PROBLEM SET # 8
MATH 261A

Due March 20.

1. (a) Check that $\mathfrak{sp}(4)$ and $\mathfrak{so}(5)$ (resp. $\mathfrak{sl}(4)$ and $\mathfrak{so}(6)$) have the same root systems.

(b) Let $V = \mathbb{C}^4$ denote the defining representation of $\mathfrak{sp}(4)$ and $\mathfrak{sl}(4)$. Use the representation of these Lie algebras in the second exterior square of V to prove the isomorphisms $\mathfrak{sp}(4) \simeq \mathfrak{so}(5)$ and $\mathfrak{sl}(4) \simeq \mathfrak{so}(6)$.

2. (Dual roots system). Let Δ be a reduced root system and

$$\Delta' := \{\frac{2\alpha}{(\alpha,\alpha)} \,|\, \alpha \in \Delta\}.$$

Prove that Δ' is a root system whose Cartan matrix is the transpose of the Cartan matrix of Δ and the Dynkin diagram of Δ' is obtained from that of Δ by inverting arrows.

3. (Root system F_4) Let e_1, e_2, e_3, e_4 be an orthonormal basis in a 4-dimensional Euclidean space E. Let $Q \subset E$ be the lattice generated by e_1, e_2, e_3, e_4 and $\frac{1}{2}(e_1 + e_2 + e_3 + e_4)$ and

$$\Delta = \{\alpha \in Q | \alpha \neq 0, (\alpha,\alpha) \leq 2\}.$$

(a) Prove that Δ is a root system.

(b) Find a base of Δ and check that its Dynkin diagram is F_4,.

4. Let W be the Weyl group of some root system and r_1, r_2 be two simple reflections. Prove that $(r_1 r_2)^m = 1$ for some positive integer m. Explain how to find such minimal positive m using the Dynkin diagram.

Date: March 15, 2017.