two corresponding bilinear forms as

$$(f,g) \overset{\eta_1}{\mapsto} \int_0^1 f(x)\left(-\partial^3 + 2u(x)\circ\partial + 2\partial\circ u(x)\right) g(x)\; dx,$$
$$(f,g) \overset{\eta_2}{\mapsto} 4\int_0^1 f(x)\,\partial g(x)\; dx, \qquad \partial = \partial/\partial x.$$

Any linear combination of these forms is degenerate, since

$$\left(-\partial^3 + 2u(x)\circ\partial + 2\partial\circ u(x)\right)\phi_1\phi_2 = 0$$

if $\phi_1$, $\phi_2$ are solutions of $\left(-\partial^2 + u(x)\right)\phi(x) = 0$, and $\phi_1\phi_2$ is periodic if $\phi_1$, $\phi_2$ are two Bloch solutions of this equation.

Here we will make the first step in studying of the differential geometry of a bihamiltonian structure on an odd-dimensional manifold. But first let us recall some simple facts concerning the linear algebra of two skew-symmetrical forms and the geometry of a Poisson manifold.

## Intermediate theorems

**Linear algebra connected with a pair of skew-symmetrical forms.** Let $\Omega_1$, $\Omega_2$ be a pair of skew-symmetrical forms on a finite-dimensional vector space $V$.

[3]We use here quotation marks since the notion of integrable system is too vague to denote something explicit.

[4]I.e., the element of 2nd skew-symmetrical power.

**Theorem** [4]. *In the above situation*

(1) *the triple* $(V, \Omega_1, \Omega_2)$ *can be transformed by a linear change* $V \simeq W \oplus \widetilde{W}^*$ *to the triple*

$$\left(W \oplus \widetilde{W}^*, \widetilde{\Omega}_1, \widetilde{\Omega}_2\right)$$

*where the 2-forms* $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$ *can be written as*

$$\widetilde{\Omega}_i\left(\left(w_1, w_1^*\right), \left(w_2, w_2^*\right)\right) = \left\langle \varphi_i\left(w_1\right), w_2^* \right\rangle - \left\langle \varphi_{i2}\left(w\right), w_1^* \right\rangle,$$

*and* $\varphi_1$, $\varphi_2$ *are two maps* $W \to \widetilde{W}$, $w_1, w_2 \in W_1$, $w_1^*, w_2^* \in \widetilde{W}^*$, *the brackets* $\langle , \rangle$ *denoting the pairing between* $\widetilde{W}$ *and* $\widetilde{W}^*$.

(2) *Moreover, let the space* $V$ *be of an odd dimension* $2k+1$ *and the forms* $\Omega_1$, $\Omega_2$ *be in general position. Then we can choose the maps* $\varphi_1$, $\varphi_2$ *as follows:*

$$\begin{aligned} \varphi_1\left(w_i\right) &= \widetilde{w}_{i+1/2}, \quad i < k, \qquad \varphi_1\left(w_{k+1}\right) = 0; \\ \varphi_2\left(w_i\right) &= \widetilde{w}_{i-1/2}, \quad i > 0, \qquad \varphi_2\left(w_0\right) = 0 \end{aligned}$$

*in the bases* $w_i$, $i = 0, 1, \dots, k$, $\widetilde{w}_j$, $j = 1/2, 3/2, \dots, k - 1/2$, *of spaces* $W$, $\widetilde{W}$ *(i.e., such pairs of form have no parameters up to coordinate change).*

(3) *The subspace* $W \subset V$ *in the last case is canonically defined and the map*

$$i \colon (l_1 : l_2) \mapsto \operatorname{Ker}\left(l_1 \Omega_1 + l_2 \Omega_2\right) \subset W$$

*considered as a map* $P^1 \to PW$ *is a Veroneze inclusion;*

(4) *The map*

$$j \colon (l_1 : l_2) \mapsto \varphi_1\left(\operatorname{Ker}\left(l_1 \Omega_1 + l_2 \Omega_2\right)\right) = \varphi_2\left(\operatorname{Ker}\left(l_1 \Omega_1 + l_2 \Omega_2\right)\right) \subset \widetilde{W}$$

*considered as a map* $P^1 \to P\widetilde{W}$ *(if* $l_1 = 0$ *then only the second expresion has sense, if* $l_2 = 0$ *only the second) is a Veroneze inclusion;*

(5) *hence the subspace* $W$ *is a linear subspace generated by* $\operatorname{Ker}\left(\lambda_1 \Omega_1 + \lambda_2 \Omega_2\right) \subset W$ *for* $(\lambda_1 : \lambda_2) \in P^1$.

*Remark.* The example of a pair of mappings from part (2) of the theorem is nothing else as a *Kroneker pair* of operators from one space to another [7]. *All the results* of this paper are based in fact on a careful study of this pair.

*Remark.* Note, how far is this situation from that in an even-dimensional space in general position, where the form $\Omega_1$ is non-degenerated, and $A = \Omega_1^{-1}\Omega_2$ is the canonically defined map $V \to V$ in general position[5]. All the invariants of this pair are the eigenvalues of $A$. On the contrary, in the odd-dimensional case there is no invariant at all!

*Remark.* As it is easy to see from the description of Veroneze inclusions as $\mathrm{SL}(\mathcal{S})$-modules (here $\mathrm{SL}(\mathcal{S})$ denotes a special linear group of $\mathcal{S}$), the another description of maps $\varphi_{1,2}$ can be obtained from considering the only $\mathrm{SL}(\mathcal{S})$-invariant map

$$S^k\mathcal{S} \otimes \mathcal{S} \to S^{k-1}\mathcal{S}.$$

**The geometry of a Poisson structure.** In what follows we will use only two facts from the geometry of a Poisson or symplectic manifold: the existence of a foliation with symplectic leaves and the geometry of a lagrangian foliation.

[5] Among maps with the double spectrum.

**Example.** Let $(Y,\omega)$ be a symplectic manifold, $\omega \in \Gamma\left(\Lambda^2 T^*Y\right)$. Then we can consider $\omega_y$ as a map $T_yY \to T_y^*Y: v \mapsto \omega_y(v,\cdot)$. Analogously the map $\omega_y^{-1}: T_y^*Y \to T_yY$ can be considered as a bivector from $\Lambda^2 T_yY$. We claim that this bivector field on $Y$ is a Poisson structure.

Inversely, let us consider a Poisson manifold $(Y,\eta)$. Then on an open subset of $Y$ where $\operatorname{rk}\eta$ is constant there exists a foliation $\mathcal{F}$ of dimension $\operatorname{rk}\eta$ and a symplectic structure $\omega_L$ on any leaf $L$ of $\mathcal{F}$ such that $i_*\left(\omega_L^{-1}\right) = \eta|_L$. Here $\omega_L^{-1} \in \Gamma\left(\Lambda^2 TL\right)$, $i_*$ is a map associated with the inclusion $i: L \hookrightarrow Y$, $|_L$ sends $\Gamma\left(Y, \Lambda^2 TY\right)$ into $\Gamma\left(L, i^*\left(\Lambda^2 TY\right)\right)$. That means, in particular, that the Poisson bracket associated with $\eta$

$$\{f,g\} = \eta\,(df, dg)$$

can be reduced to the Poisson bracket on leaves of $\mathcal{F}$:

$$\{f,g\}\,(y) = \{f\circ i_y, g\circ i_y\}_{L_y}(y)\,;$$

here $i_y: L_y \hookrightarrow Y$ is the inclusion of the containing $y$ leaf $L_y$, $\{\cdot,\cdot\}_{L_y}$ is the Poisson structure on the symplectic manifold $L_y$.

In what follows we will consider a family of Poisson structures. To denote a dependence of the foliation $\mathcal{F}$ on this variable structure we will denote it as $\mathcal{F}_\eta$.

