DESARGUES CONFIGURATIONS AND GROUPOIDS

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ABSTRACT. We show how to reduce statements on existence of Desargues configurations in a projective plane to more mainstream questions of math such as structure of homogeneous spaces. The reduction goes through the groupoid of perspective transformations of lines.

CONTENTS

0. PROJECTIVE AND AFFINE PLANES

A *pre-projective plane* is a set Π together with a collection $\mathcal{L}$ of subsets of Π (called *lines*) such that any two distinct lines intersect at exactly one point, and there is exactly one line (denoted as (PP')) passing through an arbitrary pair P, P' of points of Π. A pre-projective plane is a *projective plane* if at least 2 lines exist, and any line contains at least 3 points.

One can easily show that a pre-projective plane which is not a projective plane is either empty, or $\mathcal{L} = \{\Pi\}$, or there is a point $P \in \Pi$ such that $\mathcal{L}$ consists of $\Pi \smallsetminus \{P\}$ and of all subsets $\{P, Q\}$, $Q \neq P$.

Given two projective planes Π and Π', a *pre-homomorphism* $\Pi \to \Pi'$ is a set map $f\colon \Pi \to \Pi'$ which sends a line of Π to a line of Π'. If $f(P) = f(Q)$, consider a line l with $P \in l$, $Q \notin l$, and a line m with $P \notin m$, $Q \in m$. Consideration of $f(l \cap m)$ implies $f(l) = f(m)$; therefore $f(\Pi \smallsetminus (PQ)) \subset f(l)$. This implies $f(\Pi) = f(l)$. A pre-homomorphism is a *homomorphism* if its image is not contained in a line; then it is bijective; one can immediately see that a preimage of a line must be a line, thus any homorphism is invertible. *Endomorphisms* of a plane Π (i.e., homomorphisms

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$\Pi \to \Pi$) coincide with *automorphisms* (i.e., invertible homomorphisms); they form a group denoted by $\operatorname{Aut}\Pi$.

Consider a projective plane Π, fix a line l_∞ on Π. An *affine line* is a line distinct from l_∞; an *affine plane* Π_{fin} is $\Pi \smallsetminus l_\infty$. The line l_∞ is called the *horizon* of the affine plane. Two affine lines are parallel, if they intersect on the horizon.

Sometimes it is more convenient to consider an affine line as a subset of Π_{fin}; in what follows, this would never lead to confusion.

1. The groupoid of perspective transformations and line movements

Given a projective plane Π, two lines l, l' on Π, and a point O not on l, l', projection from O identifies l with l'. Call such mappings $l \to l'$ a (simple) perspective transformation (with center at O).

A composite perspective transformation is a composition $l \to l_1 \to l_2 \to \cdots \to l_K \to l'$ of simple perspective transformation. Moreover, given a subset L of the set $\mathcal{L}$ of lines in Π, and subset S of Π, one may consider similar chains of simple perspective transformations such that all the lines are are in L, and all the centers are in S. Call the resulting mappings $l \to l'$ (L, S)-composite perspective transformations. We obtain a category with the set of objects L, and arrows being the (L, S)-composite perspective transformations; call it $\operatorname{Persp}_{(L,S)}$. It is automatically a grouppoid. One obtains a group $\operatorname{Persp}_{(L,S)}(l, l)$ with a faithful action on l, for any l on L.

One can connect studying particular cases of this group action with existence of certain Desargues configurations. Choose a line m on Π, and a point O. Take as L the set of lines passing through O (and distinct from m provided m contains O), and $S = m$ (one can replace S by $m \smallsetminus \{O\}$; this would not change the category even if O is on m). Denote this particular case of $\operatorname{Persp}_{(L,S)}$ by $\operatorname{PencilPersp}_{(O,m)}$; denote the group $\operatorname{PencilPersp}_{(O,m)}(l, l)$ by $\operatorname{PER}_{(O,m)} l$; here a line l contains O.

Obviously, the action of $\operatorname{PER}_{(O,m)} l$ is transitive on $l_0 := l \smallsetminus (\{O\} \cup (l \cap m))$. Denote by $\mathcal{M}_{(O,m)} l$ the automorphism group of the $\operatorname{PER}_{(O,m)} l$-set l_0; in other words, it consists of mappings $f \colon l_0 \to l_0$ such that $f(gX) = g(f(X))$ for any X in l_0 and g in $\operatorname{PER}_{(O,m)} l$; call such mappings O-movements along l w.r.t. m. As a corollary, the action of $\mathcal{M}_{(O,m)} l$ is free on l_0. Moreover, the action of $\operatorname{PER}_{(O,m)} l$ on l_0 is free iff the action of $\mathcal{M}_{(O,m)} l$ on l_0 is transitive.

