Solutions of homework problems.
Math 113
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10 (8.3) First, we will show that $H$ is contained in some Sylow subgroup. Let $K = G/H$ and consider the natural homomorphism $\pi : G \rightarrow K$. Let $S$ be a Sylow $p$-group in $K$, then $P = \pi^{-1}(S)$ is a Sylow $p$-group of $G$ and $H \subseteq P$.

Now let $P'$ be another Sylow $p$-group. Then by the second Sylow theorem $P' = gP_{g^{-1}}$ for some $g \in G$. But then $gHg^{-1} \subseteq P'$. Since $H$ is normal $gHg^{-1} = H$, and thus $H \subseteq P'$.

13 (8.3) Let $N_1, \ldots, N_k$ be Sylow subgroups and $|N_i| = p_i^{m_i}$. Use induction on $k$. For $k = 1$ the statement is trivial. Since all $N_i$ are normal, $K = N_1N_2 \cdots N_{k-1}$ is a normal subgroup of $G$, each $N_i$ for $i \leq k - 1$ is a Sylow subgroup of $K$ and therefore by inductions assumption $K = N_1 \times \cdots \times N_{k-1}$, which implies $|K| = |N_1| \cdots |N_{k-1}| = p_1^{m_1} \cdots p_{k-1}^{m_{k-1}}$. Then $|K|$ and $|N_k|$ are relatively prime, and hence $K \cap N_k = \{e\}$. Then $KN_k$ is a subgroup of $G$, $|KN_k| = |K||N_k|$ = $|G|$. Thus $G = KN_k$ and $K \cap N_k = \{e\}$, therefore $G = K \times N_k = N_1 \times \ldots \times N_k$.

17(8.3) The possible number of Sylow 3-subgroups is either 1 or 7. However, $G$ has only one Sylow 7-subgroup. Thus, if $G$ has one Sylow 3-subgroup and one Sylow 7-subgroup By the previous exercise $G \cong Z_7 \times Z_3 \cong Z_{21}$. Since it is given that $G$ is noncyclic, the only possible answer is 7.

20 (8.3) Let $s$ be the number of Sylow $p$-groups. By the third Sylow theorem $s|m$ and $s \equiv 1 \pmod{p}$. Since $m < p$, we obtain $s = 1$, and $G$ contains a normal Sylow $p$-subgroup.

19 (8.4) $K$ is normal in $N(N(K))$ and $K$ is a unique Sylow $p$-subgroup of $N(K)$. Suppose that $g \in N(N(K))$, then $gKg^{-1} \subseteq N(K)$, and $gKg^{-1}$ is another Sylow subgroup of $N(K)$. But $K$ is unique Sylow $p$-subgroup, therefore $gKg^{-1} = K$, which implies $g \in N(K)$.

22 (8.4) Use complete induction on $n$. For abelian group the statement follows from the fundamental theorem of abelian groups. So we may assume that $G$ is not abelian. Let $|G| = p^n$, $K = G/Z(G)$. By theorem 8.5$|K| = p^k$, with $k < n$, and therefore by induction assumption there is a normal subgroup $N$ of $K$ of order $p^{k-1}$. Let $\pi : G \rightarrow K$ be the natural homomorphism, then $\pi^{-1}(N)$ is a normal subgroup of $G$ of order $p^{n-1}$.

17(8.3) By exercise 17(8.3) $G$ have a normal 7-subgroup. If its 3-subgroup is also normal then $G$ is isomorphic to $Z_{21}$. Assume now that $G$ has seven 3-subgroups, i.e. $G$ is not abelian. Let $N$ be a normal 7-subgroup, and choose a generator $a$ in $N$. Let $b$ be a generator of some 3-group. We have $a^7 = e$, $b^3 = e$, and every element of the group can be written as $b^ma^n$ for $m = 0, 1, 2$ and $n = 0, \ldots, 6$. Furthermore we have $bab^{-1} = a^k$, $a = b^3ab^{-3} = a^3$, which implies $k^3 \equiv 1 \pmod{7}$, i.e. $k = 2$ or 4. Note that if $k = 4$, we can choose $b^2$ instead of $b$ and get $k = 2$. Therefore without loss of generality one may assume $k = 2$. Then the multiplication law in the group is the
following
\[ b^m a^n b^p a^l = b^{m+k} b^{-p} a^n b^p a^l = b^{m+k} a^{n^2+2^{-l}}. \]
Thus, there is one non-abelian group of order 21 up to isomorphism. To realize this
group consider the subgroup of upper triangular matrices in \( \text{PSL}_2(\mathbb{Z}_7) \).

9 (10.1) Solve the equation \( a\sqrt{2} + b (\sqrt{2} + i) + c (\sqrt{3} - i) = 0 \) for real \( a, b, c \).
Possible solution \( b = c = 1, a = -1 - \frac{\sqrt{2}}{\sqrt{3}}. \)

23 (10.1) (a) Write \( a + b\sqrt{2} + c\sqrt{3} = 0 \). You have to show that if \( a, b, c \in \mathbb{Q} \), then
\( a = b = c = 0 \). If \( c = 0, b \neq 0 \), then \( \sqrt{2} = \frac{a}{b} \), which is impossible. If \( c \neq 0 \) one may
assume that \( c = 1 \). Then
\[ \sqrt{3} = -a - b\sqrt{2} \implies 3 = a^2 + 2b^2 + 2\sqrt{2}ab \implies a = 0, 2b^2 = 3 \text{ or } b = 0, a^2 = 3 \]
Both cases are impossible.
(b) Using (a) we obtain \( [\mathbb{Q}(\sqrt{2}, \sqrt{3} : \mathbb{Q}(\sqrt{2})] = 2 \), and therefore \( [\mathbb{Q}(\sqrt{2}, \sqrt{3} : \mathbb{Q}] = 4. \)
Furthermore, the basis of \( \mathbb{Q}(\sqrt{2}, \sqrt{3}) \) over \( \mathbb{Q} \) is
\[ \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\} \]

5 (10.2) In each case the number is a root of a polynomial with rational coefficients.
The polynomials are
\[ x^2 - 6x + 34, x^4 + 4x^2 + 5, x^3 - 3x^2 + 3x - 3 \]

9 (10.2) \( \sqrt{\pi} \) is a root of polynomial \( x^2 - \pi. \)

17 (10.2) Answers:
\[ x^4 - 2x^2 - 4, x^4 + 2x^2 + 25 \]

18 (10.2) Over \( \mathbb{Q} \): \( p(x) = x^4 - 2x^2 + 9 \), over \( \mathbb{R} \): \( p(x) = x^2 - 2\sqrt{2}x + 3. \)

11 (10.3) (a) \( F \subset F(u) \subset F(u, v) \) implies \( m| [F(u, v) : F] \). In the same way \( F \subset F(v) \subset F(u, v) \) implies \( n| [F(u, v) : F] \).
Since \( (m, n) = 1 \), we have \( mn| [F(u, v) : F] \).
On the other hand, \( [F(u, v) : F(u)] \leq m \). Therefore \( [F(u, v) : F] = mn. \)
(b) Let \( F = \mathbb{Q}, p(x) = x^2 - 2, q(x) = x^2 - 2x - 1. \) Then \( F(u, v) = F(u). \)
(c) 6 because of (a).