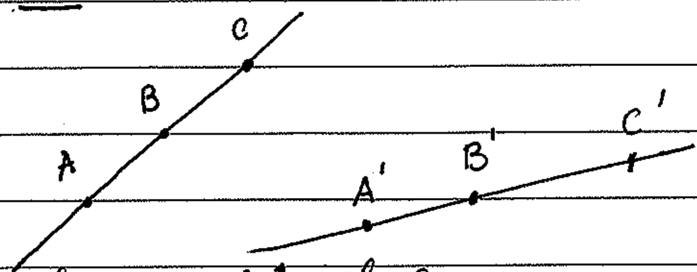
①

16.1.



m = slope $\qquad m_1$ = slope

$$d^2(AB) = (1+m^2)(b_1-a_1)^2$$
$$d^2(BC) = (1+m^2)(c_1-b_1)^2$$
$$d^2(AC) = (1+m^2)(c_1-a_1)^2$$

the same for A', B', C'

$$(1+m_1^2)(b_1'-a_1')^2 = (1+m^2)(b_1-a_1)^2$$
$$(1+m_1^2)(c_1'-b_1')^2 = (1+m^2)(c_1-b_1)^2$$

$$\cancel{\frac{b_1-a_1}{b_1'-a_1'} = \frac{c_1-b_1}{c_1'-b_1'} = \alpha}$$

$$\cancel{c_1'-a_1' = c_1'-b_1'+b_1'}$$

$$\frac{c_1'-b_1'}{c_1-b_1} = \frac{b_1'-a_1'}{b_1-a_1} = \alpha \qquad \alpha^2 = \frac{1+m^2}{1+m_1^2}$$

$$\frac{c_1'-a_1'}{c_1-a_1} = \frac{c_1'-b_1'+b_1'-a_1'}{c_1-b_1+b_1-a_1} = \alpha$$

We have $(1+m_1^2)(c_1'-a_1')^2 = (1+m^2)(c_1-a_1)^2$

$$d^2(AC) = d^2(A'C')$$

(2)

16.6. Using rigid motions make the center of a circle to the origin and line to a vertical line.
Then the equation of the line is $x = a$ and the equation of the circle is $x^2+y^2=r^2$
The line meets the circle in two points if $y^2 = r^2 - a^2$ has two solutions.
That happens if $r^2 - a^2 > 0$ or $r^2 > a^2$
If $r^2 > a^2$ the point $(a,0)$ is on a line and inside the circle. If $r^2 \le a^2$ the closest point to the center has coordinates $(a,0)$ and is outside or on the circle.

16.9. $\cos 72^\circ = \dfrac{\sqrt{5}-1}{4}$, $\sqrt{5} = \sqrt{1+2^2} \in \Omega$

$$\sin 72^\circ = \sqrt{1 - \frac{(\sqrt{5}-1)^2}{4^2}} =$$

$$= \sqrt{\frac{16-5-1+2\sqrt{5}}{16}} = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$= \frac{\sqrt{(1+\sqrt{5})^2+2^2}}{4} = \frac{\sqrt{1+ (1+\sqrt{5})^2}}{2} \in \Omega$$

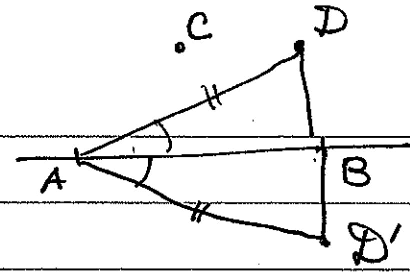
since $1+\sqrt{5} \in \Omega$

③

17.3 (a) $\Phi(A) = A$

$\Phi(B) = B$

$\Phi(C) = C$



Let $D' = \Phi(D)$

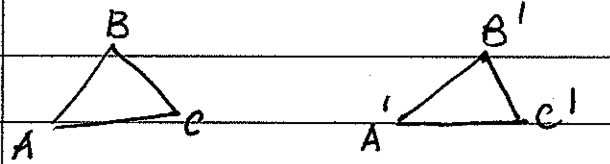
then $\angle DAB \cong \angle D'AB$ and $DA \simeq D'A$

In addition (D and C) and (C and D') are on the same side of AB.

Hence D and D' are on the same side and by (C1) (C4) $\Rightarrow D = D'$

Let $\Phi(\triangle ABC) = \triangle A'B'C'$

(b) First a reflection r_1 in perpendicular bisector to AA' moves A to A'



~~Then~~ $\Phi \circ r_1(A') = A'$

Let $\Phi \circ r_1(B) = B''$

Consider the reflection r_2 in the angle bisector of $\angle B'A'B''$ moves B' to B''

$r_2 \circ \Phi \circ r_1 (A') = A'$

$r_2 \circ \Phi \circ r_1 (B) = B'$

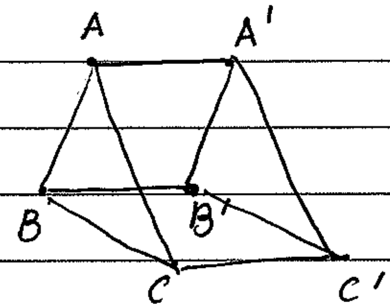
$r_2 \circ \Phi \circ r_1 = id$ or $r_2 \circ \Phi \circ r_1$ is the reflection r_3 in line $A'B'$. Then we have

$\Phi = r_2 \circ r_1$

$\Phi = r_2 \circ r_3 \circ r_1$

(4)

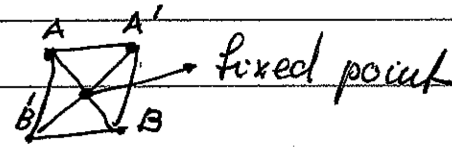
17.5 (a) Construct Φ as follows.



