SAMPLE FINAL
MATH 54

Whereabouts: Final exam will take place on Friday, December 18, 11:30-2:30 in RSF Field house. Approximately $\frac{1}{3}$ of the exam will be on linear algebra and $\frac{2}{3}$ on differential equations. Calculators, books and notes may not be used during the test. There is no possibility of make up final. Don't miss it!

In addition to the sample exam below I recommend to do the following problems from the textbook.

Linear algebra. Supplementary exercises.

Chapter 1: 13,19,23.
Chapter 2: 4,7,10,12,18.
Chapter 3: 14,15.
Chapter 4: 10,11,12,13,14,15,16.
Chapter 5: 7,8,9,10,12.
Chapter 6: 13,14,15.
Chapter 7: 2,4.

Differential equations.

Chapter 4: review exercises 4,21,33,38.
Chapter 6: p.483, 34,35. Review exercises 2,4,7.
Chapter 9: p.559, 25. Review exercises 9,11,15.
Chapter 10: p.593, 28, p. 623, 10, p.635, 20.

Date: December 9, 2015.

Practice test

1. Let A be an $n \times n$ matrix such that $A^2 = 0$.
(a) Show that $\operatorname{Col} A$ is contained in $\operatorname{Nul} A$.
(b) Show that rank $A \leq \frac{n}{2}$.

2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that

$$T(\mathbf{v})\cdot\mathbf{u} = -\mathbf{v}\cdot T(\mathbf{u})$$

for all $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.
(a) Show that if λ is a real eigenvalue of T, then $\lambda = 0$.
(b) Show that $\operatorname{Nul} T$ is the orthogonal complement to $\operatorname{Range} T$.
(c) Show that if A is the matrix of T in the standard basis, then $A^T = -A$.

3. Let A be a matrix such that $A^T = -A$. Show that e^{At} is an orthogonal matrix for any t.

4. Solve the initial value problem:

$$y''' - y'' + y' - y = e^x, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 1.$$

5. Let $f(x)$ and $g(x)$ be two continuous differentiable functions defined on the whole real line. Prove that if the Wronskian

$$\det \begin{bmatrix} f(x_0) & g(x_0) \\ f'(x_0) & g'(x_0) \end{bmatrix} \neq 0$$

for some x_0, then $f(x)$ and $g(x)$ are linearly independent.

6. Find the fundamental matrix for the system:

$$\mathbf{x}'(t) = \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \mathbf{x}(t).$$

7. Find a particular solution of the system using the method of variation of parameters:

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \tan t \\ 0 \end{bmatrix}.$$

8. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Find eigenvalues of A and the basis consisting of generalized eigenvectors.
(b) Find the general solution of the system

$$\mathbf{x}'(t) = A\mathbf{x}(t).$$

9. Compute the Fourier sine series for the function $f(x) = x^2$ on the interval $[0, \pi]$. Does this series converge to $f(x)$?

10. Find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$

satisfying the boundary conditions

$$u(0,t) = u(\pi,t) = 0$$

and the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = \sin 2x.$$

Solutions.

1.

(a) Let $A = [\mathbf{a_1}, \dots \mathbf{a_n}]$, where $\mathbf{a_1}, \dots \mathbf{a_n}$ are the column vectors of A. Then we have $A\mathbf{a_i} = \mathbf{0}$ for all $i = 1, \dots, n$. Hence $\mathbf{a_i}$ lies in $\operatorname{Nul} A$. Hence the span of $\mathbf{a_1}, \dots \mathbf{a_n}$ lies in $\operatorname{Nul} A$.

(b) By (a) we know that

$$\dim \operatorname{Col} A \leq \dim \operatorname{Nul} A.$$

Since

$$\dim \operatorname{Col} A + \dim \operatorname{Nul} A = n,$$

we have

$$2 \dim \operatorname{Col} A \leq n \quad \text{or} \quad \dim \operatorname{Col} A \leq \frac{n}{2}.$$

2.

