PROBLEM SET # 1

Due February 5.

1. Let $SU(n)$ be the group of unitary complex $n \times n$-matrices with determinant 1:

$$SU(n) = \{U \in \mathrm{Mat}(n) \mid U\bar{U}^T = I_n, \det U = 1\}.$$

(a) Show that $SU(n)$ is compact and connected.

(b) Describe $\mathrm{Lie}\, SU(n)$ as the subspace in $\mathrm{Mat}(n)$.

2. Consider the quadratic form in $\mathbb{R}^4$ given by the formula:

$$q(\mathbf{x}) = x_1^2 - x_2^2 - x_3^2 - x_4^2.$$

The Lorentz group $O(1,3)$ is the group of linear transformation in $\mathbb{R}^4$ preserving q

$$O(1,3) = \{A \in GL(4) \mid q(A\mathbf{x}) = q(\mathbf{x})\}.$$

(a) Describe $\mathrm{Lie}\, O(1,3)$ and compute its dimension.

(b) How many connected compenents does the Lorentz group have? Hint: consider the action of $O(1,3)$ on the surface $q(\mathbf{x}) = 1$.

3. Consider the action of $SL(2,\mathbb{C})$ on the space H_2 of Hermitian 2×2-matrices defined by $AX\bar{A}^T$ for any $A \in SL(2,\mathbb{C})$ and $X \in H_2$.

(a) Show that det defines a quadratic form on H_2 and $SL(2,\mathbb{C})$ preserves this form.

(b) Use (a) to construct a homomorphsim $\phi : SL(2,\mathbb{C}) \to O(1,3)$.

(c) Compute the kernel and the image of ϕ.

4. Check that the exponential map $\mathrm{Lie}\, H \to H$

(a) is surjective for $H = SO(3)$,

(b) is not surjective for $H = SL(2,\mathbb{R})$.

Hint: In the first case every element is a rotation. In the second case $\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not in the image.

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