*Remark.* Let us show how to construct this foliation. The bivector $\eta_y$ can be considered as a skew-symmetrical form on the cotangent space $T_y^*Y$. Therefore $\operatorname{Ker}\eta_y \subset T_y^*Y$. The conormal bundle to this foliation is nothing else but the field of kernels of the 2-vector field $\eta$. Being of constant rang on an open subset, this field of subspaces is therefore a bundle. Orthogonal complements to these spaces form a distribution $\widetilde{\mathcal{F}}$ of subspaces in $TY$. To prove that this distribution is integrable (i.e., of the form $T\mathcal{F}$) it is sufficient to apply the Frobenius theorem. Indeed, if $v_1, v_2 \in \widetilde{\mathcal{F}}_y$, then there exist two functions $f_1$, $f_2$ such that

$$v_i = \widetilde{\eta}_y\left(df_i|_y\right), \quad i = 1,2.$$

Here $\widetilde{\eta}_y$ is a mapping

$$\widetilde{\eta}_y: T_y^*Y \to T_yY, \qquad \left\langle \widetilde{\eta}_y(\omega_1), \omega_2\right\rangle = \eta_y(\omega_1,\omega_2), \quad \omega_1,\omega_2 \in T_y^*Y.$$

However, $\operatorname{Im}\eta_z = (\operatorname{Ker}\eta_z)^\perp$, so vector fields $v_i(z) = \widetilde{\eta}_z\left(df_i|_z\right)$, $i = 1,2$ are tangent to the distribution $\widetilde{\mathcal{F}}$, and the commutator of these fields

$$(*) \qquad \{v_1, v_2\}|_y = \widetilde{\eta}_y\left(d\{f_1,f_2\}|_y\right)$$

is also tangent to $\widetilde{\mathcal{F}}$, therefore the Frobenius theorem is applicable[6].

A submanifold $L$ of a symplectic manifold $(Y,\omega)$ is called a *lagrangian submanifold* if $\omega|_{TL} = 0$, $2\dim L = \dim Y$. A foliation $\mathcal{F}$ on a symplectic manifold is called a *lagrangian foliation* if any leaf of $\mathcal{F}$ is a lagrangian submanifold.

One of the main reasons to study lagrangian foliation is to find what additional information is contained in the symplectic structure on conormal manifold $Y = T^*X$ comparing with a generic symplectic manifold. The obvious additional structure is the projection $T^*X \to X$ with lagrangian fibers and the zero section $X \to T^*X$. It appears that this is almost all the additional information.

[6]The formula (*), that establishes the correspondence between the Poisson bracket on functions and the commutator of vector fields, is an easy reformulation of the Jacoby identity.

**Theorem.** *Let $\mathcal{F}$ be a lagrangian foliation on $Y$. Then*

(1) *on leaves of $\mathcal{F}$ a canonical affine structure (i.e., a local identification with an open subset in an affine space) is defined.*
(2) *If $\widetilde{L}$ is a lagrangian submanifold of $Y$ locally transversal to $\mathcal{F}$, then some neighborhood $U$ of $\widetilde{L}$ is canonically identified with an open subset of $T^{*}\widetilde{L}$. This identification preserves the symplectic structures. The leaves of the foliation are identified with the fibers of this bundle, the affine structures being the same.*
(3) *If a map $f: Y \to Y$ preserves any leaf of the foliation $\mathcal{F}$, then the restriction of $f$ on a leaf is a translation in the corresponding affine structure, provided that $f$ preserves the symplectic form $\omega$. If $\widetilde{L}$ is a transversal to $\mathcal{F}$ lagrangian submanifold, then the vectors of the translations can be considered as a section of $T^{*}\widetilde{L}$ (see previous assertion of this theorem). For a leafwise translation $f$ to preserve the symplectic structure this section should be closed as an 1-form on $\widetilde{L}$.*

*Proof.* Although this theorem is standard in symplectic geometry, we give here some hints about it's proof. The reason is that in what follows there appears a lot more complicated version of the same arguments, so it is useful to discuss them in an easier case beforehand.

Let $B$ be a *local base* of the foliation $\mathcal{F}$. We mean that $Y$ can be represented locally as a direct product $Y = F \times B$ with leaves of $\mathcal{F}$ being the fibers of the projection on the second argument. Let $\pi: Y \to B$ be a corresponding projection. Then the condition of the foliation being lagrangian implies that the form $\omega$ considered as a map $T_{y}Y \to T_{y}^{*}Y$ sends $T_{y}\mathcal{F}$ to $T_{\pi(y)}^{*}B$ isomorphically. Hence for $b \in B$ all the tangent spaces to a leaf $L = \pi^{-1}(b)$ of $\mathcal{F}$ are identified. Hence, on $L$ is defined a flat connection $\nabla$.

To show that this connection has no torsion, consider two covectors $\alpha_{1}, \alpha_{2} \in T_{b}^{*}B$. Then for $y \in L$ the map $\omega_{y}^{-1}: T_{b}^{*}B \to T_{y}L$ sends them to vectors $v_{1}(y)$, $v_{2}(y)$, that form two vector fields on $L$. These fields being constant respective to $\nabla$, their commutator corresponds to the value of the torsion on these vector fields.

To compute this commutator, we use again the correspondence (*) between the Poisson bracket on functions and the commutator of vector fields. Let $\varphi_{1}$, $\varphi_{2}$ be two functions on $B$ satisfying

$$d\varphi_{i}|_{b} = \alpha_{i}, \quad i = 1, 2.$$

Then for $y \in L$

$$[v_{1}, v_{2}]|_{y} = \omega_{y}^{-1}\{\pi \circ \varphi_{1}, \pi \circ \varphi_{2}\}|_{y}.$$

Applying again the condition of the foliation being lagrangian, we find out that the last Poisson bracket is 0. Hence the connection $\nabla$ determines an affine structure on $L$. Evidently, the associated vector space coincides with $T_{b}^{*}B$.

The choice of a transversal submanifold marks a point in an affine space of any leaf $L$. Hence taking this point as 0 we can identify locally $L$ and $T_{x}^{*}\widetilde{L}$, $x$ being $L \cap \widetilde{L}$. We used a natural identification of $\widetilde{L}$ and $B$. So locally $Y$ is identified with $T^{*}\widetilde{L}$.

Since $\widetilde{L}$ is a lagrangian submanifold, the symplectic forms on $Y$ and $T^{*}\widetilde{L}$ coincide in the points of $\widetilde{L}$. Really, for arbitrary transversal in any point they coincide (being both 0) on tangent to $\mathcal{F}$ (or fibers) vectors, coincide by construction of an affine structure on pairs of vectors one of which is tangent to $\mathcal{F}$ (indeed, we can change the second vector to a tangent to $B = \widetilde{L}$ vector). Moreover, in points of $\widetilde{L}$ they coincide (being both 0) on tangent to $\widetilde{L}$ (or the zero section) vectors, what proves the assertion.

Now we can proliferate this coincidence to any point of $Y$ using *Hamiltonian flows.* Let $f$ be a function on a symplectic manifold $Z$. Then the vector field $\omega^{-1}(df)$ preserves the form $\omega$:

$$\mathcal{L}_{\omega^{-1}(df)} \cdot \omega = 0.$$

Therefore the *flow* $T^{t}$ of this vector field also preserves $\omega$:

$$\left(T^{t}\right)^{*} \omega = \omega.$$

Taking as $f$ pushed-up from the same function on $\widetilde{L}$ functions on $Y$ and $T^*\widetilde{L}$ we conclude that the corresponding vector fields coincide, hence their flows also coincide. Hence the set of points on $Y = T^*\widetilde{L}$ where two symplectic forms coincide is stable respective to these flows along the foliation. Therefore they coincide everywhere. In fact we have defined an identification of $T^*B$ and $Y$ basing on a lagrangian transversal to $\mathcal{F}$.

If $\widetilde{L}_1$, $\widetilde{L}_2$ are two transversal lagrangian submanifolds, then we get two identifications of $T^*B$ and $Y$, hence an automorphism of $(Y, \omega)$ sending $\widetilde{L}_1$ onto $\widetilde{L}_2$. Inversely, any automorphism of $(Y, \omega)$ sends a lagrangian submanifold to a lagrangian submanifold.

