PROBLEM SET # 10

Due April 9.

In this set $\mathfrak{g}$ is a semisimple Lie algebra P is the weight lattice and Q is a root lattice and Q^+ denotes the positive cone of Q.

1. Let $\varphi : M(\lambda) \to M(\mu)$ be a non-trivial homomorphism of Verma modules. Prove that $\mu - \lambda \in Q^+$ and φ is injective.

2. Let $L(\lambda)$ denote the simple finite-dimensional $\mathfrak{g}$-module with highest weight λ.

(a) Prove that the dual module $L(\lambda)^*$ is isomorphic to $L(-w_0\lambda)$, where w_0 is the longest element of the Weyl group.

(b) If the Dynkin diagram of $\mathfrak{g}$ has no symmetries, then every simple finite-dimensional $\mathfrak{g}$-module is isomorphic to its dual.

3. Prove that the order of the finite group P/Q equals the determinant of the Cartan matrix.

4. Prove that for $\mathfrak{g} = \mathfrak{sl}(n+1)$ the finite abelian group P/Q is cyclic of order $n+1$ by checking that the class of the first fundamental weight is a generator of P/Q.

Date: April 3, 2017.