(a) Let $\mathbf{u}$ be an eigenvector of T with eigenvalue λ, then

$$\lambda \mathbf{u} \cdot \mathbf{u} = T(\mathbf{u}) \cdot \mathbf{u} = -\mathbf{u} \cdot T(\mathbf{u}) = -\lambda \mathbf{u} \cdot \mathbf{u}.$$

This implies $\lambda = -\lambda = 0$.

(b) Let $\mathbf{u}$ be a vector in $\operatorname{Nul} T$. We show that $\mathbf{u}$ is orthogonal to any vector $\mathbf{v}$ in $\operatorname{Range} T$. Indeed, $\mathbf{v} = T(\mathbf{w})$ and we have

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot T(\mathbf{w}) = -T(\mathbf{u}) \cdot \mathbf{w} = \mathbf{0} \cdot \mathbf{w} = 0.$$

On the other hand, if $\mathbf{u}$ is orthogonal to any $\mathbf{v}$ in $\operatorname{Range} T$, then

$$\mathbf{u} \cdot T(\mathbf{w}) = -T(\mathbf{u}) \cdot \mathbf{w} = 0$$

for any vector $\mathbf{w}$. Take $\mathbf{w} = T(\mathbf{u})$. Then

$$T(\mathbf{u}) \cdot T(\mathbf{u}) = 0.$$

Hence $T(\mathbf{u}) = 0$.

(c) Using

$$T(\mathbf{e}_i) = \sum_{j=1}^{n} a_{ij} \mathbf{e}_j,$$

we obtain

$$a_{ij} = T(\mathbf{e}_i) \cdot \mathbf{e}_j = -\mathbf{e}_i \cdot T(\mathbf{e}_j) = -a_{ji}.$$

3. Let $U = e^{At}$. Using properties of the matrix exponential we have

$$U^t U = (e^{At})^T e^{At} = e^{A^T t} e^{At} = e^{-At} e^{At} = I.$$

4. We start with solving the auxiliary equation

$$\lambda^3 - \lambda^2 + \lambda - 1 = (\lambda - 1)(\lambda^2 + 1) = 0.$$

There is one real root $\lambda = 1$ and two complex roots $i, -i$. Hence we can take $\{e^t, \cos t, \sin t\}$ as a fundamental set of solutions of the homogeneous equation.

Now we need to find a particular solution. We use the method of undetermined coefficients. Since 1 is a root of the auxiliary equation we look for particular solution in the form cte^t. We get the equation

$$c(te^t + 3e^t - te^t - 2e^t + te^t + e^t - te^t) = 2ce^t = e^t, \quad c = \frac{1}{2}.$$

That gives a particular solution

$$y_p(t) = \frac{1}{2}te^t,$$

and the general solution

$$y(t) = \frac{1}{2}te^t + C_1e^t + C_2\cos t + C_3\sin t.$$

It remains to find C_1, C_2, C_3 for the given initial conditions. We have

$$C_1 + C_2 = 0, \quad \frac{1}{2} + C_1 + C_3 = 1, \quad 1 + C_1 - C_2 = 1,$$

and we get $C_1 = C_2 = 0$, $C_3 = \frac{1}{2}$. Therefore

$$y(t) = \frac{1}{2}te^t + \frac{1}{2}\sin t.$$

5. It is sufficient to show that if $f(x)$ and $g(x)$ are linearly dependent, then the Wronskian is the zero function. Indeed, we have without loss of generality $g(x) = Cf(x)$ for some constant C. Therefore

$$\det\begin{bmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{bmatrix} = \det\begin{bmatrix} f(x) & Cf(x) \\ f'(x) & Cf'(x) \end{bmatrix} = 0.$$

6. The characteristic equation

$$(3-\lambda)^2 - 16 = (\lambda - 7)(\lambda + 1) = 0$$

gives eigenvalues 7 and -1. The corresponding eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Therefore the fundamental set of solutions is

$$\left\{ e^{7t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

and the fundamental matrix

$$\begin{bmatrix} e^{7t} & e^{-t} \\ e^{7t} & -e^{-t} \end{bmatrix}.$$

7. We start with finding the fundamental matrix. Since the roots of the characteristic equation are $\pm i$, the fundamental matrix is

$$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

Next we solve the linear system

$$\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \begin{bmatrix} \tan t \\ 0 \end{bmatrix}.$$