Therefore to prove the last part of the theorem it suffices to prove that the graph of an 1-form $\alpha$ (i.e., the submanifold $\{(b, \alpha|_b)\} \subset T^*B$) is a lagrangian submanifold iff $\alpha$ is closed. However, it is a special case of a more general formula[7]

$$\Phi_\alpha^*\omega = d\alpha, \qquad \Phi_\alpha: B \to T^*B: b \mapsto \alpha|_b \in T_b^*B.$$

□

*Remark.* This theorem can easily be restated for the case of a Poisson manifold. For this we should change a lagrangian foliation to a foliation whose leaves are contained in the symplectic leaves and whose restriction on any symplectic leaf is a lagrangian foliation. A lagrangian transversal submanifold changes to an *coisotropic* transversal, i.e., a submanifold $\widetilde{L}$ such that the form $\eta$ is 0 on any two conormal to $\widetilde{L}$ covectors. In what follows we will freely use this generalization.

## Two maps

**Reduction of a bihamiltonian structure.** Let us consider a bihamiltonian structure on a manifold $Y$. In each cotangent space $T_y^*Y$ a pair of skew-symmetrical bilinear forms is defined. Let us call a point $y \in Y$ a regular point, if the values of Poisson structures in this point $\eta_1|_y, \eta_2|_y \in \Lambda^2 T_y Y$ are in general position as a pair of bilinear skewsymmetrical forms on $T_y^*Y$. The regular points form an open subset of $Y$.

From now on we suppose that $\dim Y$ is odd, $\dim Y = 2k+1$. We will consider the subset of regular points only[8], so we suppose that *all* $Y$ consists of regular points only. Call such a bihamiltonian structure *a structure in general position.* In this case the theorem on linear algebra shows that in any cotangent space $T_y^*Y$ there is a canonically defined subspace $W_y$.

This subspace is generated by $\operatorname{Ker}(l_1\eta_1 + l_2\eta_2)$, hence its orthogonal complement $W_y^\perp$ is the intersection of different $\operatorname{Ker}(l_1\eta_1 + l_2\eta_2)^\perp$. But the last spaces are tangent spaces to the foliations associated with Poisson structures $l_1\eta_1 + l_2\eta_2 = \eta_\lambda$, $\lambda = (l_1, l_2)$. Moreover, since one-dimensional subspaces $\operatorname{Ker}(l_1\eta_1 + l_2\eta_2)$ form a Veroneze curve, any $k+1$ of them corresponding to $k+1$ different values $\lambda_1, \lambda_2, \dots, \lambda_{k+1} \in P\mathcal{S}$ of $(l_1 : l_2)$ form a basis in $W_x$. Therefore the corresponding foliations $\mathcal{F}_{\lambda_1}, \mathcal{F}_{\lambda_2}, \dots, \mathcal{F}_{\lambda_{k+1}}$ intersect themselves transversally and the tangent space to this intersection $\overline{\mathcal{F}}$ at $y$ is $W_y^\perp$. We come to the following

**Theorem.** *The subspaces $W_y \subset T_y^*Y$ form orthogonal complements to a foliation $\overline{\mathcal{F}}$. On the base of this foliation a Veroneze web is canonically defined.*

*Proof.* The first part is already established. To prove the second it suffices to note that $\overline{\mathcal{F}}$ being a sub-foliation of any foliation $\mathcal{F}_\lambda$, $\mathcal{F}_\lambda$ can be pulled down to a foliation on the (local) base $X$ of the foliation $\overline{\mathcal{F}}$. The resulting foliation on $X$ will be denoted by the same letter—this must not lead to a misunderstanding. If we fix a point $y \in Y$ then for its image $x = \pi(y)$ on $X$ (here $\pi$ being the projection $Y \to X$) the space $T_x^*X$ is canonically isomorphic to $W_y$. Hence the orthogonal complements to the leaves of foliations $\mathcal{F}_\lambda$ on $X$ correspond by this isomorphism to $\operatorname{Ker} \eta_\lambda$. Therefore they form a Veroneze curve. □

[7]In what follows we find several more formulae with similar sense. Such formulae can be, therefore, a part of some general formalism.
[8]Note, that in general case this subset can be empty.

**Definition.** This locally defined Veroneze web $(X, \mathcal{F}_\lambda)$ is called *the reduction of bihamiltonian manifold* $(Y, \eta_\lambda)$, $(X, \mathcal{F}_\lambda) = \mathfrak{R}_Y = \mathfrak{R}_{(Y, \eta_\lambda)}$.

**A reverse map.** Here we will show how to construct a bihamiltonian structure basing on a Veroneze web $X$.

Recall, that $\mathrm{SL}(\mathcal{S})$ denotes a special linear group of the space $\mathcal{S}$.

**Lemma.** *On any tangent to $X$ space $T_x X$ the action of* $\mathrm{SL}(\mathcal{S})$ *is canonically defined.*

*Proof.* This is already discussed in the section on Veroneze curves. □

Now we are going to introduce a notion of *associated bundles* to the tangent bundle to $X$. Begin with a definition of an operation over vector spaces.

Let $V$ be an irreducible $\mathrm{SL}(\mathcal{S})$-module with the highest weight $k$ and $l \in \mathbb{Z}$, $l \geq -k$. Let $V^{(l)}$ denote the only $\mathrm{SL}(\mathcal{S})$-irreducible component of $V \otimes S^{|l|}\mathcal{S}$ with the highest weight $k + l$. It is clear that $V^{(0)} \simeq V$ canonically and that there exists a canonical morphism $i_l : V \otimes S^{|l|}\mathcal{S} \to V^{(l)}$.

**Definition.** Denote a vector bundle formed by the vector spaces $T_x^{*(l)} X \stackrel{\text{def}}{=} (T_x^* X)^{(l)}$ as $T^{*(l)} X$.

In fact we have considered the bundle $T^* X$ as a $\mathrm{SL}(\mathcal{S})$-bundle and realized a *change of a fiber* using the described above explicit construction[9].

We will define the underlying manifold of the bihamiltonian structure corresponding to a Veroneze web $(X, \mathcal{F}_\lambda)$ as follows.

**Definition.** Let $Y_X = T^{*(-1)} X$ as a manifold. (Two Poisson structures on $Y$ will be defined later.)

To define the Poisson structures on $Y_X$, let us first note that since a Poisson structure on the cotangent bundle *is* canonically defined, on a cotangent bundle $T^* \mathcal{F}$ for a foliation $\mathcal{F}$ (which is a union of cotangent spaces to foliation leaves) a Poisson structure is also defined as on a union of Poisson manifolds (see the section on Poisson structures). Therefore a Poisson structure $\widetilde{\eta}_\lambda$ is defined on $T^* \mathcal{F}_\lambda$. It remains only to establish a connection between $T^* \mathcal{F}_\lambda$ and $T^{*(-1)} X$.

Let $V$ be an $\mathrm{SL}(\mathcal{S})$-module, $0 \neq \lambda \in \mathcal{S}$ and $\widetilde{\lambda}$ be the corresponding point in $P\mathcal{S}$, $B = \mathrm{Stab}\,\widetilde{\lambda}$ be a Borel subgroup in $\mathrm{SL}(\mathcal{S})$ and $w \in V$ is a $B$-highest-weight vector. Then the map

$$V \to V^{(-1)} : v \mapsto i_{-1}(v \otimes \lambda), \quad \text{where } i_l : V \otimes S^{|l|}\mathcal{S} \to V^{(l)},$$

sends $V/w$ to $V^{(-1)}$ isomorphically. Therefore for every $\lambda$ the space $T_x^* \mathcal{F}_\lambda$ is mapped to the space $T_x^{*(-1)} X$ as a factor of $T_x^* X$ by the space spanned by a highest weight vector.

This identification gives us a Poisson structure on $T^{*(-1)} X$ for any fixed $\lambda \in V$. Since this identification is of homogeneity degree 1 in $\lambda$, the corresponding structure $\eta_\lambda$ is of homogeneity degree 1 in $\lambda$. Therefore, if $\alpha_1$, $\alpha_2$ are fixed, then $\eta_\lambda(\alpha_1, \alpha_2)$ is a function on $\mathcal{S} \setminus \{0\}$ of homogeneity degree 1. Being algebraic, $\eta_\lambda$ is a linear function in $\lambda$. So we get a bihamiltonian structure on $Y_X = T^{*(-1)} X$.