There are two distinct cases to consider: the “little” case, when O is on m, and the “fixed-point” case, when O is not on m. In the first case l_0 is a line without a point, in the second l_0 is a line without 2 points. Considering m as horizon of an affine plane $\Pi \smallsetminus m$, in the first case we consider compositions of parallel projections between a pencil of parallel lines; in the second case we consider parallel projections between lines passing through a “fixed point” O.

Example 1.1. One can immediately see that the action of $\operatorname{PER}_{(O,m)} l$ and of $\mathcal{M}_{(O,m)} l$ on l_0 is free transitive if Π is a projective plane over a field k. The corresponding groups are abelean, and $\operatorname{PER}_{(O,m)} l = \mathcal{M}_{(O,m)} l$. In the “little” case these groups coincide with the additive group of k acting by translation on an affine line; in the "fixed point" case these groups coincide with the multiplicative group of k acting by expansion on an affine line with a fixed point O.

In the "little" case, when $O = m \cap l$, $\mathcal{M}_{(O,m)}l$ is denoted by $\mathrm{Trans}_m\, l$; its elements are called *translations* of l. In the remaining ("fixed point") case elements of $\mathcal{M}_{(O,m)}l$ are called *O-dilations* of l.

2. Desargues configurations

Theorem 2.1. *The action of* $\mathrm{PER}_{(O,m)}\, l$ *(and/or of* $\mathcal{M}_{(O,m)}l$ *on* l_0*) is free transitive iff the Desargue theorem with center* O *and axis* m *holds.*

Proof. It is enough to show that the action of $\mathrm{PER}_{(O,m)}\, l$ on l_0 is free iff the Desargues theorem with center O and axis m holds. On the other hand, the former statement is equivalent to the fact that given two distinct lines l, l' through O (both distinct from m), and points P, P' on l_0, l'_0 correspondingly, there is at most one composite perspective transformation $l \to l'$ in $\mathrm{PencilPersp}_{(O,m)}\,(l, l')$ which sends P to P'. On the gripping hand, there is always a simple perspective transformation which sends P to P'; therefore the theorem reduces to

Lemma 2.2. *The set of composite perspective transformations in* $\mathrm{PencilPersp}_{(O,m)}\,(l, l')$ *coincides with the set of simple perspective transformations* $l \to l'$ *with center on* m *(for any 2 distinct line* l, l' *through* O*, distinct from* m*) iff the Desargues theorem with center* O *and axis* m *holds.*

Proof. Assume that the Desargues theorem with center O and axis m holds. Consider 3 distinct lines l, l', l'' through O, and two simple perspective transformations $l \xrightarrow{\pi} l' \xrightarrow{\pi'} l''$ with centers P, P' on the line m. If the centers are not distinct, the composition is a simple perspective transform with the same center; assume that the centers are distinct. We claim that the composition is again a simple perspective transformation with a center on m. Indeed, take two distinct points A_1, A_2 on l_0 and not on m; let $B_k = \pi A_k$, $C_k = \pi' B_k$, $k = 1, 2$.

Recall that what the Desargues theorem claims is that the lines (A_1C_1) and (A_2C_2) intersect at a point of m. In particular, (A_2C_2) passes through the point $P'' := (A_1C_1) \cap m$; since this point does not depend on A_2, we conclude that the composition $l \to l''$ is a simple perspective with center P''.

Likewise, assume that the composition $l \to l''$ is a simple perspective with center P''. It is enough to show that $P'' \in m$. Note that $\pi' \circ \pi$ sends the points $Q = m \cap l$ to $Q'' = m \cap l''$; thus P'' is collinear with Q and Q''; if $O \notin m$, $Q \neq Q''$, thus P'' is on the line $(QQ'') = m$. If $O \in m$, and $P'' \notin m$, then applying the same arguments to representation of π as a composition: $\pi = (\pi')^{-1} \circ (\pi' \circ \pi)$ leads to a contradiction.

Now consider an arbitrary element of $\mathrm{PencilPersp}_{(O,m)}\,(l_1, l_K)$ with $l_1 \neq l_K$; consider its shortest representation as composition of simple perspective transformations $l_1 \to$

$l_2 \to \cdots \to l_K$; by definition, $l_m \neq l_{m+1}$; if the "collapse" rule above holds, one must have $l_m = l_{m+2}$. Therefore, if $K > 1$, then the chain starts as $l_1 \to l_2 \to l_1 \to l_2 \to \dots$. Take any line l' passing through O distinct from l_1, l_2 and m (such a line may not exit if there are 3 points on a projective line; however, there is exactly one such plane, and this exceptional case is easy to check). Split projection $l_2 \to l_1$ in the chain to composition $l_2 \to l' \to l_2$ of projections with the same centers. Then the chain becomes $l_1 \to l_2 \to l' \to l_1 \to l_2$, which collapses to $l_1 \to l_2$; contradiction with assumption $K > 1$.

