PRACTICE FINAL EXAM
MATH 130

1. Let $ABCD$ be a rhombus in Euclidean plane, P, Q, R and S be the midpoints of the sides AB, BC, CD and DA. Show that $PQRS$ is a rectangle.

2. In Euclidean Plane given two circles C_1 and C_2 and a point P, which does not belong to either C_1 or C_2. Using ruler and compass construct a circle C passing through P and perpendicular to C_1 and C_2. (Hint: Inversion can help.)

3. Let $\mathbb{Z}^2$ denote the set of all points in $\mathbb{R}^2$ with integer coordinates. A line is defined as a non-empty subset of $(x, y) \in \mathbb{Z}^2$ satisfying the equation $ax + by + c = 0$ for some $a, b, c \in \mathbb{Z}$, a or $b \neq 0$. Betweenness and congruence are defined as in $\mathbb{R}^2$. Determine which of the axioms $(I1)$-$(I3)$, $(B1)$-$(B4)$, $(C1)$-$(C6)$, (P) hold for $\mathbb{Z}^2$.

4. Let Π be an incidence plane satisfying Playfair's axiom. Let every line in Π contain exactly 4 points. Show that the total number of points in Π is not greater than 16.

5. Show that in a semielliptic plane (i.e., a Hilbert plane with sum of angles of triangle larger than two RA) the midline of Saccheri quadrilateral is greater than either of two congruent sides.

6. In a Poincaré plane given three P-lines l, m, n such that l is perpendicular to m and n. Show that m and n are P-parallel but not limiting parallel (i.e., the corresponding lines/circles cannot intersect on the absolute).

7. Give a ruler and compass construction of a triangle in Poincare plane with all three angles equal to $30°$. The ruler and compass construct the lines/circles in the ambient Euclidean plane (as opposed to P-lines and P-circles).

8. (a) Show that the set F of numbers of the form $a + b\sqrt{5}$, $a, b \in \mathbb{Q}$, is a field which allows an ordering $\prec$ such that $0 \prec a + b\sqrt{5}$ if and only if $0 > a - b\sqrt{5}$.

(b) Show that the coordinate plane F^2 may be equipped with two distinct notions of betweenness.

Solutions.

1. Note that the reflection in AC moves B to D, P to S, Q to R. Thus $\angle \mathrm{SP}\, Q \cong \angle PSR$, $\angle PQR \cong \angle QRS$. In the same way the reflection in BD moves A to C, P to Q and S to R. Hence $\angle QRS \cong \angle PSR$, $\angle SPQ \cong \angle PQR$. Therefore all four angles of the quadrilateral $PQRS$ are congruent, and their sum is $4RA$ in Euclidean plane. Hence the angles of $PQRS$ are right.

2. Draw any circle Γ with center at P. Perform the inversion ρ_Γ on C_1 and C_2 and obtain the circles C_1' and C_2'. Draw the line l through the centers of C_1' and C_2'. Construct $\gamma = \rho_\Gamma(l)$. Since l is perpendicular to C_1' and C_2' by construction and ρ_Γ preserves angles, then γ is perpendicular to C_1 and C_2.