**Definition.** Let $e_1, e_2$ be a basis of $\mathcal{S}$. Then a *corresponding to $X$ bihamiltonian structure* $Y_X$ is defined as $\left(T^{*(-1)} X, \eta_{e_1}, \eta_{e_2}\right)$.

*Remark.* It is easy to establish $\eta_\lambda$ being algebraic in $\lambda$, since it depends only on the $m$-jet of Veroneze web, where $m$ is finite. Since the theorem on linear algebra allows the field extension, the above construction can be applied over the algebraic closure of the field in question. Therefore this algebraic function has no singularity.

Thus, to every Veroneze web on a manifold $X$ of dimension $k + 1$ we associate a bihamiltonian structure on manifold $Y_X$ of dimension $2k + 1$. It is clear that the foliation on $Y_X$ associated with the Poisson bracket $\eta_\lambda$ is the pull-up of the foliation $\mathcal{F}_\lambda$ on $X$. Therefore the reduction of this bihamiltonian structure is the original Veroneze web on $X$. Hence we have constructed a right inverse map to the reduction map $Y \to \mathfrak{R}_Y$. Our next task is to

[9]This explicit construction make it possible to define geometric objects on the bundle with a changed fiber.

show that locally this map is completely inverse to reduction, i.e., every bihamiltonian manifold of odd dimension in general (in the specified above sense) position can be (locally) obtained basing on this construction.

*Remark.* This being a key definition of this article, we wanted to give a clearer construction. However, we couldn't. So this definition remains mystique. If a reader isn't satisfied, he can try to construct these bivector field in a local frame. We couldn't do this since it is difficult to understand which local frame is appropriate for consideration of a Veroneze web.

We can easily define this two bivector fields in points on the zero section[10]. To define them in the points outside this section we could consider the translations along the fibers of this bundle that preserve the bihamiltonian structure we want to define[11]. However, although we know that there is sufficiently many such translations, we cannot justify if *right now.*

## Double complex

**The bicomplex.** The identification of $T^{*(-1)}X$ with $T^{*}\mathcal{F}_{\lambda}$ enables us to construct a remarkable complex of differential operators on $X$. To do this let us consider $\lambda\in\mathcal{S}$ and the foliation $\mathcal{F}_{\lambda}$ on $X$. The identification

$$T^{*(-1)}X\simeq T^{*}\mathcal{F}_{\lambda}$$

of homogeneity degree $-1$ in $\lambda$ gives rise to an identification

$$\Lambda^{m}T^{*(-1)}X\simeq\Omega^{m}\mathcal{F}_{\lambda}$$

of homogeneity degree $-m$ in $\lambda$. (Here $\Omega^{m}\mathcal{F}_{\lambda}$ denotes $\Lambda^{m}T^{*}\mathcal{F}_{\lambda}$.) Therefore the operator $d$ of exterior differentiation induces a first order differential operator

$$d_{\lambda}:\Gamma\left(\Lambda^{m}T^{*(-1)}X\right)\to\Gamma\left(\Lambda^{m+1}T^{*(-1)}X\right)$$

of homogeneity degree 1 in $\lambda$. Hence the above consideration shows that this operator must be linear in $\lambda$. Since $d_{\lambda}^{2}\equiv 0$,

$$d_{\lambda}d_{\mu}+d_{\mu}d_{\lambda}=\left(d_{\lambda}+d_{\mu}\right)^{2}-d_{\lambda}^{2}-d_{\mu}^{2}=0$$

for any $\lambda,\mu\in\mathcal{S}$. Therefore for fixed $\lambda$ and $\mu$ we get a bicomplex

$$\begin{array}{ccccccc}
\dots & & \dots & & \dots & & \\
\uparrow{\scriptstyle d_{\mu}} & & \uparrow{\scriptstyle d_{\mu}} & & \uparrow{\scriptstyle d_{\mu}} & & \\
\Lambda^{2}T^{*(-1)}X & \xrightarrow{d_{\lambda}} & \Lambda^{3}T^{*(-1)}X & \xrightarrow{d_{\lambda}} & \Lambda^{4}T^{*(-1)}X & \xrightarrow{d_{\lambda}} & \dots \\
\uparrow{\scriptstyle d_{\mu}} & & \uparrow{\scriptstyle d_{\mu}} & & \uparrow{\scriptstyle d_{\mu}} & & \\
\Lambda^{1}T^{*(-1)}X & \xrightarrow{d_{\lambda}} & \Lambda^{2}T^{*(-1)}X & \xrightarrow{d_{\lambda}} & \Lambda^{3}T^{*(-1)}X & \xrightarrow{d_{\lambda}} & \dots \\
\uparrow{\scriptstyle d_{\mu}} & & \uparrow{\scriptstyle d_{\mu}} & & \uparrow{\scriptstyle d_{\mu}} & & \\
\Lambda^{0}T^{*(-1)}X & \xrightarrow{d_{\lambda}} & \Lambda^{1}T^{*(-1)}X & \xrightarrow{d_{\lambda}} & \Lambda^{2}T^{*(-1)}X & \xrightarrow{d_{\lambda}} & \dots
\end{array}.$$

This bicomplex will play a crucial role in what follows.

[10]Really, the tangent space to a linear bundle at such a point is a canonical direct sum of a tangent space to the base and of the fiber space. We can denote the tangent space to the base as $V$. Then the tangent space to the bundle is $W=V\oplus\left(V^{*}\right)^{(-1)}=V\oplus\left(V^{(-1)}\right)^{*}$. Hence $\Lambda^{2}W=\Lambda^{2}V\oplus V\otimes\left(V^{(-1)}\right)^{*}\oplus\Lambda^{2}\left(V^{*}\right)^{(-1)}$. The both bivectors lie in the middle addend, hence they correspond to maps $V\to V^{(-1)}$. If we consider this pair of bivectors as an element of $\mathcal{S}\otimes\Lambda^{2}V$, then it corresponds to a natural projection map $\mathcal{S}\otimes V\to V^{(-1)}$.

[11]For example, it is one of the simplest possible ways to define a Poisson (or symplectic) structure on a cotangent bundle $TZ$. In this case the allowed translations "correspond" to closed 1-forms on $Z$—see section on the Poisson geometry.

**Objects of differential geometry.** A notion of *an object of differential geometry* lingered for a long time and was introduced (more or less explicitly) in the works [5].

**Definition.** Let $\mathfrak{D}$ be a $\mathbb{Z}$-graded superalgebra[12] with only three graded components: $\mathfrak{D}_{-1}$, $\mathfrak{D}_0$ and $\mathfrak{D}_1$. *An object of differential geometry* is a $\mathbb{Z}$-graded representation of $\mathfrak{D}$.

**Example.** The crucial example of an object of differential geometry is the de Rham complex $\Omega^*$ for a manifold $Z$. Here $\mathfrak{D}_1$ is generated by a single element $d$, which acts as the exterior differentiation on $\Omega^*$. Both spaces $\mathfrak{D}_{-1}$ and $\mathfrak{D}_0$ are identified with the space $\mathrm{Vect}\,(Z)$ of global vector fields on $Z$. $\mathfrak{D}_{-1}$ acts on $\Omega^*$ by the inner multiplication and $\mathfrak{D}_0$ acts by Lie differentiation. The Leibnitz identity

$$\mathcal{L}_v = i_v \circ d + d \circ i_v,$$

where $\mathcal{L}_v$ is Lie differentiation and $i_v$ is the inner multiplication together with

$$i_{v_1} i_{v_2} + i_{v_2} i_{v_1} = 0, \quad d^2 = 0$$

shows that $\mathfrak{D}$ with the structure of Lie superalgebra inherited from action on $\Omega^*$ does satisfy the conditions of definition.

The notion of an object of differential geometry is an alternative to the description of geometry based on the ring structure on the space of (global) functions. The latter structure seems exceeding in the case, say, of the variational calculus where the functions in question (i.e., integrals of local functionals) don't allow the multiplication.

