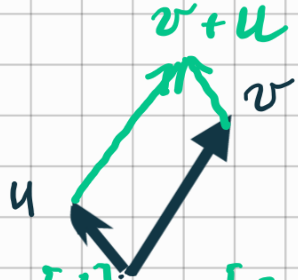
Vectors in $\mathbb{R}^n$

Sections 1.3 – 1.4

A vector in $\mathbb{R}^n$ is a column of length n.

$\mathbb{R}^2$

$v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$



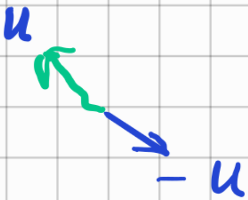
$u = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$v + u$

$$v + u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 3+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

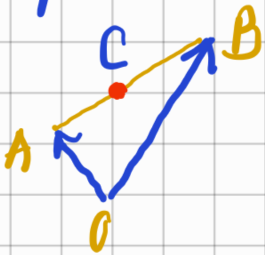
$-u$

$v - u$



$\mathbb{R}^4 \quad \begin{bmatrix} 1 \\ 3 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$

Midpoint



$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

<u>midpoint</u> of the segment AB

$$\vec{OC} = \frac{1}{2}\left(\vec{OA} + \vec{OB}\right)$$

Works for $\mathbb{R}^n$

Addition $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$

Multiplication by scalar $c \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c x_1 \\ \vdots \\ c x_n \end{bmatrix}$ $\mathbb{R}$

Properties

$u + v + w$

- $u + v = v + u$ commutativity
- $(u + v) + w = u + (v + w)$ associativity
- $u + \underline{0} = \underline{0} + u = u$ $\quad \underline{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$
- $c(u + v) = cu + cv$ distributivity
- $(c+d)u = cu + du \Rightarrow u + (-u) = \underline{0}$

  $-u = -1 u$
- $c(du) = (cd)u$
- $1 u = u$ $\quad 0 u = \underline{0}$

Linear combination $v_1, \dots, v_p$ vectors in $\mathbb{R}^n$

$u = c_1 v_1 + \dots + c_p v_p$, where $c_1, \dots, c_p$ some numbers

u is called a linear combination of $v_1, \dots, v_p$ with weights $c_1, \dots, c_p$

Example $2\begin{bmatrix}1\\1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}2\\2\\0\end{bmatrix} + \begin{bmatrix}0\\3\\3\end{bmatrix} = \begin{bmatrix}2\\5\\3\end{bmatrix}$

linear combination

Problem

Gi ven two vectors

$$v_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}.$$

Determine whether

$$u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is a linear combination of v_1, v_2

$$x_1 v_1 + x_2 v_2 = u$$

vector equation

$$x_1 \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x_1 + 2x_2 = 1$$

$$-3x_1 - x_2 = 1$$

$$5x_1 + 4x_2 = 1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 1 \\ 5 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 4 \\ 0 & -6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & -6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Inconsistent

u is not a linear combination of v_1, v_2.

Vector equation $x_1 v_1 + \ldots + x_n v_n = \underline{b}$

$v_1, \ldots, v_n$ are some vectors $\mathbb{R}^m$

$\underline{b}$ is also a vector in $\mathbb{R}^m$

Vector equation has a solution if and only $\underline{b}$ is a linear combination of $v_1, \ldots, v_n$.

Span.

- Span$\{v_1, \ldots, v_p\}$ is the collection of all linear combinations of $v_1, \ldots, v_p$

A vector u lies in the span$\{v_1, \ldots, v_p\}$ if and only if vector equation

$x_1 v_1 + \ldots + x_p v_p = u$ has a solution.

