Sample Final.
1. Show that the equation
\[ x^2 - 5y^2 = 7 \]
does not have integer solutions.
2. Let \( U_{34} \) denote the group of all units of the ring \( \mathbb{Z}_{34} \). Is \( U_{34} \) cyclic? Justify your answer.
3. Determine which of the following polynomial rings are integral domain? Justify your answer.
   \( (a) \mathbb{Q} [x] / (x^3 - 3) \)
   \( (b) \mathbb{Q} [x] / (x^4 - 4) \)
   \( (c) \mathbb{Z}_{15} [x] \)
   \( (d) \mathbb{C} [x] / (x^2 + 1) \)
4. Factor the polynomial \( x^4 + x^2 + 1 \) into product of irreducibles in the ring \( \mathbb{Z}_3 [x] \).
5. Let \( \mathbb{C}^* \) denote the group of all non-zero complex numbers with operation of multiplication.
   \( (a) \) Describe all finite subgroups of \( \mathbb{C}^* \).
   \( (b) \) Show that if \( H \) is a finite subgroup of \( \mathbb{C}^* \), then the quotient group \( \mathbb{C}^* / H \) is isomorphic to \( \mathbb{C}^* \).
6. Is any ideal in \( \mathbb{Z}_2 [x] \) principal? The same question for \( \mathbb{Z}_4 [x] \).
7. Let \( R \) denote the quotient ring \( \mathbb{Z}_3 [x] / (x^2 + 1) \). Find the elementary divisors of the additive group \( R \) and the multiplicative group \( U \) of all units of \( R \).
8. List all Sylow subgroups of the group \( D_9 \).
9. Show that a group of order 33 is cyclic.
10. Show that the number \( 3 \sqrt{3} + 1 \) is algebraic over \( \mathbb{Q} \). Find its minimal polynomial.
Solutions.
1. First solve the equation in \( \mathbb{Z}_7 \). We obtain \( \left( \frac{x}{y} \right)^2 = 5 \) or \( x = y = 0 \). Since the former equation does not have solutions in \( \mathbb{Z}_7 \), we obtain \( x \equiv y \equiv 0 \pmod{7} \). Then the left hand side of the equation \( x^2 - 5y^2 \) is divisible by 49, and could not be equal 7.
2. Since \( \mathbb{Z}_{34} \cong \mathbb{Z}_{17} \times \mathbb{Z}_2 \), \( U_{34} \cong U_{17} \), and \( U_{17} \) is a cyclic group of order 16 by Theorem 7.15.
3. \( (a) \) Yes, because \( x^3 - 3 \) is irreducible over \( \mathbb{Q} \).
   \( (b) \) No, since \( x^4 - 4 = (x^2 - 2)(x^2 + 2) \).
   \( (c) \) No, since \( \mathbb{Z}_{15} \) is not an integral domain.
   \( (d) \) No, because \( x^2 + 1 \) is reducible over \( \mathbb{C} \).
4. \( x^4 + x^2 + 1 = (x + 1)^2 (x - 1)^2 \)
5. Any finite subgroup of \( \mathbb{C}^* \) must be cyclic. Therefore for every \( n > 0 \) there exists a unique subgroup of order \( n \) consisting of all \( n^{th} \) roots of unity. Let \( H \) be such a subgroup. Consider the homomorphism \( f : \mathbb{C}^* \to \mathbb{C}^* \) defined by \( f(z) = z^n \). Then \( f \) is surjective and the kernel of \( f \) is \( H \). Therefore \( \mathbb{C}^* \) is isomorphic to \( \mathbb{C}^*/H \) by the first isomorphism theorem.

6. Every ideal in \( F[x] \) is principal if \( F \) is a field. In particular when \( F = \mathbb{Z}_2 \). For \( \mathbb{Z}_4[x] \) this is not so. Counter example: the ideal generated by 2 and \( x \).

7. \( R \) is isomorphic to \( \mathbb{Z}_3 \oplus \mathbb{Z}_3 \), \( U \) is cyclic, because \( R \) is a field and therefore \( U \) is isomorphic to \( \mathbb{Z}_8 \). Therefore elementary divisors of \( R \) are 3 and 3, elementary divisor of \( U \) is 8.

8. There is one Sylow 3-subgroup isomorphic to \( \mathbb{Z}_9 \). It is the subgroup of all rotations. There are 9 Sylow 2-subgroups, each is generated by one flip.

9. Let \( m \) be the number of Sylow 3-subgroups and \( n \) be the number of Sylow 11-subgroups. Then \( m \mid 11 \) and \( m \equiv 1 \pmod{3} \), that implies that \( m = 1 \). In the same way \( n \mid 11 \) and \( n \equiv 1 \pmod{11} \) implies that \( n = 1 \). Both Sylow subgroups are normal and cyclic, and the whole group is isomorphic to \( \mathbb{Z}_3 \times \mathbb{Z}_{11} \cong \mathbb{Z}_{33} \).

10. The number \( 3\sqrt{3} + 1 \) is a root of polynomial \( p(x) = x^3 - 3x^2 + 3x - 4 \). Note that \( p(x) \) is irreducible over \( \mathbb{Q} \) because it does not have rational roots. Therefore \( p(x) \) is the minimal polynomial of \( 3\sqrt{3} + 1 \).