Though the spaces $\mathfrak{D}_{-1}$ and $\mathfrak{D}_1$ are completely symmetrical in the above definition, they are very different in the example: $\mathfrak{D}_{-1}$ is infinite-dimensional and the $\mathfrak{D}_1$ is one-dimensional. It is easy to see that the procedure described in the previous section gives an example of an object of differential geometry with *the two-dimensional* space $\mathfrak{D}_1$. Indeed, let $\mathfrak{D}_{-1}$ be the space of $T^{(-1)}X = \left(T^{*(-1)}X\right)^*$ sections and let $\mathfrak{D}_1$ be generated by $d_\lambda$ and $d_\mu$. Then they are both acting on $\Lambda^* T^{*(-1)}X$: the first one by inner multiplication, the second one by differentiation. Since

$$i_{v_1} i_{v_2} + i_{v_2} i_{v_1} = 0, \quad d_\lambda^2 = d_\mu^2 = d_\lambda d_\mu + d_\mu d_\lambda = 0,$$

these spaces of operators in $\Lambda^* T^{*(-1)}X$ generate together a $\mathbb{Z}$-graded superalgebra with components in the gradings $-1$, 0 and 1 only. Thus the structure of a differential geometry object is defined on $\Lambda^* T^{*(-1)}X$.

**Double cohomology.**

**Definition.** A double complex is a graded linear space $\oplus A^i$, $i \in \mathbb{Z}$, and a differential $d_\lambda$ on $A^*$ that depends linearly on a vector $\lambda \in \mathcal{S}$. Here $\mathcal{S}$ is a fixed two-dimensional space.

*Remark.* The identity $d_\lambda^2 = 0$ implies that

$$d_\mu d_\lambda + d_\lambda d_\mu = 0.$$

We define double cohomology of a double complex $(A^*, d_\lambda)$ as

$$\mathbb{H}^i\,(A) = \frac{\mathrm{Ker}\, d_{\nu_1} d_{\nu_2} \colon A^i \to A^{i+2} \otimes \Lambda^2 \mathcal{S}^*}{\mathrm{Im}\, d_\nu \colon A^{i-1} \otimes \mathcal{S} \to A^i}.$$

[12] It means that the even and odd parts of $\mathfrak{D}$ are the sum of components with even and odd grading:

$$\mathfrak{D}^+ = \sum_i \mathfrak{D}_{2i}, \quad \mathfrak{D}^- = \sum_i \mathfrak{D}_{2i+1}.$$

*Remark.* There is another related definition of double cohomology:

$$\widetilde{\mathbb{H}}^{i}(A) = \frac{\operatorname{Ker} d_{\nu}\colon A^{i} \to A^{i+1} \otimes \mathcal{S}^{*}}{\operatorname{Im} d_{\nu_1} d_{\nu_2}\colon A^{i-2} \otimes \Lambda^{2}\mathcal{S} \to A^{i}}.$$

The definition we have chosen seems a little more useful.

Clearly for fixed $\lambda \in \mathcal{S}$ there is a natural map $d_{\lambda} : \mathbb{H}^{i-1}(A) \to \widetilde{\mathbb{H}}^{i}(A)$. If $(A^{*}, d_{\lambda})$ for every $\lambda \neq 0$ is exact in the degree $i$, then this map is isomorphism for this $i$. Really, if $d_{\lambda}a^{i} = 0$, $\lambda \in \mathcal{S}$, then for any $0 \neq \nu \in \mathcal{S}$ there exists $b_{\nu}^{i-1}$ such that $d_{\nu}b_{\nu}^{i-1} = a^{i}$. Since $d_{\lambda}d_{\mu}b_{\nu}^{i-1} = 0$ for $\lambda, \mu, \nu \in \mathcal{S}$, the considered map is surjective. The injectivity of this map is obvious.

## Twist of a bihamiltonian structure

We have built a bihamiltonian structure associated to a Veroneze web on a manifold. In fact, there exists a whole family of such structures, all of them having the same reduction. To construct a member of this family it will be rather useful to introduce the notion of *a twisted bihamiltonian structure*.

**Definition.** Let $(Y, \eta_{\lambda})$ be a bihamiltonian manifold. Let $X = \mathfrak{R}_{Y}$ be its reduction and let $f$ be a function on $X$ considered as a section of $\Lambda^{0}T^{*(-1)}X$. Suppose $f$ be a "double cocycle" in the complex associated with the above bicomplex, i.e., $d_{\lambda}d_{\mu}f = 0$. Let us call the bihamiltonian manifold $\left(Y, \lambda_{1}\eta_{1} + \lambda_{2}\mathfrak{T}_{d_{\mu}f*}\eta_{2}\right)$ *the twist of $Y$ with the cocycle $f$*. Here $\eta_{1}$, $\eta_{2}$ are two Poisson structures associated with the basis of $\mathcal{S}$, $\lambda_{1}$, $\lambda_{2}$ are the corresponding coordinate functions in this basis, $\mathfrak{T}_{\phi}$ for a section $\phi$ of a linear bundle $\mathcal{B}$ denotes the translation on this section in the bundle:

$$\mathfrak{T}_{\phi}\left((x, v)\right) = \left(x, v + \phi(x)\right), \qquad v \in \mathcal{B}_{x}.$$

*Remark.* To prove that both bivectors fields form Poisson structures it is sufficient to note that the direct image of a Poisson structure is a Poisson structure again. However, to prove the agreement of these structures between themselves is much more difficult. The proof will be given in the next section, after an extensive study of the geometry of a bihamiltonian manifold.

## Geometry of a bihamiltonian structure

First, we will study geometry of a leaf of the foliation we have defined on a bihamiltonian manifold.

**An affine structure.** The first target is to prove the following

**Theorem.** *On each leaf of foliation $\widetilde{\mathcal{F}}$ on bihamiltonian manifold $Y$ a canonical affine structure can be defined*[13].

*Proof.* Since $\eta_{\lambda}$ maps the conormal space $N_{y}^{*}\widetilde{\mathcal{F}}_{y}$ for the leaf $\widetilde{\mathcal{F}}_{y}$ of foliation $\widetilde{\mathcal{F}}$ which contains point $y \in Y$ into the tangent space $T_{y}\widetilde{\mathcal{F}}_{y}$ for this leaf:

$$\eta_{\lambda}\colon N_{y}^{*}\widetilde{\mathcal{F}}_{y} \to T_{y}\widetilde{\mathcal{F}}_{y}$$

(compare with the theorem on linear algebra) and since $N_{y}^{*}\widetilde{\mathcal{F}}_{y}$ coincides with $T_{x}^{*}X$ (where $x = \pi(y)$, $\pi\colon Y \to \mathfrak{R}_{Y} = X$), the tangent spaces to $\widetilde{\mathcal{F}}_{y}$ at different points are all identified with $T_{x}^{*}X/\operatorname{Ker}(\eta_{\lambda})$, where $\eta_{\lambda}$ is considered as a map $T_{x}^{*}X \to T_{y}\widetilde{\mathcal{F}}_{y}$. Since this kernel is just $N_{x}^{*}\mathcal{F}_{\lambda}$ (here $\mathcal{F}_{\lambda}$ is considered earlier foliation on $X = \mathfrak{R}_{Y}$) and doesn't depend on the point on $\widetilde{\mathcal{F}}_{y} = \pi^{-1}(x)$, we obtain a canonical identification of these tangent spaces at different points of $\widetilde{\mathcal{F}}_{y}$. Thus a flat affine connection on $\widetilde{\mathcal{F}}_{y}$ is canonically defined. Hence the only thing to prove now is to show that the torsion of this connection equals to 0.

