PROBLEM SET # 2
MATH 261A

Due February 12/

1. Let G be a Lie group,

$$\Phi : G \to GL(V), \quad \Psi : G \to GL(W)$$

two representations of G.

(a) Check that the formula

$$g(v \otimes w) = gv \otimes gw, \quad g \in G,\ v \in V,\ w \in W$$

defines a representation $\Gamma : G \to GL(V \otimes W)$.

(b) Show that the corresponding representation of $\operatorname{Lie} G$ in $V \otimes W$ is defined by the formula

$$X(v \otimes w) = Xv \otimes w + v \otimes Xw, \quad X \in \operatorname{Lie} G,\ v \in V,\ w \in W.$$

2. Let G, H be connected Lie groups and $\Phi : G \to H$ be a homomorphism. Show that

(a) Φ is surjective if and only if $d\Phi : \operatorname{Lie} G \to \operatorname{Lie} H$ is surjecive.

(b) $\operatorname{Ker} \Phi$ is a discrete subgroup of G if and only if $d\Phi$ is injective.

3. Assume that G, H and Φ are the same as in the previous problem. Assume that the image $\Phi(H)$ is normal in G. Check that $d\Phi(\operatorname{Lie} H)$ is an *ideal* in $\operatorname{Lie} G$, i.e. if $X \in \operatorname{Lie} G, Y \in d\Phi(\operatorname{Lie} H)$ then $[X, Y] \in d\Phi(\operatorname{Lie} H)$.

4. Let G be a connected Lie group and Z be its center. Show that $\operatorname{Lie} Z$ coincides with the *center* of $\operatorname{Lie} G$. (The center of a Lie algebra $\mathfrak{g}$ is the set of all $X \in \mathfrak{g}$ such that $[X, \mathfrak{g}] = 0$.)