PROBLEM SET # 11

Due April 17.

In this homework $\mathfrak{g}$ is isomorphic to the symplectic Lie algebras $\mathfrak{sp}(2n)$ over complex numbers and V denotes the standard $2n$-dimensional representation of $\mathfrak{g}$.

1. Prove that S^2V is isomorphic to the adjoint representation $\mathfrak{g}$.

2. Let $L_1, \dots, L_n$ be the fundamental representations of $\mathfrak{g}$ and L_0 denote the trivial representation.

(a) Show for $2 \leq k \leq n$ the k-th exterior power $\Lambda^k V$ is isomorphic to the direct sum $L_k \oplus \Lambda^{k-2}V$.

(b) Let $\omega \in \Lambda^2$ denote the $\mathfrak{g}$-invariant bivector. Define the operators $e, f, h : \Lambda V \to \Lambda V$ by

$$e(x) := \omega \wedge x, \quad f = e^*, \quad h(x) = (n-k)x$$

for any $x \in \Lambda^k V$. Prove that e, f, h commute with the action of $\mathfrak{g}$ and generate the Lie algebra isomorphic to $\mathfrak{sl}(2)$.

(c) Prove that

$$\Lambda V \simeq \bigoplus_{k=0}^{n} L_k \otimes V_{n-k},$$

where V_{n-k} is the irreducible representation of $\mathfrak{sl}(2)$ with highest weight $n-k$.

3. Let $\langle \cdot, \cdot \rangle$ denote a $\mathfrak{g}$-invariant symplectic form on V. Define the Weyl algebra A_n as the quotient of the tensor algebra $T(V)$ by the ideal generated by $[x, y] - \langle x, y \rangle$ for $x, y \in V$. Prove that the span of $\{xy + yx \mid x, y \in V\}$ form a Lie subalgebra of A_n isomorphic to $\mathfrak{g}$.

Date: April 10, 2017.