Let $f$, $g$ be two functions on $X$. Then

$$\left\{\pi^{*}(f), \pi^{*}(g)\right\}_{\lambda} = 0$$

[13]It is this structure that allows considering such a manifold as an integrable system.

since $d(\pi^*(f))$ and $d(\pi^*(g))$ both lie in the space $W$ from the theorem on linear algebra, which is isotropic relative to any bilinear form. Therefore the vector fields $L^\lambda_{\pi^*(f)}$, $L^\lambda_{\pi^*(g)}$ (where

$$L^\lambda_\varphi \cdot \psi = \{\varphi, \psi\}_\lambda, \quad \varphi \text{ and } \psi \text{ being functions on } Y)$$

commute. By the above consideration these vector fields are constant along $\widetilde{\mathcal{F}}_y$ relative to the considered connection, hence this commutator is the value of the torsion on these vectors. Since the vectors of the specified type fill the whole tangent space to $\widetilde{\mathcal{F}}_y$, the torsion is identically 0, what proves the theorem, if we show that the defined structure doesn't depend on $\lambda \in \mathcal{S}$.

Anyway, for fixed $\lambda$ on the reduction $X = \mathfrak{R}_Y$ is defined a canonical bundle with an affine fiber. Let us consider the associated vector bundle $\mathfrak{L}$.

**Lemma.** *This vector bundle is canonically isomorphic to $T^{*(-1)}X$. This isomorphism doesn't depend on $\lambda$.*

*Proof.* Fix a point $y \in Y$, let $x = \pi(y)$. Then $\eta_\lambda$ determines a map $T^*_x X \otimes \mathcal{S} \to T_y \widetilde{\mathcal{F}}$. For any $\lambda \in \mathcal{S}$ the subspace $\operatorname{Ker}(\eta_\lambda) \otimes \lambda$ lies in the kernel of this map. Moreover, the kernel is generated by these subspaces. Consider the spaces $T^*_x X$, $\mathcal{S}$ and $T_y \widetilde{\mathcal{F}}$ as $\mathrm{SL}(\mathcal{S})$-modules. The structures of a module on $T^*_x X$ and on $T_y \widetilde{\mathcal{F}}$ are defined by the Veroneze curves in these spaces: $\lambda \mapsto \operatorname{Ker}(\eta_\lambda)$ in the first and $\lambda \mapsto \operatorname{Im}(\operatorname{Ker}(\eta_\lambda) \otimes \mathcal{S})$ in the second. A remark after the theorem on linear algebra shows that the map $T^*_x X \otimes \mathcal{S} \to T_y \widetilde{\mathcal{F}}$ is $\mathrm{SL}(\mathcal{S})$-covariant. Since the domain of this map is a sum of two irreducible components, this map is essentially the projection on one of this components – $(T^*_x X)^{(-1)}$, i.e., there is an identification

$$(T^*_x X)^{(-1)} \simeq T_y \mathcal{F}.$$

The above discussion shows that this identification taken at different points of $\pi^{-1}(x)$ leads to the same affine structure on $\pi^{-1}(x)$. Hence there is an identification $(T^*_x X)^{(-1)} \to \mathfrak{L}$. □

This shows that affine structure on leaves really doesn't depend on $\lambda \in \mathcal{S}$. □

This identification transfers the bihamiltonian structure on $(T^*X)^{(-1)}$ to a bihamiltonian structure on $\mathfrak{L}$.

We can apply this construction to the example of a bihamiltonian structure—to the total space of the bundle $T^{*(-1)}X$. It is easy to see that in this case we get nothing new[14]:

**Lemma.** *The considered above affine structure on the fibers of the bundle $T^{*(-1)}X$ is associated with the linear structure on this bundle.*

In fact now we have shown

(1) how to construct a new bihamiltonian manifold $T^{*(-1)}\mathfrak{R}_Y$ basing on a bihamiltonian manifold $Y$;
(2) that this operation is an idempotent operation;
(3) that the old manifold is connected with the new as an bundle with an affine fiber is connected with the associated vector bundle.

In fact we want to show that these bihamiltonian manifolds are isomorphic. Both this manifold are affine foliations over the same base. To identify them it is sufficient to choose a section of $Y \to X$ that corresponds to the zero section of $T^{*(-1)}X \to X$.

**A connection with lagrangian foliations.** Let us first fix $\lambda \in \mathcal{S}$ and consider *one* Poisson structure $\eta_\lambda$. Any leaf of the corresponding foliation $\mathcal{F}_\lambda$ is fibered (by $\widetilde{\mathcal{F}}$) over $X$ (with the corresponding foliation denoted by the same symbol $\mathcal{F}_\lambda$). Since any two functions on $X$ are in involution and the required restraints on dimensions are satisfied, the fibers of this foliation are lagrangian inside leaves of the first foliation. It is known that Hence these (lagrangian) fibers are equipped with a canonical affine structure.

[14] Let us recall that $\mathfrak{R}_{T^{*(-1)}X} = X$.

Now we are going to show that in fact these two defined on leaves of $\widetilde{\mathcal{F}}$ affine structures coincide. To do this, it is sufficient to note that the first identification connects $T_y\widetilde{\mathcal{F}}_y$ with $T_x^*X/\operatorname{Ker}(\eta_\lambda)$, where $\eta_\lambda$ is considered as a map $T_x^*X \to T_y\widetilde{\mathcal{F}}_y$, $y = \pi(x)$. The second one connects this space with $T_x^*\mathcal{F}_{\lambda,x}$, (here $\mathcal{F}_{\lambda,x}$ is the containing the point $x$ leaf of the foliation $\mathcal{F}_\lambda$ on $X$). These two spaces are obviously canonically isomorphic and this isomorphism makes the diagram

$$\begin{array}{ccc} T_y\widetilde{\mathcal{F}}_y & \longrightarrow & T_x^*X/\operatorname{Ker}(\eta_\lambda) \\ \Big\| & & \downarrow \\ T_y\widetilde{\mathcal{F}}_y & \longrightarrow & T_x^*\mathcal{F}_{\lambda,x} \end{array}$$

commutative, which proves the assertion.

**Corollary.** *Let us fix two points $\lambda, \mu \in \mathcal{S}$. Then on a leaf of the foliation $\widetilde{\mathcal{F}}$ we can consider two affine structures, associated with structures of lagrangian foliations on $\mathcal{F}$ respective to Poisson structures $\eta_\lambda$ and $\eta_\mu$. These two affine structures coincide.*

*Remark.* We have proved this corollary by describing this affine structure in independent of $\lambda$ terms. However, we cannot prove this miraculous fact more directly.

**Translations and Schouten—Nijenhuis bracket.** Darboux theorem (on the straightening of a symplectic structure) shows that there exist a lagrangian submanifold transversal to given foliation. Similar arguments make evident the existence of a transversal for a Poisson manifold in general position (which is foliated on symplectic manifolds).

Let us consider two Poisson structures $\eta_\lambda$, $\eta_\mu$, $\lambda, \mu \in \mathcal{S}$. The above arguments show that we can find two transversal to a foliation $\mathcal{F}$ submanifold, one coisotropic respective to $\eta_\lambda$, another respective to $\eta_\mu$. The theorem on Poisson structures, applied to the first of them, identify $(Y, \eta_\lambda)$ with $\left(T^{*(-1)}X, \eta_\lambda\right)$. The second transversal corresponds to a section $t$ of $T^{*(-1)}X$ under this identification. Hence,

$$(Y, \eta_\lambda, \eta_\mu) \simeq \left(T^{*(-1)}X, \eta_\lambda, \mathfrak{T}_{t*}\eta_\mu\right),$$

where $\mathfrak{T}_t$ is the shift on the section $t$ on the vector bundle $T^{*(-1)}X$:

$$\mathfrak{T}_t\colon (x, \alpha) \mapsto (x, a + t(x)).$$

The next problem to consider is the following

**Theorem.** *For a Veroneze web $X$ and sections $t_1$, $t_2$ of $T^{*(-1)}X$ the manifold*

$$\left(T^{*(-1)}X, \mathfrak{T}_{t_1*}\eta_\lambda, \mathfrak{T}_{t_2*}\eta_\mu\right)$$

*with two Poisson structures is bihamiltonian iff the sections $t_{1,2}$ satisfy the equation*

$$d_\lambda d_\mu (t_1 - t_2) = 0,$$

*here $d_\lambda$ and $d_\mu$ are two differential in the considered above bicomplex.*