Similarly, the former condition of lemma implies the case of composition $l \to l' \to l''$ considered above; thus it implies the corresponding case of Desargues theorem. □

This finishes the proof of the theorem. □

3. Plane movements

Consider the pencil of lines passing through a point $O \in \Pi$. We saw that given a line m on Π, one can associate a group $\mathcal{M}_{(O,m)}l$ to each of these lines; this group acts freely on the line with removed point(s) O and $\{m \cap l\}$.

One can immediately see that these groups are canonically identified: given a point P on m, the projection from P identifies any two lines l, l' of the pencil which do not pass through P. This identification associates to $\mathcal{M}_{(O,m)}l$ a certain group G' of transformations of l'.

Lemma 3.1. *The action of G' on l' coincides with the action of $\mathcal{M}_{(O,m)}l'$. Thus obtained identification of $\mathcal{M}_{(O,m)}l$ with $\mathcal{M}_{(O,m)}l'$ does not depend on the choice of P.*

Proof. Consider $g \in \mathcal{M}_{(O,m)}l$, denote by g' the corresponding element of g. Denote by π the P-projection from l' to l; then $g' = \pi^{-1} g \pi$. We want to show that g' commutes with the action of $\operatorname{PER}_{(O,m)} l'$.

Given any element of $\rho' \in \operatorname{PER}_{(O,m)} l'$, consider $\rho = \pi \rho' \pi^{-1}$; it is manifestly an element of $\operatorname{PER}_{(O,m)} l$. Since g commutes with ρ, g' must commute with ρ'. Hence $g' \in \mathcal{M}_{(O,m)}l'$.

Given a different projection $\widetilde{\pi}\colon l' \to l$ with center on m, $\pi\widetilde{\pi}^{-1}$ is an element of $\operatorname{PER}_{(O,m)} l$. Thus $g' = \pi^{-1} g \pi = \pi \left(\pi\widetilde{\pi}^{-1}\right) g \left(\pi\widetilde{\pi}^{-1}\right)^{-1} \pi^{-1} = \widetilde{\pi}^{-1} g \widetilde{\pi}$. □

An element $g \in \mathcal{M}_{(O,m)}l$ acts on $l \smallsetminus (\{O\} \cup (l \cap P))$; extend it to O and to $l \cap m$ as identity. Given $g \in \mathcal{M}_{(O,m)}l$, we constructed a permutation of points of any line passing through O (and distinct from m). Since these permutations are compatible on O, they induce a permutation of points $\Pi \smallsetminus m$; extend it to m as identity. Call such permutations of Π the (O,m)-*motions of* Π.

Theorem 3.2. *An (O,m)-motion of Π is an automorphism of Π; in other words, it sends a line to a line.*

Proof. The statement is obvious for the line m, and for any line passing though O. Let n be a line not passing through O and distinct from m. Since motions are invertible, it is enough to show that the image of n is contained in a line passing through $P = n \cap m$.

Consider lines l, l' passing through O and distinct from m; consider $Q = l \cap n$; it is distinct from O and $l \cap m$. Identify l and l' using projection π from P; let Q be identified with Q'. Consider $g \in \mathcal{M}_{(O,m)} l$ and the corresponding movement g' of l'; it is enough to show that points P, $g'Q'$ and gQ are collinear, but this immediately follows from g being idenfied with g' via π. □

4. Coordinatization and commutation

A *coordinate system* OXY is a collection of three non-collinear points O, X, Y on a projective plane Π. The projection from Y identifies $\Pi \smallsetminus (XY)$ with $(OX) \smallsetminus \{X\}$; the projection from X identifies $\Pi \smallsetminus (XY)$ with $(OY) \smallsetminus \{Y\}$. Obviously, two projections taken together identify the affine plane $\Pi \smallsetminus (XY)$ with the product of two affine lines $(OX) \smallsetminus \{X\}$ and $(OY) \smallsetminus \{Y\}$; this identification is called *OXY-coordinate system.*

Consider a translation in $\mathrm{Trans}_{(XY)}(OX)$ and the corresponding motion g of Π. Since g preserves lines passing through X, the projection to (OY) is not changed by g.

