PROBLEM SET # 12

Due April 24.

1. Let G be an algebraic group over $\mathbb{C}$ and $\mathcal{O}(G)$ be the Hopf algebra of regular functions on G. Let V be a G-module. Check that the action map $G \times V \to V$ induces the comodule map

$$\rho : V \to V \otimes \mathcal{O}(G)$$

which satisfies the conditions

$$(\rho \otimes \mathrm{id}_{\mathcal{O}(G)}) \circ \rho = (\mathrm{id}_V \otimes \Delta) \circ \rho,$$
$$(\mathrm{id}_V \otimes \varepsilon) \circ \rho = \mathrm{id}_V,$$

here Δ is the comultiplication and ε is the counit.

2. In notation of problem one, consider the G-module $\mathcal{O}(G)$ with action induced by the right multiplication on G. Prove that for any $f \in \mathcal{O}(G)$ there exists a finite-dimensional G-submodule containing f.

3. Let G be a connected compact real Lie group. Prove that left and right invariant volume forms on G coincide.

4. Prove that a real form of $\mathfrak{sl}(n, \mathbb{C})$ is isomorphic to one of the following:

(1) $\mathfrak{sl}(n, \mathbb{R})$;
(2) $\mathfrak{su}(p, q)$, the Lie algebra of traceless matrices which preserve the Hermitian form of signature (p, q), where $p + q = n$;
(3) $\mathfrak{sl}(\frac{n}{2}, \mathbb{H})$, the traceless $\mathbb{H}$-linear operators in $\mathbb{H}^{\frac{n}{2}}$ for even n.

Date: April 17, 2017.