*In this case the bihamiltonian structure coincides (after a diffeomorphism of $T^{*(-1)}X$) with the standard bihamiltonian structure on $T^{*(-1)}X$ twisted with $t_2 - t_1$:*

$$\left(T^{*(-1)}X, \mathfrak{T}_{t_1*}\eta_\lambda, \mathfrak{T}_{t_2*}\eta_\mu\right) \simeq \mathfrak{T}_{t_1}\left(T^{*(-1)}X, \eta_\lambda, \mathfrak{T}_{(t_2-t_1)*}\eta_\mu\right).$$

*For* $t_1 - t_2 = -d_\lambda f_1 + d_\mu f_2$[15]*,* $f_{1,2}$ *being sections of* $\Lambda^0 T^{(-1)}X$ *(this space coincides with functions on* $X$*), this bihamiltonian structure coincides (after a diffeomorphism of* $T^{*(-1)}X$*) with the standard bihamiltonian structure on* $T^{*(-1)}X$*:*

$$\left(T^{*(-1)}X, \mathfrak{T}_{t_1*}\eta_\lambda, \mathfrak{T}_{t_2*}\eta_\mu\right) \simeq \mathfrak{T}_{t_1+d_\lambda f_1}\left(T^{*(-1)}X, \eta_\lambda, \eta_\mu\right).$$

*Proof.* The last assertion is almost obvious. As it was proved in the section on Poisson structures, the translation on the differential of the function in conormal bundle to a manifold

$$(x, \xi) \mapsto (x, \xi + df(x))$$

preserves the symplectic (hence also the Poisson) structure.

Clearly, the same is true for a conormal bundle to a foliation (with the change of symplectic and Poisson structures). Hence the isomorphism of $(T^{*(-1)}X, \eta_\lambda)$ and $(T^*\mathcal{F}_\lambda, \eta_\lambda)$ (together with the corresponding isomorphism with a change $\lambda$ and $\mu$) shows that

$$\mathfrak{T}_{d_\lambda f_1*}\eta_\lambda = \eta_\lambda, \quad \mathfrak{T}_{d_\mu f_1*}\eta_\mu = \eta_\mu,$$

what proves the last formula of the theorem.

Inversely, the translation along the affine bundle clearly doesn't change the field of 2-vectors' kernels.

Let $\Phi$ be a diffeomorphism such that

$$\left(T^{*(-1)}X, \mathfrak{T}_{t_1*}\eta_\lambda, \mathfrak{T}_{t_2*}\eta_\mu\right) \simeq \Phi\left(T^{*(-1)}X, \eta_\lambda, \eta_\mu\right).$$

Since the projection $T^{*(-1)}X \to X$ is defined by the bihamiltonian structure $(T^{*(-1)}X, \eta_\lambda, \eta_\mu)$, $\Phi$ sends a leaf of the foliation $\widetilde{\mathcal{F}}$ on $T^{*(-1)}X$ to a leaf. Hence $\Phi$ induces the diffeomorphism $\Phi_X$ of the Veroneze web on $X$. We can suppose this diffeomorphism to be identical since

$$\widetilde{\Phi_X}\left(T^{*(-1)}X, \eta_\lambda, \eta_\mu\right) \simeq \left(T^{*(-1)}X, \eta_\lambda, \eta_\mu\right)$$

and $\Phi$ can be changed to $\Phi \circ \widetilde{\Phi_X}^{-1}$, $\widetilde{\Phi_X}$ being the induced by $\Phi_X$ diffeomorphism of $(T^{*(-1)}X, \eta_\lambda, \eta_\mu)$.

The $\Phi$-image of the zero section must be a coisotropic respective to any form $\eta_\lambda$ transversal to this foliation submanifold. Fixing $\lambda$, we get an identification of $\mathfrak{T}_{t_1}^{-1}\Phi\,(0\text{-section})$ and the differential along the foliation $\mathcal{F}_\lambda$ of a function $f_1$:

$$\mathfrak{T}_{t_1}^{-1}\Phi\,(0\text{-section}) = \Gamma_{d_\lambda f_1},$$

here $\Gamma_\varphi$ being a graph for a section $\varphi$ of a bundle, $\Gamma_\varphi = \{(x, \varphi(x))\}$. The same can be repeated for a fixed $\mu$:

$$\mathfrak{T}_{t_2}^{-1}\Phi\,(0\text{-section}) = \Gamma_{d_\mu f_2}.$$

Hence

$$t_1 + d_\lambda f_1 = t_2 + d_\mu f_2.$$

Now it is easy to see that we have found the functions $f_1$, $f_2$ required in the theorem.

We have considered the last assertion of the theorem. Let us consider now the first assertion. In fact we will prove much more stronger assertion:

$$[\mathfrak{T}_{t_1}\eta_\lambda, \mathfrak{T}_{t_2}\eta_\mu] \simeq d_\lambda d_\mu (t_1 - t_2), \qquad d_\lambda d_\mu\colon \Lambda^1 T^{*(-1)}X \to \Lambda^3 T^{*(-1)}X.$$

[15] Such $t$ clearly satisfies the above equation.

Here the sign $\simeq$ denotes that the left-hand side is an image of the tangent to the fibers 3-vector field in the right-hand side in the space of 3-vectors in the total space of the bundle. Application of $\mathfrak{T}_{t_1}^{-1}$ allows us to to reduce the general case to a case $t_1 = 0$, $t_2 = t$.

The Schouten—Nijenhuis bracket

$$[\eta_\lambda, \mathfrak{T}_{t*}\eta_\mu]$$

has a decomposition connected with the filtration

$$0 \subset T_y\widetilde{\mathcal{F}} \subset T_yT^{*(-1)}X, \qquad T_yT^{*(-1)}X/T_y\widetilde{\mathcal{F}} \simeq T_x^*X, \quad y \in T_x^{*(-1)}X.$$

The corresponding filtration on $\Lambda^3 T_yT^{*(-1)}X$ has associated factors

$$\Lambda^3 T_y\widetilde{\mathcal{F}}, \quad \Lambda^2 T_y\widetilde{\mathcal{F}} \otimes T_x^*X, \quad T_y\widetilde{\mathcal{F}} \otimes \Lambda^2 T_x^*X, \quad \Lambda^3 T_x^*X.$$

It is easy to see that *for any* section $t$ the components of $[\eta_\lambda, \mathfrak{T}_{t*}\eta_\mu]$ in the last three factors are zeros.

Really, to simplify the expression

$$2\,[\eta_\lambda, \mathfrak{T}_{t*}\eta_\mu] = \operatorname*{Alt}_{ijm} \eta_\lambda^{il} \mathfrak{T}_{t*}\eta_{\mu,l}^{jm} + \operatorname*{Alt}_{ijm} \mathfrak{T}_{t*}\eta_\mu^{il} \eta_\lambda^{jm}{}_{,l},$$

it is useful to choose a coordinate system $(x_i, i = 1, \dots, k)$ on $X$ such that the foliation $\mathcal{F}_\nu$ (for an appropriate $\nu$) is $x_k =$ const and that $\eta$ has constant coefficients. After that we can choose on $T^{*(-1)}X$ the coordinate system associated with the identification of this bundle and $T^{*(-1)}\mathcal{F}_\nu$.

The 2-vector $\mathfrak{T}_{t*}\eta_\mu - \eta_\mu$ at a fixed point of $T_x^{*(-1)}X$ is a tangent to $\widetilde{\mathcal{F}}$ 2-vector. (It is clear for $t(x) = 0$ or $t = d_\mu f$ and consequently for an arbitrary $t$.)

