PROBLEM SET # 2

Due February 6.

1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that for any $\xi \in \mathfrak{g}$

$$\exp \operatorname{ad}(\xi) = \operatorname{Ad} \exp(\xi).$$

2. Let G be a connected compact complex Lie group. Prove that G is abelian. (Hint: observe that $\operatorname{Ad} : G \to \operatorname{End}_{\mathbb{C}}(\mathfrak{g})$ is a holomorphic map and use Liouville's theorem.)

3. Consider the action of $SL(2, \mathbb{C})$ on the complex projective line $\mathbb{P}^1$ given in local coordinates by linear fractional maps

$$x \mapsto \frac{ax + b}{cx + d}.$$

Describe the image of the Lie algebra $sl(2)$ in the Lie algebra of vector fields on $\mathbb{P}^1$ in local coordinates.

4. Consider the representation of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ in the space of matrices $M(2, \mathbb{C})$ defined by

$$\rho(A, B)(X) = AXB^{-1} \text{ for all } (A, B) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C}), X \in M(2, \mathbb{C}).$$

(a) Find the kernel of ρ.

(b) Prove that the action of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ on $M(2, \mathbb{C})$ preserves the quadratic form $\det(X)$.

(c) Prove that the Lie algebras of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ and $SO(4, \mathbb{C})$ are isomorphic.

Date: January 26, 2017.