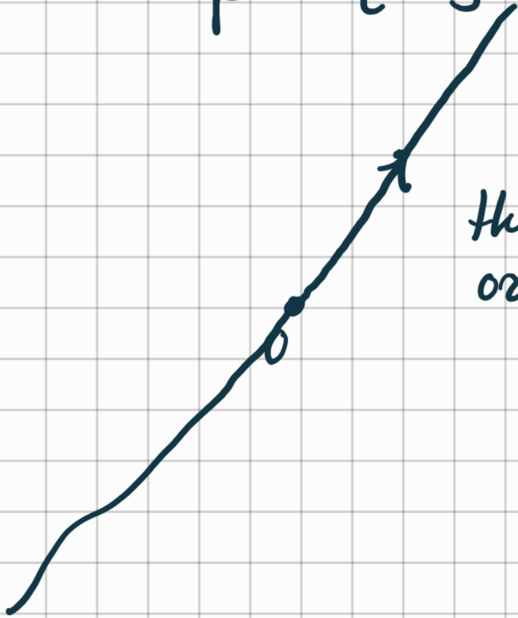
$$\text{span}\{v_1, \ldots, v_p\} = \{c_1 v_1 + \ldots + c_p v_p\}$$

$c_1, \ldots, c_p$ run all real numbers

Span$\{u\}$ is a line, $u \neq \underline{0}$

Span$\{u, v\}$

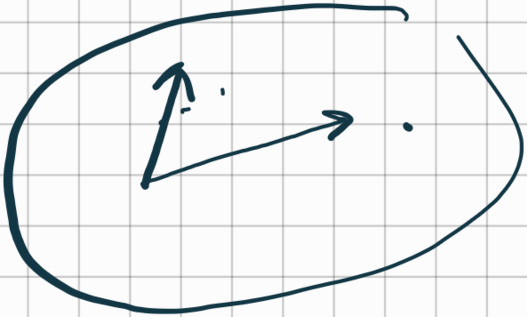
Span$\{u\} = \{cu\}$



line through origin.

Span$\{\underline{0}\} = \{\underline{0}\}$

$u = cv$ a line



a plane containing origin.

If u, w lie in Span $\{v_1, \ldots, v_p\}$ then $au + bv$ lie in Span $\{v_1, \ldots, v_p\}$

If u, w lie in Span $\{v_1, \ldots, v_p\}$

$u = c_1 v_1 + \ldots + c_p v_p$

$w = d_1 v_1 + \ldots + d_p v_p$

$u + w = c_1 v_1 + \ldots + c_p v_p + d_1 v_1 + \ldots + d_p v_p =$

$= c_1 v_1 + d_1 v_1 + c_2 v_2 + d_2 v_2 + \ldots + c_p v_p + d_p v_p$

$= (c_1 + d_1) v_1 + (c_2 + d_2) v_2 + \ldots + (c_p + d_p) v_p$

Matrix-vector product

A is $m \times n$ matrix, $A = [\underline{a}_1, \dots, \underline{a}_n]$
collection of vectors in $\mathbb{R}^m$.

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is a vector in $\mathbb{R}^n$

<u>Def</u> $A x = x_1 \underline{a}_1 + \dots + x_n \underline{a}_n$ a vector in $\mathbb{R}^m$.

$$\begin{array}{cc} & n \\ m & \left[\qquad\qquad\right] \end{array} \left[\;\;\right] n = \left[\;\;\right] m$$

$$\begin{bmatrix} 3 & 5 & 7 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix} = -\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 0.5 \begin{bmatrix} 5 \\ -2 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \end{bmatrix} =$$

$$= \begin{bmatrix} -3 \\ -1 \end{bmatrix} + \begin{bmatrix} 2.5 \\ -1 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 6.5 \\ 1 \end{bmatrix}$$

Second way to do matrix vector multiplication

Row-vector rule

$$\begin{bmatrix} 3 & 5 & 7 \\ +1 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 6.5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix} = -3 + 2.5 + 7 = 6.5$$

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.5 \\ 1 \end{bmatrix} = -1 - 1 + 3 = 1$$

Properties of matrix vector product

$$A(u+v) = Au + Av$$

$$A(cv) = cAv$$

A is $m \times n$ matrix, u, v are in $\mathbb{R}^n$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad A = \begin{bmatrix} \underline{a}_1 & \dots & \underline{a}_n \end{bmatrix}$$

$$(u_1 \underline{a}_1 + \dots + u_n \underline{a}_n) + (v_1 \underline{a}_1 + \dots + v_n \underline{a}_n)$$

$$= (u_1 + v_1)\underline{a}_1 + \dots + (u_n + v_n)\underline{a}_n$$

$$= A(u+v)$$