PROBLEM SET 4

Due February 26.

1. Let $\mathfrak{g}$ be a simple Lie algebra over algebraically closed field of characteristic 0. Show that any invariant symmetric form on $\mathfrak{g}$ is proportional to the Killing form.

2. Let $\mathcal{H}$ be the 3-dimensional Heisenberg Lie algebra, i.e. the algebra with a basis X, Y, Z such that $Z = [X, Y], [Z, X] = [Z, Y] = 0$. Let $M = k[x]$. Prove that M is a simple $\mathcal{H}$-module with $\mathcal{H}$-action defined by

$$Xf = \frac{d}{dx}f,\ Yf = xf,\ Zf = f$$

for any $f \in M$.

3. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Show that $\mathfrak{g}$ is nilpotent (resp., solvable, semisimple) if and only if $\mathfrak{g}_{\mathbb{C}}$ is nilpotent (resp., solvable, semisimple)

4. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. A linear map $\varphi : \mathfrak{g} \to \mathfrak{g}$ is an *automorphism* of $\mathfrak{g}$ if

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$$

and a *derivation* of $\mathfrak{g}$ if

$$\varphi([X, Y]) = [\varphi(X), Y] + [X, \varphi(Y)].$$

Let $\mathrm{Aut}(\mathfrak{g})$ denote the group of automorphisms of $\mathfrak{g}$.

(a) Show that $\mathrm{Aut}(\mathfrak{g})$ is a closed subgroup of $GL(\mathfrak{g})$ and hence a Lie group.

(b) Show that $\mathrm{Lie}\,\mathrm{Aut}(\mathfrak{g})$ coincides with the Lie algebra of all derivations of $\mathfrak{g}$.