Sample midterm

1. Find the rank and the dimension of the null space of the matrix A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix},$$

2. Let $\mathbb{P}_3$ denote the space of all polynimials of degree less or equal than 3. Let T be a linear transromation from $\mathbb{P}_3$ to itself defined by

$$T(p(t)) = p''(t) + p'(t) + p(t).$$

(a) Find the matrix $[T]_{\mathcal{B}}$ for the basis $\mathcal{B} = \{1, t, t^2, t^3\}$.

(b) Find eigenvalues and eigenspaces of T.

(c) Is T diagonalizable?

3. Let T and S be linear transformations in $\mathbb{C}^n$ such that $ST = TS$. Prove that T and S have common eigenvector, i. e. there exists a vector v in $\mathbb{C}^n$ such that

$$T(v) = \lambda v, \quad S(v) = \mu v$$

for some complex numbers λ and μ.

4.

(a) Find the orthogonal basis in Col A for

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

(b) Find the orthogonal projection of $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ onto Col A.

5. Let A be a symmetric $n \times n$ matrix. Show that Nul A is the orthogonal complement to Col A.

6. Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Find a diagonal matrix D and an orthogonal matrix P such that $A = PDP^T$.

Solutions

1. The rank is 2 since we have the relation $2\mathbf{a}_2 = \mathbf{a}_1 + \mathbf{a}_3$, and $\mathbf{a}_1, \mathbf{a}_3$ are linearly independent. The dimension of Nul A equals $1 = 3 - 2$ (see Theorem 14 page 221).

2. (a)

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) The eigenvalue is 1 (see Theorem 1 page 239). The eigenspace is 1-dimensional spanned by 1.

(c) T is not diagonalizable since it does not have 4 linearly independent eigenvectors.

3. Over complex numbers every linear transformation has an eigenvalue. Let λ be an eigenvalue of T and W be the corresponding eigenspace. Then for any vector u in W we have

$$T(S(u)) = S(T(u)) = S(\lambda u) = \lambda S(u).$$

Therefore $S(u)$ also lies in W. Consider now S as a linear transformation $W \to W$. Then S has an eigenvector v in W. Then v is an eigenvector for both S and T.

4. Using Gram–Schmidt algorithm get

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

For orthogonal projection use Theorem 8 page 294. To get

$$\hat{y} = \frac{4}{3}v_1 - \frac{1}{2}v_2 = \begin{bmatrix} 0 \\ \frac{3}{2} \\ \frac{3}{2} \\ 1 \end{bmatrix}.$$

5. The easiest way is to use Theorem 3 on page 281 and the fact that $A^T = A$. Indeed,

$$(\text{Col}\, A^T)^\perp = (\text{Col}\, A)^\perp = \text{Nul}\, A.$$

6. Eigenvalues are -1 and 3 and eigenvectors are

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

They are orthogonal and to make them orthonormal we divide by $\sqrt{2}$. Hence

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}.$$