First we want to prove an explicit formula for this difference

$$\mathfrak{T}_{t*}\eta_\mu - \eta_\mu = d_\mu t, \qquad d_\mu\colon \Lambda T^{*(-1)}X \to \Lambda T^{*(-1)}X.$$

Since here we need to consider only one of two Poisson structures $\eta_\mu$, we can use the above coordinate system with $\nu = \mu$ and the corresponding identification of $T^{*(-1)}X$ and $T^*\mathcal{F}_\mu$. If $\phi$ is a section of the cotangent bundle for a manifold $Z$ and $\eta$ is the Poisson structure on this bundle, then

$$(\mathfrak{T}_\phi)_* \eta - \eta = -d\phi,$$

where $d\phi$ is considered as a section of $\Omega^2 Z = \Lambda^2 (T^*Z) \subset \Lambda^2 T(T^*Z)$. Indeed, it is already proved in the case $d\phi = 0$, and the proof in the general case is completed after considering $\phi$ such that $\phi(z) = 0$. In this case

$$\mathfrak{T}_{\phi*}|_{(z,\zeta)} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \widetilde{d\phi} & \mathbf{1} \end{pmatrix}, \qquad (z,\zeta) \in T_z^*Z$$

in the coordinate system $(z^i, \zeta_i)$ on $T^*Z$. Here $\left(\widetilde{d\phi}\right)_{ij} = \phi_{i,j}$. Hence $\Lambda^2\mathfrak{T}_{\phi*}$ has also the lower-diagonal block structure and it is easy to see that the only important for the calculation of $\Lambda^2 (\mathfrak{T}_\phi)_* \cdot \eta$ block is

$$\widetilde{d\phi} \wedge \mathbf{1} = -d\phi \wedge \mathbf{1}, \quad \left(\widetilde{d\phi}\right)_{ij} = -\phi_{i,j} + \phi_{j,i},$$

$$\left(\widetilde{d\phi} \wedge \mathbf{1}\right)\left(\frac{\partial}{\partial z^j} \wedge \frac{\partial}{\partial \zeta_j}\right) = \left(\widetilde{d\phi}\right)_{ij} \frac{\partial}{\partial \zeta_i} \wedge \frac{\partial}{\partial \zeta_j}.$$

Since

$$(d\phi \wedge \mathbf{1}) \cdot \eta = d\phi,$$

the assertion is proved.

The arguments for the case of cotangent bundle to a foliation are exactly the same. This shows that

$$[\eta_\lambda, \mathfrak{T}_{t*}\eta_\mu] = [\eta_\lambda, \mathfrak{T}_{t*}\eta_\mu - \eta_\mu] = -[\eta_\lambda, d_\mu t]$$

(we have used the identification of operator $d$ on $T^*\mathcal{F}_\mu$ and operator $d_\mu$ on $T^{*(-1)}X$).

In this formula again only one of Poisson structures appears, hence we can again consider the above identification of $T^{*(-1)}X$ and (this time) $T^*\mathcal{F}_\lambda$ (i.e., now $\nu = \lambda$). The section $d_\mu t$ of $\Lambda^2 T^{*(-1)}X$ corresponds to a 2-form $\omega$ along the foliation $\mathcal{F}_\lambda$ (so $\omega \in \Lambda^2 T^*\mathcal{F}_\lambda$) and consequently to a tangent to fibers 2-vector field $\widetilde{\omega}$ on $T^*\mathcal{F}_\lambda$, $\widetilde{\omega}\,(x,\xi) = \omega_{ij}\frac{\partial}{\partial\zeta_i} \wedge \frac{\partial}{\partial\zeta_j}$ if $\omega = \omega_{ij} dx^i \wedge dx^j$. We claim that on a symplectic manifold $Z$

$$[\eta, \widetilde{\omega}] = -\widetilde{d\omega},$$

where $\eta$ is the Poisson structure, $\omega$ is an arbitrary 2-form on $Z$, $\widetilde{\omega}$ is the corresponding 2-vector field on $T^*Z$, $\widetilde{d\omega}$ is a 3-vector field on $T^*Z$ that corresponds to $d\omega$ in the same way as $\widetilde{\omega}$ corresponds to $\omega$. The extension of this formula to the case of foliation will complete the proof of the theorem.

To prove this formula consider a local coordinate system on $Z$. Let us choose the associated coordinate system on $T^*Z$, $\left(y^1, \dots, y^{2k}\right) = \left(z^1, \dots, z^k, \zeta_1, \dots, \zeta_k\right)$. In this frame the tensor field $\eta$ is constant, so

$$[\eta, \widetilde{\omega}] = \operatorname*{Alt}_{ijm} \eta^{il} \widetilde{\omega}^{jm}_{,l} + \operatorname*{Alt}_{ijm} \widetilde{\omega}^{il} \eta^{jm}_{,l} = \operatorname*{Alt}_{ijm} \eta^{il} \widetilde{\omega}^{jm}_{,l}.$$

The 2-vector $\widetilde{\omega}$ is an image of a tangent to a fiber 2-vector. Let us consider a filtration on $\Lambda^3 T_{(z,\zeta)} T^*Z$ that is connected with the filtration

$$0 \subset T_{(z,\zeta)}\mathfrak{F} \subset T_{(z,\zeta)}T^*Z, \qquad T_{(z,\zeta)}T^*Z / T_{(z,\zeta)}\mathfrak{F} \simeq T_z Z, \quad (z,\zeta) \in T^*Z$$

on $T_{(z,\zeta)}T^*Z$ (here $\mathfrak{F}$ is a vector bundle consisting of tangent to fibers of projection $T^*Z \to Z$ vectors). It has associated factors

$$\Lambda^3 T_{(z,\zeta)}\mathfrak{F}, \quad \Lambda^2 T_{(z,\zeta)}\mathfrak{F} \otimes T_z Z, \quad T_{(z,\zeta)}\mathfrak{F} \otimes \Lambda^2 T_z Z, \quad \Lambda^3 T_z Z.$$

It is easy to see that *for any* section $\omega$ of $\Omega^2 Z$ the components of $[\eta, \widetilde{\omega}]$ (where $\widetilde{\omega}$ is the corresponding section of $\Lambda^2 T\left(T^{*(-1)}Z\right)$) in the last three factors vanish. Indeed, before the anti-symmetrization in the formula

$$[\eta, \widetilde{\omega}] = \operatorname*{Alt}_{ijm} \eta^{il} \widetilde{\omega}^{jm}_{,l}$$

only members with $j$, $m$ in the direction of fibers (i.e., $j, m \geq k+1$) and $l$ in the direction of the base (i.e., $l \leq k$) remain, hence only members with $i$, $j$, $m$ in the direction of the fibers remain. The anti-symmetrization evidently preserves this property.

To bring the prove to an end we can note that the resulting formula for $[\eta, \widetilde{\omega}]$

$$[\eta, \widetilde{\omega}] = \operatorname*{Alt}_{\substack{ijm \\ i,j,m \geq k+1}} \sum_{l \leq k} \left(\delta^{i+k,l} - \delta^{i-k,l}\right) \widetilde{\omega}^{jm}_{,l} = - \operatorname*{Alt}_{\substack{ijm \\ i,j,m \geq k+1}} \widetilde{\omega}^{jm}_{,i-k}$$

coincides with the formula for exterior differentiation up to a sign and that the generalization to the case of a cotangent bundle to a foliation doesn't meet any obstacles. □

## Double cohomology of a double complex

As we have already seen, the classification problem for bihamiltonian manifolds with a given reduction (which is a Veroneze web) is reduced to a linear problem: find all the solutions of

$$d_\lambda d_\mu t = 0, \qquad t \in \Gamma\left(T^{*(-1)}X\right),$$

modulo

$$t = d_\lambda \varphi_1 + d_\mu \varphi_2, \qquad \varphi_1, \varphi_2 \in \Gamma\left(\Lambda^0 T^{*(-1)}X\right) \simeq \Gamma\left(\mathcal{O}\right).$$

**Hypothesis.** *For a point $x$ on the Veroneze web $X$ there exists a neighborhood $U$ such that*

$$\mathbb{H}^i\left(U\right) = 0 \quad \textit{for } i \geq 1.$$

*Here $\mathbb{H}^i$ denotes the double cohomology of the double complex of sub-differential forms on $X$.*

In the case of analytical manifolds category we know a proof of this hypothesis. However, in the case of $C^\infty$-category the situation concerning many cohomological invariants is, as we know, quite different comparing with analytical case. So in the former case it is better to call this hypothesis a question.

Anyway, in the case of analytical manifolds this hypothesis (the proof of which we will write elsewhere) allows as to prove the following

**Theorem.** *An analytical bihamiltonian manifold of odd dimension in general position is defined locally by its bihamiltonian reduction (which is a Veroneze web) up to a local diffeomorphism.*

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