PROBLEM SET # 5

Due March 4.

Assumptions: all Lie algebras are finite-dimensional and the ground field has characteristic zero.

1. Show that the classical Lie algebras $\mathfrak{sl}(n, \mathbb{C})$ for $n \geq 2$, $\mathfrak{so}(n, \mathbb{C})$ for $n \geq 3$ and $\mathfrak{sp}(2n, \mathbb{C})$ for $n \geq 1$ are semisimple. (The algebra $\mathfrak{sp}(2n, \mathbb{C})$ is the Lie algebra of the group which preserves a skew-symmetric non-degenerate form in $\mathbb{C}^{2n}$.)

2. Prove that the following conditions on a Lie algebra $\mathfrak{g}$ are equivalent:

(1) The adjoint $\mathfrak{g}$-module is semisimple.
(2) $[\mathfrak{g}, \mathfrak{g}]$ is a semisimple Lie algebra.
(3) $\mathfrak{g}$ is a direct sum of a semisimple Lie algebra and an abelian Lie algebra.

A Lie algebra satisfying these conditions is called *reductive*.

3. Show that a Lie algebra of a compact Lie group is reductive.

4. Assume that a Lie algebra $\mathfrak{g}$ is a direct sum (as a vector space) of a subalgebra $\mathfrak{h}$ and an ideal $\mathfrak{n}$. Furthermore assume that H and N are the corresponding simply connected connected Lie groups. Define a semidirect product $G = H \ltimes N$ in such a way that $\mathfrak{g} = \operatorname{Lie} G$. (Hint: lift the homomorphism of Lie algebras $\mathfrak{h} \to \operatorname{Der}(\mathfrak{n})$ to a homomorphism $H \to \operatorname{Aut}(N)$.)

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