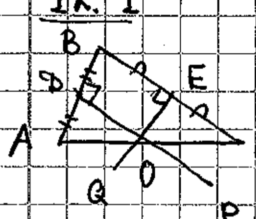
12.1 We claim that DP and EQ which



are perpendicular bisectors to AB and BC respectively meet.

Indeed if $DP \parallel EQ$, ~~then by (I.29)~~ then by (P) ~~$\angle BEQ \cong RA$ so B~~ DP intersects BC and is perpendicular to BC by (I.29).

Then $DP \perp BC$ and $DP \perp AB$, by (I.28) $AB \parallel BC$. Contradiction.

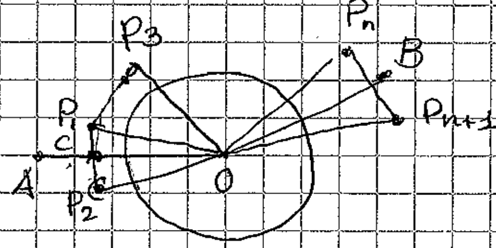
Now we have $OA \cong OB \cong OC$ (C6), hence O is the center of circumscribed circle.

12.6. One has to use the following standard definition of segment connected set.

S is segment connected if for any $A, B \in S$ there exists $A_1, \dots, A_n \in S$ such that $\overline{AA_1}, \overline{A_1A_2}, \dots, \overline{A_nB}$ belong to S. For definition in 11.1 the statement is incorrect.

Now let A and B be outside Γ, O be the center of Γ.

Choose C between A and O so that $OC < OA$, $OC < OB$.



But C is outside Γ. Let $CP_1 \perp OA$, $CP_1 + OC < OA$ and OB. Using (C1) and (C4) construct P_2 so that

$\triangle OP_2C \cong \triangle OP_1C$, then P_3 so that $\triangle OP_1P_3 \cong \triangle OP_2P_1$.

Then $\triangle OP_3P_4 \cong \triangle OP_1P_3$ etc. Use Lemma 35.1 to check that for some n $\angle COP_{n+1} > \angle AOB$.

Then $\overrightarrow{OB}$ is inside $\angle P_nOP_{n+1}$,

then $AP_1, P_1P_3, \dots, P_nB$ lie outside Γ.

13.2. We ha to show that if a, b are in a standard form, then $a+b, a-b, ab, \sqrt{a}, \frac{1}{a}$ can be written in a standard form. For the first 3 elemens the statement is obvious. So I do the last two. $a = rA = \frac{p}{q}A \Rightarrow \sqrt{a} = \frac{\sqrt{pqA}}{q}$ a standard form. The case of $\frac{1}{a}$ is more tricky. Clearly, it sufficent to write $\frac{1}{A}$ in the standard form. Note that A is obtained as a solution of quadratic equation $x^2 + Bx + C$ where B and C are in a standard form. If $\bar{A}$ is the second root then $\frac{1}{A} = \frac{\bar{A}}{A\bar{A}} = \frac{\bar{A}}{C}$.
No we may proceed by induction on the number of quadratic equations involved to obtain A.

13.3
$$\frac{\sqrt{5}+1}{\sqrt{10+2\sqrt{5}}} = \frac{\sqrt{5}+1}{\sqrt{2\sqrt{5}(1+\sqrt{5})}} =$$
$$= \frac{\sqrt{\sqrt{5}+1}}{\sqrt{2\sqrt{5}}} = \frac{\sqrt{2\sqrt{5}}\sqrt{\sqrt{5}+1}}{2\sqrt{5}} =$$
$$= \frac{\sqrt{10+2\sqrt{5}}}{2\sqrt{5}} = \frac{\sqrt{50+10\sqrt{5}}}{10}$$

13.8 Consider the regular tetrahedron in $\mathbb{R}^4$ with verticer $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$. The center of the circumbribed sphere is $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$. In this case the side $a = \sqrt{2}$, the radius of the sphere $r = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{\sqrt{3}}{2}$.
So $\frac{a}{r} = \frac{\sqrt{2}}{\sqrt{3}/2} = \sqrt{\frac{8}{3}}$. If radius is 1 the side is $\sqrt{\frac{8}{3}}$.

13.10. $a+b\sqrt{2} = (p+q\sqrt{2})^2 = p^2+2q^2+2\sqrt{2}pq$

$p,q \in \mathbb{Q},\ a,b \in \mathbb{Z}$ $\quad p^2+2q^2 = a \in \mathbb{Z}$

$a-b\sqrt{2} = (p-q\sqrt{2})^2$

$a^2-2b^2 = (p^2-2q^2)^2 \in \mathbb{Z} \Rightarrow p^2-2q^2 \in \mathbb{Z}$

$2p^2 \in \mathbb{Z} \Rightarrow p \in \mathbb{Z}$

$p^2-2q^2 \in \mathbb{Z} \Rightarrow 2q^2 \in \mathbb{Z} \Rightarrow q \in \mathbb{Z}$

13.20. the side of pentagon d, the side of hexagon 1, the side of decagon e. Have to show that $e^2+1^2 = d^2$. By Hartshorne $d^2 = \dfrac{5-\sqrt{5}}{2}$

$e^2 = 2-2\cos 36^\circ$.

$2\cos^2 36 - 1 = \cos 72^\circ$.

$\cos 72^\circ = \dfrac{2-d^2}{2} = \dfrac{\sqrt{5}-1}{4}$

$2\cos^2 36^\circ = 1+\cos 72^\circ = \dfrac{3+\sqrt{5}}{4}$

$\cos^2 36^\circ = \dfrac{3+\sqrt{5}}{8} = \dfrac{6+2\sqrt{5}}{16} = \dfrac{(1+\sqrt{5})^2}{16}$

$\cos 36^\circ = \dfrac{1+\sqrt{5}}{4}$

$e^2+1 = 2-2\cos 36^\circ + 1 = 3 - \dfrac{1+\sqrt{5}}{2} = \dfrac{5-\sqrt{5}}{2} = d^2.$