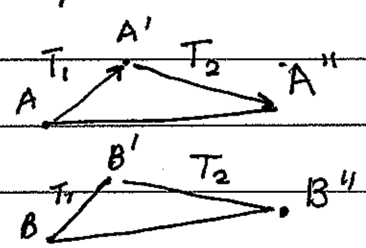
$\Phi(B) = B'$
so that $BB' \parallel AA'$
and $AB \parallel A'B'$
By SAS Φ preserves congruence of segment.
Hence Φ is a rigid motion.

(b) $AA' \simeq BB'$ (condition)
$AB \simeq A'B'$ (rigid motion)
$ABB'A'$ is a parallelogramm
Moreover, AB does not meet $A'B'$ (otherwise the intersection point is fixed by translation which is impossible.

Impossible case



(c)



$T_1(A) = A'$ $T_1(B) = B'$
$T_2(A') = A''$ $T_2(B') = B''$

By SAS $AA'' \cong BB''$.

Hence $T_2 \circ T_1$ is again translation.

~~(d) Yes. Let Φ be any rigid motion~~,
T be a translation.
Let ~~Then~~ $\Phi \circ T \circ \Phi^{-1}(A) = A'$, $\Phi \circ T \circ \Phi^{-1}(B) = B'$
Let $A = \Phi(A'')$, $B = \Phi(B'')$
Then $\Phi \circ T \circ \Phi^{-1}(A) = \Phi \circ T(A'')$

⑤

Yes.

17.5 (d) Let T be a translation and and Φ be a rigid motion. We have to show that $\Phi \circ T \circ \Phi^{-1}$ is a translation. Let $A' = \Phi \circ T \circ \Phi^{-1}(A)$ and $B' = \Phi \circ T \circ \Phi^{-1}(B)$

Let $A'' = T \circ \Phi^{-1}(A)$, $B'' = T \circ \Phi^{-1}(A)$.

Then $A''' = \Phi^{-1}(A'')$, $B''' = \Phi^{-1}(B)$

Since T is a translation,

$A''A''' \cong B''B'''$

Since Φ is a rigid motion

$A'A \cong B'B$

Therefore $\Phi \circ T \circ \Phi^{-1}$ is a translation by definition in (a).

17.10 Solution repeats 17.3(b).

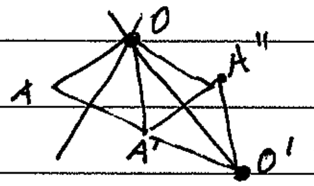
17.11 If S_1 is a rotation on angle α_1 and S_2 is a rotation on α_2, for any line ℓ, $S_1(\ell)$ and ℓ has angle α_1 and $S_2 \circ S_1(\ell)$ and $S_1(\ell)$ has angle $\beta = \alpha_1 \pm \alpha_2$.

Thus, if $S = S_2 \circ S_1$, and A and B are fixed by S, then the whole line AB is fixed but then S is a reflection. That is impossible because the angle between ℓ and $S(\ell)$ depends on ℓ. So if S has a fixed point, then S is a rotation.

Assume that S does not have fixed points.

(6)

Now consider the case when g does not have fixed points. Let $g(A) = A'$ and $g(A') = A''$. Note that $A'' \neq A$ (otherwise midpoint of AA' is fixed). If A, A' and A'' are not collinear, then there is a unique O such that $OA \simeq OA' \simeq OA''$.



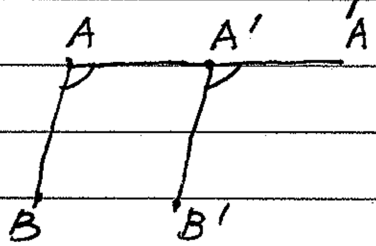
Then $g(O) = O$ or $g(O) = O'$, where O' is a point symmetric to O with respect to $A'A''$.

But the latter case is impossible since then $OA' \parallel O'A''$ but AA' is not parallel to $A'A''$ and we proved before that the angle between l and $g(l)$ does not depend on l.

On the other hand, $g(O) = O$ also impossible since g does not have fixed points.

That proves A, A', A'' are collinear.

Take another point B, let $B' = g(B)$.



Then $AA' \parallel B'B$ (otherwise the point of intersection is fixed), $\angle BAA' \cong B'A'A''$ so $AB \parallel A'B'$. Then $AA' \simeq BB'$ and that show T is a translation?

17.12. Note that the composition of a translation and a rotation is a translation (proof as in 17.11).

The composition of two reflections is either rotation or translation. So $r_1 r_2 \dots r_{2k-1} r_{2k} = r_1' \dots r_{2l+1}'$ implies $r_1 \dots r_{2k} r_1' \dots r_{2l}' = r_{2l+1}'$. LHS is a rotation or translation, RHS is a reflection. Contradiction.