Sample midterm

1. Are vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

linearly independent in $\mathbb{R}^4$?

2. Determine if the following matrices are invertible and find the inverse when possible.

(a)

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix},$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{bmatrix}.$$

3. Find the area of the triangle in $\mathbb{R}^2$ whose vertices have coordinates $(1, 2), (3, 5), (4, 1)$.

4. Which of the following sets are subspaces in $\mathbb{R}^4$

(a) the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $x_1 + x_2 = 2(x_3 + x_4)$,

(b) the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $x_1 x_2 = x_3 x_4$,

(c) the set of all vectors $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ such that $x_1 + x_2 = x_3 + x_4 + 1$?

5. Let A be a matrix of size 7×5 and $T : \mathbb{R}^5 \to \mathbb{R}^7$ be the linear transformation defined by $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^5$. Assume that the sum of all entries in each row is zero.

(a) Is T onto?

(b) Is T one-to-one?

Justify you answers.

Solutions

1. Yes. The condition

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 = 0$$

implies

$$x_1 = 0, -x_1 + x_2 = 0, -x_2 + x_3 = 0.$$

This system has only one solution $x_1 = x_2 = x_3 = 0$.

2. The first matrix is not invertible because the columns are linearly dependent. Indeed

$$\begin{bmatrix}1\\4\\7\end{bmatrix} - 2\begin{bmatrix}2\\5\\8\end{bmatrix} + \begin{bmatrix}3\\6\\9\end{bmatrix} = 0.$$

The second matrix is invertible, its determinant is -6. The inverse matrix is

$$\begin{bmatrix}1 & \frac{1}{3} & -\frac{1}{3}\\ \frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{6} & \frac{1}{3}\end{bmatrix}.$$

3. This area is the half of the area of parallelogram defind by the vectors $\begin{bmatrix}2\\3\end{bmatrix}$ and $\begin{bmatrix}3\\-1\end{bmatrix}$. Hence the area is

$$|\frac{1}{2}\det\begin{bmatrix}2 & 3\\3 & -1\end{bmatrix}| = \frac{11}{2}.$$

4.

(a) Yes, the set is the null space of the matrix $\begin{bmatrix}1 & 1 & -2 & -2\end{bmatrix}$.

(b) No, because $u = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and $v = \begin{bmatrix}2\\2\\1\\4\end{bmatrix}$ are in the set but their sum $u + v = \begin{bmatrix}3\\3\\2\\5\end{bmatrix}$ is not.

(c) No, does not contain $\mathbf{0}$.

5.

(a) No. The maximal number of pivot positions is 5 which is less than the number of rows 7.

(b) No. The vector $\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$ is in the null space.