This yields the solution

$$v_1'(t) = \sin t, \quad v_2'(t) = \sin t \tan t = \sec t - \cos t.$$

We integrate to find $v_1(t)$ and $v_2(t)$:

$$v_1(t) = -\cos t + C_1, \quad v_2(t) = \ln|\sec t + \tan t| - \sin t + C_2.$$

A particular solution

$$\mathbf{x}_p(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} -\cos t \\ \ln|\sec t + \tan t| - \sin t \end{bmatrix} = \begin{bmatrix} -1 + \sin t \ln|\sec t + \tan t| \\ \cos t \ln|\sec t + \tan t| \end{bmatrix}.$$

8. The matrix A has one eigenvalue 1. We check that

$$(A - I)^3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 = 0.$$

Hence any vector is a generalized eigenvector with eigenvalue 1. We can take the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Now we find three fundamental solutions

$$\mathbf{x}_1(t) = e^t \mathbf{e}_1,$$

$$\mathbf{x}_2(t) = e^t(\mathbf{e}_2 + (A - I)t\mathbf{e}_2) = e^t(\mathbf{e}_2 + t\mathbf{e}_1),$$

$$\mathbf{x}_3(t) = e^t(\mathbf{e}_3 + (A - I)t\mathbf{e}_3 + \frac{t^2}{2}(A - I)^2\mathbf{e}_3) = e^t(\mathbf{e}_3 + t\mathbf{e}_1 + t\mathbf{e}_2 + \frac{t^2}{2}\mathbf{e}_1).$$

That gives the general solution

$$\mathbf{x}(t) = C_1 \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} te^t \\ e^t \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} (\frac{t^2}{2} + t)e^t \\ te^t \\ e^t \end{bmatrix}.$$

9. We calculate coefficients

$$b_n = \frac{2}{\pi} \int_0^\pi x^2 \sin nx dx.$$

The integral is calculated by parts (twice)

$$\int x^2 \sin nx dx = -x^2 \frac{\cos nx}{n} + \int 2x \frac{\cos nx}{n} dx,$$

$$\int 2x \frac{\cos nx}{n} dx = 2x \frac{\sin nx}{n^2} - 2 \int \frac{\sin nx}{n^2} dx = 2x \frac{\sin nx}{n^2} + 2 \frac{\cos nx}{n^3}.$$

Thus, we obtain

$$\int_0^\pi x^2 \sin nx dx = \left(\frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right)_0^\pi = \frac{(-1)^{n+1}\pi^2}{n} + 2\frac{(-1)^n - 1}{n^3},$$

and hence

$$b_n = \begin{cases} \frac{(-1)^{n+1}2\pi}{n} & \text{for even} \quad n \\ \frac{(-1)^{n+1}2\pi}{n} - \frac{8}{\pi n^3} & \text{for odd} \quad n. \end{cases}$$

The Fourier sine series is

$$x^2 \sim 2\pi \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \sin nx - \frac{8}{\pi} \sum_{k=0}^\infty \frac{1}{(2k+1)^3} \sin(2k+1)x.$$

The series converges to x^2 in all points except $x = \pi$, where it converges to 0.

10. We look for solution in the form

$$u(x,t) = \sum_{n=1}^\infty (a_n \cos 2nt + b_n \sin 2nt) \sin nx.$$

The initial conditions give

$$\sum_{n=1}^\infty a_n \sin nx = 0, \quad \sum_{n=1}^\infty 2nb_n \sin nx = \sin 2x.$$

We obtain $a_n = 0$ and $b_n = 0$ if $n \neq 2$, $b_2 = \frac{1}{4}$. The solution is given by the formula

$$u(x,t) = \frac{1}{4} \sin 4t \sin 2x.$$