PROBLEM SET # 7

Due March 13.

1. Let $\mathfrak{g}$ be a semisimple Lie algebra over algebraically closed field. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and $\Delta \subset \mathfrak{h}^*$ denote the set of roots. Prove that $\mathfrak{g}$ is simple iff Δ is indecomposable, i.e. not a disjoint union of two orthogonal root systems.

2. Describe the root system of the Lie algebra $\mathfrak{so}(2n+1)$ for $n \geq 2$. (List all roots in terms of an orthonormal basis of $\mathfrak{h}^*$.)

3. Prove that the space of homogeneous polynomials $k[x, y]$ of degree $n \geq 0$ has the structure of an $\mathfrak{sl}(2)$-module with the action given by

$$e = x\frac{\partial}{\partial y}, \quad h = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad f = y\frac{\partial}{\partial x}.$$

Then prove that this is a simple module with highest weight n.

4. Let V_n denote a simple module $\mathfrak{sl}(2)$-module with highest weight n. Show that for $n \geq m \geq 0$

$$V_n \otimes V_m \simeq V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m+2} \oplus V_{n-m}.$$

Date: March 3, 2017.