# PROBLEM SET # 9

Due April 3.

**1.** Let $\mathfrak{g}$ be a simple Lie algebra with generators $\{e_i, f_i, h_i \mid i = 1, \dots, n\}$ and simple roots $\alpha_1, \dots, \alpha_n$. Check that for any $h$ in the Cartan subalgebra:

$$\exp(\operatorname{ad} e_i) \exp(-\operatorname{ad} f_i) \exp(\operatorname{ad} e_i)(h) = h - \alpha_i(h) h_i.$$

**2.** Compute dimensions of the exceptional Lie algebras $F_4, E_6, E_7, E_8$.

**3.** Obtain the Lie algebra of type $B_n$, $C_n$ and $F_4$ as the set of fixed points of a suitable automorphisms of the Lie algebra $A_{2n}$, $A_{2n+1}$ and $E_6$ respectively.

**4.** Let $\mathfrak{g}_0$ be a simple Lie algebra with the Killing form $B$, $W$ and $W^*$ be simple $\mathfrak{g}_0$-modules dual to each other. Assume that there are non-zero $\mathfrak{g}_0$-invariant maps

$$T : \Lambda^3 W \to k, \quad T' : \Lambda^3 W^* \to k.$$

(a) Check that the maps $\eta : \Lambda^2 W \to W^*$, $\theta : \Lambda^2 W^* \to W$ defined by

$$\langle \eta(u \wedge v), w \rangle = T(u, v, w), \quad u, v, w \in W,$$
$$\langle \theta(u' \wedge v'), w' \rangle = T'(u', v', w'), \quad u', v', w' \in W^*,$$

and $\zeta : W \otimes W^* \to \mathfrak{g}_0$ defined by

$$B(\zeta(v \otimes w'), x) = \langle w', xv \rangle, \quad x \in \mathfrak{g}_0, v \in W, w' \in W^*$$

are homomorphisms of $\mathfrak{g}_0$-modules.

(b) Let $\mathfrak{g} = \mathfrak{g}_0 \oplus W \oplus W^*$. Define the bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ by setting

(1) $[x, y]$ is the same as in $\mathfrak{g}_0$ for $x, y \in \mathfrak{g}_0$,
(2) $[x, v] = xv$ for all $x \in \mathfrak{g}_0$ and $v \in W$,
(3) $[x, v'] = xv'$ for all $x \in \mathfrak{g}_0$ and $v' \in W^*$,
(4) $[v, w] = a\eta(v \wedge w)$ for all $v, w \in W$,
(5) $[v', w'] = b\theta(v' \wedge w')$ for all $v', w' \in W^*$,
(6) $[v, w'] = c\zeta(v \otimes w')$ for all $v \in W$ and $w' \in W^*$,

where $a, b, c \in k$. Show that one can choose $a, b, c \neq 0$ so that $\mathfrak{g}$ is a Lie algebra.

**5.** Apply the construction of Problem 5 in the following cases:

(a) $\mathfrak{g}_0 = \mathfrak{sl}(3)$, $W$ is the standard 3-dimensional $\mathfrak{sl}(3)$-module to obtain the Lie algebra of type $G_2$.

(b) $\mathfrak{g}_0 = \mathfrak{sl}(9)$, $W = \Lambda^3 V$, where $V$ is the standard 9-dimensional $\mathfrak{sl}(9)$-module to obtain the Lie algebra of type $E_8$.

*Date*: March 29, 2017.