3. $(I1)$ holds. If (x_0, y_0) and $(x_1, y_1) \in \mathbb{Z}^2$, then they are solutions of $ax+by+c=0$, where $b = x_0 - x_1$, $a = y_1 - y_0$, $c = x_1y_0 - x_0y_1$. $(I2)$ holds. If (x_0, y_0) belongs to a line $ax + by + c = 0$, then $(x_0 - b, y_0 + a)$ belongs to the same line. $(I3)$ holds since (0,0),(1,0) and (0,1) are non-collinear. $(B1)$ and $(B3)$ hold since they hold in $\mathbb{R}^2$ and $\mathbb{Z}^2$ is a subset of $\mathbb{R}^2$. $(B2)$ holds. If $A = (a_1, a_2)$, $B = (b_1, b_2)$, put $C = (2b_1 - a_1, 2b_2 - a_2)$. $(B4)$ does not hold, counterexample $A = (0, 2)$, $B = (0, 0)$, $C = (1, 0)$, $D = (0, 1)$, $E = (1, -1)$. Then $l =$DE meets BC at $\left(\frac{1}{2}, 0\right)$ which does not exist in $\mathbb{Z}^2$. $(C1)$ does not hold. Counterexample: $A = C = (0, 0)$, $B = (1, 0)$, $M = (1, 1)$, the ray CM does not contain D with integer coordinates such that $CD \cong AB$. $(C2)$,$(C3)$, $(C5)$ and $(C6)$ hold since they are true for $\mathbb{R}^2$ and $\mathbb{Z}^2$ is a subset of $\mathbb{R}^2$. $(C4)$ also holds but it is harder to prove. Indeed, any angle in $\mathbb{Z}^2$ has a rational tangent (see Section 16 for details). Let $s \in \mathbb{Q}$, AB be a ray. We will show that there exists C such that the tangent of $\angle BAC$ equals s. Using translation we may assume without loss of generality that $A = (0, 0)$. If the slope of AB is m then the slope AC can be calculated by the formula $n = \frac{s+m}{1+ms}$ (the formula for the tangent of sum of two angles). Since s and m are rational, $n = \frac{p}{q}$, and we can take $C = (q, p)$. Finally, (P) is not true. For instance, take $A = (0, 0)$, $B = (1, 0)$, $C = (0, 1)$, $D = (1, 1)$, $E = (1, -1)$. Then both CD and CE are parallel to AB.

4. Let l be a line and A be a point not on l (it exists by $I3$). We claim that there is at most 5 lines passing through A. Indeed, there is at most one line m through A parallel to l, any other line through A meets l, there are exactly 4 such lines for each point on l. Since every line passing through A has exactly three points except A, the total number of points is not greater than $3 \times 5 + 1 = 16$.

5. Let $ABCD$ be a Saccheri quadrilateral and M be the midpoint of AB, N be the midpoint of CD. Let E be the point on the ray AC such that $AE \cong MN$. Then $MANE$ is a Saccheri quadrilateral. Because the plane is semielliptic, $\angle MNE$ is obtuse. Since $\angle MNC$ is right, NE lies outside $\angle MNC$. Hence $A * C * E$ and $AE >$ AC.

6. First, m and n are parallel because the alternating angles are congruent. To see that they are not limiting parallel let l and m meet at A, l and n meet at B. If m and

n are limiting parallel, then the angle of parallelism $\alpha\,(AB)$ is right. But we know that the angle of parallelism is always acute in a hyperbolic plane. Contradiction.

7. Let Γ be the boundary of the Poincare model, O be the center of Γ. Choose P and Q on Γ so that $\angle POQ = 30°$. Construct PM and QN so that $\angle OPM = \angle OQN = 120°$. Let S be the point where PM and QN meet. Construct δ with center at S and radius SP and ε with the diameter SO. Get C and D where δ and ε meet. Let A and and B be the points where Γ meets OC and OD. Through A and B construct the lines parallel to SC and SD respectively. Let them meet at E. Construct the circle γ centered at E with radius $EA = EB$. Then γ, OP and OQ form required triangle. Proof as in exercise 39.4.

8. (This is a sketch only.)

The only non-trivial check for axioms of field is a possibility to find the inverse number. However, define $\overline{a + b\sqrt{5}}$ by $a - b\sqrt{5}$; then $f \cdot \overline{f}$ is a rational number, which shows that $1/f$ is in F if f is.

Moreover, one can immediately see that $\overline{f + f'} = \overline{f} + \overline{f'}$, likewise for multiplication. Since $<$ gives an ordering of F (induced from the ordering of $\mathbb{R}$), and $f \prec f'$ is equivalent to $\overline{f} < \overline{f'}$, the relationship $\prec$ has the required properties of compatibility with addition and multiplication.

Finally, since to describe betweenness on a coordinate plane over a field is equivalent to describing ordering of the field, the orders $<$ and $\prec$ on F give two distinct notions of betweenness.