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Corollary 4.1. *Motions g, g' of Π corresponding to an element of* $\mathrm{Trans}_m l$ *and an element of* $\mathrm{Trans}_m l'$ *commute provided $l \cap l'$ is not on m.*

Proof. Indeed, consider OXY-coordinate system on $\Pi \smallsetminus m$; here $m = (XY)$, $l = (OX)$, $l' = (OY)$. Since g sends any line through X into itself, g preserves the second coordinate; likewise, g' preserves the second coordinate. Moreover, g sends any line through Y to a line through Y; therefore g can be written in coordinates x, y as $(x, y) \mapsto (G(x), y)$; likewise g' acts as $(x, y) \mapsto (x, G'(y))$. Therefore g and g' commute on $\Pi \smallsetminus m$, thus on Π. □

5. Types of automorphisms of Π and commutation

Fix a line m in a projective plane Π. Call an automorphism $f \colon \Pi \to \Pi$ an *m-translation* if $f(X) = X$ for any $X \in m$, and either $f = \mathrm{id}$, or $f(X) = X$ implies $X \in M$. It is clear that any motion corresponding to an element of $\mathrm{Trans}_m l$, $l \neq m$, is an m-translation.

Proposition 5.1. *Consider an m-translation $f \neq id$. Consider $P \notin M$; then $f(P) \neq P$. Consider the line $l = (P f(P))$. Then f is a motion corresponding to the translation $f|_l$ along the line l.*

Proof. First of all, let $O = l \cap m$. Then $f(l) = f((PO)) = (f(P) O)$. Since $f(P)$ and O are on l, $f(l) = l$. Moreover, for any $Q \notin m$, the line $(Q f(Q))$ passes through O. (Indeed, as above, we know that $(Q f(Q))$ and l are sent into themselves by f; therefore $(Q f(Q)) \cap l$ is preserved by f. Therefore $(Q f(Q)) \cap l$ is on m, thus coincides with O.)

One can conclude that any line passing through O is sent into itself by f. Consider a sequence of simple perspectives $l \to l_1 \to \cdots \to l_K \to l$ with all lines l, l_k, $k = 1, \dots, K$, passing through O, and the centers of perspective on m. Then the lines and the centers of perspectives are preserved by f; therefore f commutes with any mapping of the chain, thus with the composition. Therefore $f|_l \in \operatorname{Trans}_m l$.

Now the fact that l sends any line not passing through O to a line which intersects m at the same point implies that restriction of f to all the lines passing through O are compatible (in the sense used in the definition of the motion). This finishes the proof of the proposition. □

Corollary 5.2. *Suppose that the group* $\operatorname{Trans}_m l'$ *contains more than one element. Then the group* $\operatorname{Trans}_m l$ *is commutative provided* $l \cap l'$ *is not on* m.

Proof. Let $O = l \cap l'$, h be a non-trivial element of $\operatorname{Trans}_m l'$, and $Y_1 = hO \in l'$. Consider $g, g' \in \operatorname{Trans}_m l$, let $X_1 = g(O) \in l$; denote by the same letters h, g, g' the corresponding motions of Π. We already know that h commutes both with g and with g'.

Consider the automorphism hg of Π. Let $X = l \cap m$, $Y = l' \cap m$; consider the OXY coordinate system. Note that hg sends any line passing through X to a line passing through X; identify the set of lines passing through X with l'; then gh induces a mapping $F\colon l' \to l'$. Using the coordinate system the same way as in the proof of Corollary 4.1, it is clear that F coincides with h. Since $\operatorname{Trans}_m l'$ acts freely on $l' \setminus \{Y\}$, $h(P) \neq P$ for any $P \in l' \smallsetminus \{Y\}$. Therefore $gh(P) \neq P$ for any $P \in \Pi \smallsetminus m$.

By the preceeding proposition, gh is a non-trivial m-translation of Π. Moreover, the arguments above show that $gh(O) \notin l$; therefore, the line $\bar{l} = (O\, gh(O))$ is distinct from l. By the proposition above, gh corresponds to a translation $(gh)|_{\bar{l}}$ along the line $\bar{l}$; by Corollary 4.1, gh commutes with g'. Since h commutes with g', $gg' = ghh^{-1}g' = gh \cdot g'h^{-1} = g' \cdot gh \cdot h^{-1} = g'g$. □

6. On commutation of translations (not needed, unfinished)

Consider an (O, m)-motion g and an (O', m')-motion g'; suppose that both are non-trivial. If they commute, then g fixes $g'O$ and points of $g'm$; therefore, $g'm = m$; thus either $m = m'$, or $O' \in m$ and $O \in m'$. Moreover, if $O \notin m$, one can conclude that $g'O$ cannot be on m (neither if $m = m'$, nor if $O \in m'$); hence $g'O = O$. If $m = m'$, then $g'O = O$ implies $O = O'$.

Thus either both motions are translations, or they are both dilations, or $O' \in m$ and $O \in m'$. Here we consider the simplest case when both transformations are translations, and $m = m'$.

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